1. There are three objects:

$SL_2(\mathbb{C})$ – ~~ea~~ algebraic group
$SU(2)$ – compact Lie group
$sl_2(\mathbb{C})$ – Lie algebra

All have the same finite dimensional representations.

~~There is a three dim~~

2. $sl(2)$ is a three dimensional ~~a~~ Lie algebra (over any field).

A basis is $\{E, H, F\}$ and the Lie bracket are

$$[E,F]=H \qquad [E,H]=-E \qquad [F,H]=F$$

This is also the Lie algebra of 2×2 matrices ~~with trace zero~~. whose trace is zero.

$$E=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For the fields $\mathbb{R}, \mathbb{C}$ ~~these of~~ exponentiating gives the one parameter subgroups

$$\exp(zE)=\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \qquad \exp(zH)=\begin{pmatrix} \exp(z) & 0 \\ 0 & \exp(-z) \end{pmatrix}$$

$$\exp(zF)=\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

3. This Lie algebra acts on the vector space $\mathbb{C}[x, y]$ by

$$E \mapsto y\frac{\partial}{\partial x} \qquad H \mapsto y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} \qquad F \mapsto y\frac{\partial}{\partial x}$$

For each $d \geq 0$ homogeneous polynomials of degree d is a representation of dimension $d+1$.

Put $v_{r,s} = \dfrac{(r+s)!}{r!\,s!} x^r y^s$

$$E v_{r,s} = (s+1) v_{r-1,s+1} \quad H v_{r,s} = (r-s) v_{r,s} \quad F v_{r,s} = (r+1) v_{r+1,s-1}.$$

4. For a group we ~~define~~ use ~~the exp~~

- coproduct $g \mapsto g \otimes g$
- counit $g \mapsto 1$
- antipode $g \mapsto g^{-1}$.

For a Lie algebra we use

- coproduct $g \mapsto 1 \otimes g + g \otimes 1$
- counit $g \mapsto 0$
- antipode $g \mapsto -g$.

5. Then we have Clebsch-Gordan decomposition

$$V(a) \otimes V(b) = \bigoplus_{i=0}^{\min(a,b)} V(a+b-2i).$$

and then 6j-symbols.

6. Every finite dimensional representation is completely reducible and can be decomposed using the Casimir

7. Quantum group. This has generators E, F, K, K^{-1} and defining relations over $\mathbb{C}(q)$

$$K.K^{-1} = 1 = K^{-1}.K \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$KEK^{-1} = q^{2} E \qquad KFK^{-1} = q^{-2} F.$$

(physics papers have $K = q^{H}$).

$$[n] = \frac{q^{n} - q^{-n}}{q - q^{-1}}.$$

~~For a~~ There is the trivial representation

$$E \mapsto 0 \quad F \mapsto 0 \quad K \mapsto 1.$$

and for ~~each~~ $a > 0$ we have <u>two</u> representations of dimension $a+1$.

$$E v_{r,s} = [s+1]\, v_{r-1,s+1} \qquad F v_{r,s} = [r+1]\, v_{r+1,s-1}$$

~~$H v_{r,s}$~~ ~~K~~ $K v_{r,s} = q^{r-s} v_{r,s}$ <u>or</u> $K v_{r,s} = q^{s-r} v_{r,s}$.

<u>Not</u> a deformation!

8. Tensor product. / Coproduct.

Always have $\Delta(K) = K \otimes K$.

i $\quad \Delta(E) = E \otimes 1 + K \otimes E \qquad \Delta(F) = F \otimes K^{-1} + 1 \otimes F$

ii $\quad \Delta(E) = E \otimes 1 + K^{-1} \otimes E \qquad \Delta(F) = F \otimes K + 1 \otimes F$.

with respective antipodes

i. $E \mapsto -K^{-1}E \qquad F \mapsto -FK \qquad K^{\pm 1} \mapsto K^{\mp 1}$.

ii. $E \mapsto -KE \qquad F \mapsto -FK^{-1} \qquad K^{\pm 1} \mapsto K^{\mp 1}$.

Still have Clebsch-Gordan decomposition.

Casimir is:

$$(q-q^{-1})^2 EF + q^{-1}K + qK^{-1}$$

$$= (q-q^{-1})^2 FE + qK + q^{-1}K^{-1}.$$

9. Temperley-Lieb

$$u_i^2 = \delta u_i$$
$$u_i u_{i\pm1} u_i = u_i$$
$$u_i u_j = u_j u_i \quad \text{if } |i-j|>1.$$

$u_1 = \substack{\cup\\ \cap}\,|\cdots|$ $\quad u_2 = |\substack{\cup\\ \cap}\,|\cdots|$ $\quad u_3 = ||\substack{\cup\\ \cap}\,|\cdots|$ etc.

