EQUIP: Research Challenges and Goals

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Enabling Quantification of
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Uncertainty for Inverse Problems

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Outline

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Motivation

- Aim: to solve $y = \mathcal{G}(u) + \text{noise}$ for u given y.
- PDEs: $\mathcal{G}(\cdot)$ defined through solution of complex PDE.
- Probability/Statistics: data y and model $\mathcal{G}$ are uncertain.
- LSQ: minimize (regularized) $\Phi(u; y) := \frac{1}{2}\|y - \mathcal{G}(u)\|^2$.
- Bayesian: (u, y) a jointly varying random variable; find $u|y$.

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Bayesian Inversion

- Unknown $u \in X$.
- Data $y \in \mathbb{R}^J$.
- Prior $u \sim \mathbb{P}(u)$.
- Likelihood $\mathbb{P}(y|u)$: $y|u \sim N(\mathcal{G}(u), \Gamma)$.
- Bayes' Theorem: $\mathbb{P}(u|y) \propto \mathbb{P}(y|u) \times \mathbb{P}(u)$.
- Posterior: $\mathbb{P}(u|y) \propto \exp\Big(-\Phi(u;y)\Big)\mathbb{P}(u)$.
- Potential: $\Phi(u;y) := \frac{1}{2}\Big\|\Gamma^{-\frac{1}{2}}\big(y - \mathcal{G}(u)\big)\Big\|^2$.

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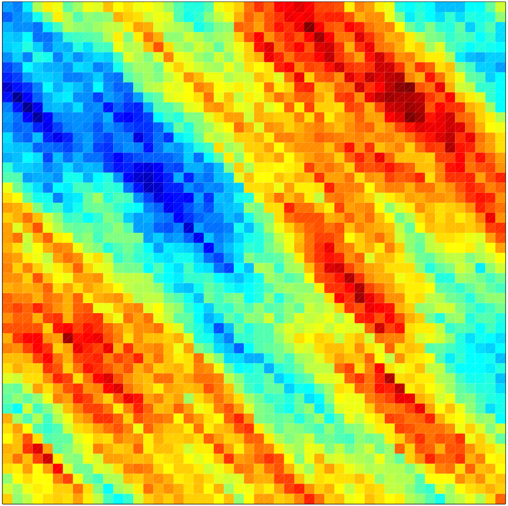
Gaussian Priors



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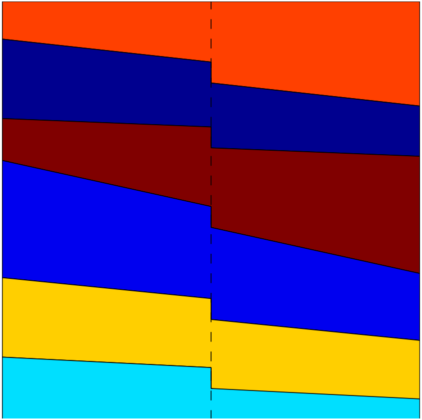
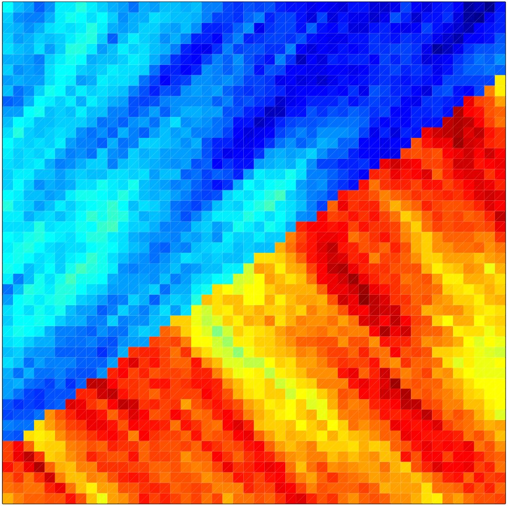
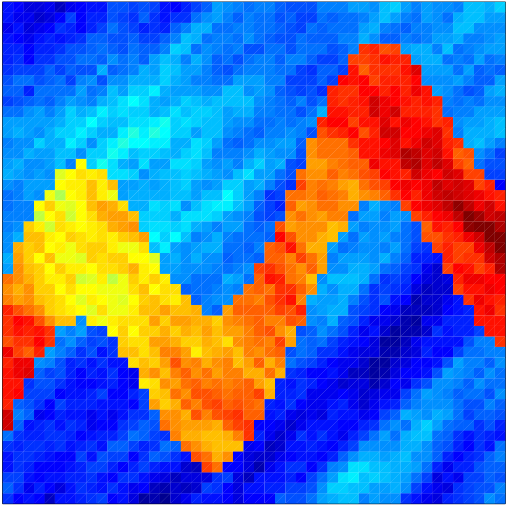
IC Fault Model