10. Put $\delta = q + q^{-1}$. V^2 the spin $\frac{1}{2}$, weight 1, dimension 2 rep.

Take $V \otimes V$ with ordered basis $++$ $+-$ $-+$ $--$.

~~This decompo~~ As a representation of $U_q(sl(2))$

$$V^2 \otimes V^2 \cong V^3 \oplus V^1.$$

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e = \frac{1}{q+q^{-1}}\, u \quad \text{is idempotent}$$

This projects to trivial representation.

$$E = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q & 0 \end{pmatrix} \quad K = \begin{pmatrix} q^{2} & & & \\ & 1 & & \\ & & 1 & \\ & & & q^{-2} \end{pmatrix}.$$

Pauli spin matrices $\underset{\sigma_x}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\underset{\sigma_y}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$ $\underset{\sigma_z}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$.

XYZ Hamiltonian

$$H = -\frac{1}{2}\sum_{j=1}^{N} J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z .$$

"is effectively a logarithmic derivative of an eight vertex transfer matrix"

XYZ Hamiltonian, put $J_y = J_x$.

$$\begin{pmatrix} J_z & & & J_x - J_y \\ & -J_z & \cancel{2} J_x + J_y & \\ & \cancel{2} J_x + J_y & -J_z & \\ J_x - J_y & & & J_z \end{pmatrix}.$$

z-measures. $\{\delta_\lambda\}$ basis indexed by partitions

$$U\delta_\lambda = \sum_{\mu=\lambda+\square} (z+c(\square))\,\delta_\mu$$

$$L\delta_\lambda = (zz'+2|\lambda|)\,\delta_\lambda$$

$$D\delta_\lambda = \sum_{\mu=\lambda-\square} (z'+c(\square))\,\delta_\mu .$$

$$[D,U]=L \qquad [L,U]=2U \qquad [L,D]=-2D.$$

$\{v_k\}$ basis indexed by $\mathbb{Z}+1/2$.

$$U v_k = (z+k+1/2)\, v_{k+1}$$

$$L v_k = (2k+z+z')\, v_k$$

$$D v_k = (z'+k-1/2)\, v_{k-1}$$

This has q-analogue

$$E v_k = [z+k+1/2]\, v_{k+1}$$

$$K v_k = q^{2k+z+z'}\, v_k .$$

$$F v_k = [z'+k-1/2]\, v_{k-1}$$

INVARIANT TENSORS FOR BINARY FORMS

BRUCE W. WESTBURY

ABSTRACT. These are notes for a talk.

1. Binary forms

1.1. Classical. For $n \geq 2$ we have an algebraic group $\mathrm{SL}(n)$ and a Lie algebra $\mathfrak{sl}(n)$. The group $\mathrm{SL}(n)$ is the group of $n \times n$ matrices whose determinant is 1 and the Lie algeba $\mathfrak{sl}(n)$ is the vector space of $n \times n$ matrices whose trace is 0. For $n = 2$ the Lie algebra has dimension 3 and a basis is given by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Lie brackets are given by

$$[E, F] = H \qquad [E, H] = -E \qquad [F, H] = F$$

These are the infintesimal generators of the three one parameter subgroups of $\mathrm{SL}(2)$

$$\exp(zE) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \qquad \exp(zF) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \qquad \exp(zH) = \begin{pmatrix} \exp(z) & 0 \\ 0 & \exp(-z) \end{pmatrix}$$

The universal enveloping algebra, U, is a cocommutative Hopf algebra. The algebra U is generated by elements E, F, H and defining relations are

$$EF - FE = H \qquad EH - HE = -E \qquad FH - HF = F$$

The coproduct is an algebra homomorphism $\Delta \colon U \to U \otimes U$ and so is determined by the map on generators. This map is

$$\Delta(E) = E \otimes 1 + 1 \otimes E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

There is a representation of $\mathrm{SL}(2)$ on the polynomial ring $\mathbb{C}[x, y]$. The associated action of $\mathfrak{sl}(2)$ is given by the following homomorphism $U \to W$ where W is a Weyl algebra

$$E \mapsto y\frac{\partial}{\partial x} \qquad F \mapsto x\frac{\partial}{\partial y} \qquad H \mapsto y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$$

For each $d \geq 0$ let $V(d)$ be the subspace of $\mathbb{C}[x, y]$ of homogeneous polynomials of degree d. Then the decomposition of vector spaces $\mathbb{C}[x, y] \cong \bigoplus_{d \geq 0} V(d)$ is also a decomposition of representations.

Furthermore, in characteristic zero, each representation $V(d)$ is irreducible and every irreducible representation is isomorphic to $V(d)$ for a unique $d \geq 0$.

These representations can be given explicitly. For $r, s \geq 0$ put $v_{r,s} = \binom{r+s}{r} x^r y^s$. Then we have

$$Ev_{r,s} = (s+1)v_{r-1,s+1} \qquad Fv_{r,s} = (r+1)v_{r+1,s-1} \qquad Hv_{r,s} = (r-s)v_{r,s}$$

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Note that $\{v_{r,s} | r+s=d\}$ is a basis of $V(d)$ for each $d \geq 0$.

1.2. **Quantum.** The quantised enveloping algebra or Drinfled-Jimbo quantum group, U_q, is a Hopf algebra over $\mathbb{Q}(q)$. The algebra U is generated by elements E, F, K, K^{-1} and defining relations are

$$KK^{-1} = 1 = K^{-1}K$$
$$KEK^{-1} = q^2 E$$
$$KFK^{-1} = q^{-2} F$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

The counit $\varepsilon \colon U_q \to \mathbb{Q}(q)$ is determined by

$$E \mapsto 0 \qquad F \mapsto 0 \qquad K^{\pm 1} \mapsto 1$$

The coproduct is an algebra homomorphism $\Delta \colon U_q \to U_q \otimes U_q$ and so is determined by the map on generators. This map is

$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$
$$\Delta(K) = K \otimes K$$

The antipode $S \colon U_q \to U_q^{\mathrm{op}}$ is determined by

$$E \mapsto -K^{-1}E \qquad F \mapsto -FK \qquad K^{\pm 1} \mapsto K^{\mp 1}$$

The coproduct is not unique. For example an alternative is

$$\Delta(E) = E \otimes 1 + K^{-1} \otimes E$$
$$\Delta(F) = F \otimes K + 1 \otimes F$$
$$\Delta(K) = K \otimes K$$

The antipode is determined by the coproduct and for this alternative is

$$E \mapsto -KE \qquad F \mapsto -FK^{-1} \qquad K^{\pm 1} \mapsto K^{\mp 1}$$

One sequence of representations is given by

$$Ev_{r,s} = [s+1]v_{r-1,s+1} \qquad Fv_{r,s} = [r+1]v_{r+1,s-1} \qquad Kv_{r,s} = q^{r-s}v_{r,s}$$

For each $d \geq 0$ let $V_+(d)$ be the vector space with basis $\{v_{r,s} | r+s=d\}$. These are irreducible representations.

Another sequence of representations is given by

$$Ev_{r,s} = [s+1]v_{r-1,s+1} \qquad Fv_{r,s} = [r+1]v_{r+1,s-1} \qquad Kv_{r,s} = q^{-r+s}v_{r,s}$$

For each $d \geq 0$ let $V_-(d)$ be the vector space with basis $\{v_{r,s} | r+s=d\}$. These are irreducible representations.

Then every finite dimensional irreducible representation is isomorphic to precisely one of these.

2. Temperley-Lieb

Let $V = V_+(1)$. Define the category of invariant tensors to have objects $\mathbb{N}$ and morphisms $\mathrm{Hom}(n,m) = \mathrm{Hom}_{U_q}(\otimes^n V, \otimes^m V)$. This category is isomorphic to the Temperley-Lieb category.

The Temperley-Lieb category is generated as a monoidal category by

Defining relations are

The r-string Temperley-Lieb algebra is generated by $\{u_i|1 \leq i \leq r-1\}$ and defining relations are

$$\begin{aligned} u_i^2 &= (q+q^{-1})u_i \\ u_i u_{i\pm1} u_i &= u_i \\ u_i u_j &= u_j u_i \text{ for } |i-j|>1 \end{aligned}$$

This is the endomorphism algebra of r in the Temperley-Lieb category.

Then there is a homomorphism from the r-string braid group to the r-string Temperley-Lieb algebra. This is given on the generators by

$$\sigma_i^{\pm1} \mapsto q^{\pm1} - u_i$$

This also satisfies the tangle relations

$$u_i u_{i\pm1} \sigma_i^{\pm1} = u_i \sigma_{i\pm1}^{\mp1} \qquad \sigma_i^{\pm1} u_{i\pm1} u_i = \sigma_{i\pm1}^{\mp1} u_i$$

There is also a solution to the Yang-Baxter equation given by

$$R_i(u) = \frac{u\sigma_i - u^{-1}\sigma^{-1}}{q-q^{-1}} = \left(\frac{uq - u^{-1}q^{-1}}{q-q^{-1}}\right) - \left(\frac{u-u^{-1}}{q-q^{-1}}\right) u_i$$

This is normalised to satisfy $R_i(1) = 1$.

This satisfies

$$R_i(u)R_{i+1}(uv)R_i(v) = R_{i+1}(v)R_i(uv)R_{i+1}(u)$$

This also satisfies

$$u_i u_{i\pm1} R_i(u) = u_i R_{i\pm1}(-u^{-1}) \qquad R_i(u) u_{i\pm1} u_i = R_{i\pm1}(-u^{-1}) u_i$$

These are the single bond transfer matrices in the six vertex model and in the Potts model.