Fault/Gaussian



Channel/Gaussian



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Random Functions

- $\{\phi_j\}_{j=0}^{\infty}$ an infinite sequence of functions in X.
- Construct function u by, for sequence $u_j = \gamma_j \xi_j$,

$$u = \phi_0 + \sum_{j=1}^{\infty} u_j \phi_j.$$

- The deterministic sequence $\gamma = \{\gamma_j\}_{j=1}^{\infty}$.
- The i.i.d. centred random sequence $\xi = \{\xi_j\}_{j=1}^{\infty}$.
- Permeability $\kappa = \exp(u)$.

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Random Geometry

- Consider permeability defined on a domain D with

$$\bigcup_i D_i = D, \quad D_i \cap D_j = \varnothing, \forall i \neq j.$$

- Set

$$\kappa(x) = \sum_{i=1}^{n} \kappa^{(i)} \chi_{D_i}(x).$$

- Parameterize the $\{D_i\}$ by a finite set of random geometric parameters $a \in \mathbb{R}^N$.
- Parameterize $\kappa^{(i)}$ as on previous slide through an infinite set of random parameters.

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Groundwater Flow Inversion

- Let $f \in H^{-1}(D)$. Find log permeability $u \in X = L^{\infty}(D)$:

$$-\nabla \cdot \left(e^{u} \nabla p\right) = f, \quad x \in D$$
$$p = 0, \quad x \in \partial D.$$

- Given, for $j = 1, \ldots, J$,

$$y_j = \ell_j(p) + \eta_j, \;\; \ell_j \in H^{-1}(D), \;\; \eta_j \text{ noise.}$$

- Abstractly: for $\mathcal{G} : X \mapsto Y = \mathbb{R}^J$ find u given

$$y = \mathcal{G}(u) + \eta, \text{ noise.}$$

Two Phase Flow

- Find log permeability $u \in X = L^{\infty}(D)$:

$$\begin{aligned} -\nabla \cdot \left[\lambda(s)e^{u}\nabla p\right] &= f_{\mathrm{P}} + f_{\mathrm{I}}(s,p) \quad \text{in } D \times [0,T], \\ \phi\frac{\partial s}{\partial t} - \nabla \cdot \left[\lambda_{w}(s)e^{u}\nabla p\right] &= g_{\mathrm{P}} + g_{\mathrm{I}}(s,p) \quad \text{in } D \times [0,T]. \end{aligned}$$

- Given, for $(j,k) = (1,1)\dots,(J,K)$,

$$y_{j,k} = f_{\mathrm{I}}\big(s(x_j,t_k),p(x_j,t_k)\big) + \eta_j, \;\; \eta_j \text{ noise.}$$

- Abstractly: for $\mathcal{G} : X \mapsto Y = \mathbb{R}^J$ find u given

$$y = \mathcal{G}(u) + \eta, \text{ noise.}$$

Black Box Software

- Find log permeability $u \in X = L^\infty(D)$:

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- Given, for $j = 1, \dots, J$,

$$y_j = \textit{FIELD DATA}.$$

- Abstractly: for $\mathcal{G} : X \mapsto Y = \mathbb{R}^J$ find u given

$$y = \mathcal{G}(u) + \eta, \text{ noise.}$$

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High Dimensional Probability Measures

- There is a prior probability measure μ_0 on X.
- The posterior measure of interest is μ^y, also on X.
- μ^y is related to μ_0 by

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z}\exp\Big(-\Phi(u)\Big).$$

- Since $\mu^y(du) = Z^{-1}\exp\big(-\Phi(u)\big)\mu_0(du)$ we have

$$\mathbb{E}^{\mu^y} f(u) = \frac{1}{Z}\mathbb{E}^{\mu_0}\Big(\exp\big(-\Phi(u)\big)f(u)\Big).$$

- We want to extract information from μ^y: most likely u, confidence bands on predictions, rare events . . .

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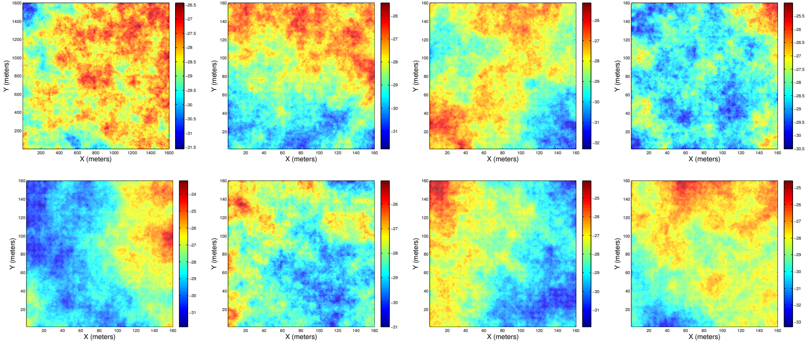
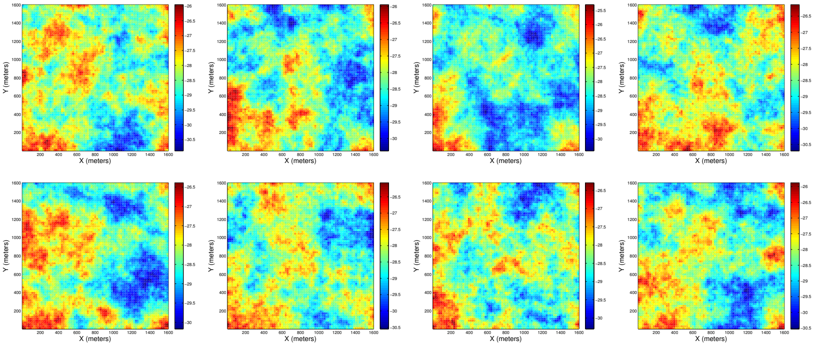
Prior Geology



Posterior Geology (include production data)



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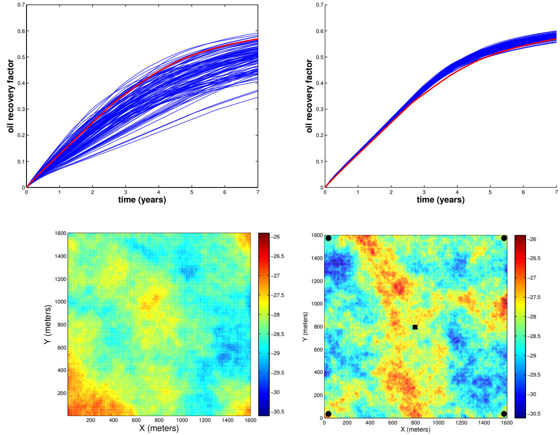
Geology with data ⇒ smaller uncertainty in prediction.
Geology without data ⇒ larger uncertainty in prediction.



Model Error

- What is forward model $\mathcal{G}$?
- Computational forward model $\mathcal{G}_{\text{comp}}$.
- True forward model $\mathcal{G}_{\text{true}}$.
- Model error = model mis-specification.
- Try to learn about model error from data.

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Ad Hoc Algorithms

- Many algorithms used in practice are not well-founded from statistical point of view.
- The primary example is the ensemble Kalman filter.
- Need to develop theory for existing such algorithms, as they are currently used.
- Need to develop modifications of such algorithms to ensure statistical realibility.

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Fundamental Research Challenges

The three challenges:

- **High Dimensional Probability Measures**
- **Model Error**
- **Ad Hoc Algorithms**

all interact. Progress requires:

- confronting each individual challenge;
- appreciation for the interaction between challenges.