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Brief paper

A quadrature-based method of moments for nonlinear filtering ☆

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ABSTRACT

According to the nonlinear filtering theory, optimal estimates of a general continuous-discrete nonlinear filtering problem can be obtained by solving the Fokker–Planck equation, coupled with a Bayesian update rule. This procedure does not rely on linearizations of the dynamical and/or measurement models. However, the lack of fast and efficient methods for solving the Fokker–Planck equation presents challenges in real time nonlinear filtering problems. In this paper, a direct quadrature method of moments is introduced to solve the Fokker–Planck equation efficiently and accurately. This approach involves representation of the state conditional probability density function in terms of a finite collection of Dirac delta functions. The weights and locations (abscissas) in this representation are determined by moment constraints and modified using the Bayes' rule according to measurement updates. As demonstrated by numerical examples, this approach appears to be promising in the field of nonlinear filtering.

1. Introduction

Our interest in filtering and prediction problems, in which the dynamics are continuous and observations are discrete, stems from our interest in orbit determination. Jazwinski (1966, 1970), Kushner (1967), Kushner and Budhiraja (2000) and Stratonovich (1959) are some of the early pioneers who studied the recursive Bayesian type nonlinear filtering technique where the probability density function (PDF) associated with states is updated and estimated using the incoming measurements.

In a classical nonlinear filtering problem, the system is modeled as an n-dimensional continuous Itô stochastic differential equation (SDE)

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), t) + \mathbf{G}(\mathbf{x}(t), t)\,\mathbf{w}(t) \quad t \geq t_0 \tag{1}$$

where $\mathbf{x} = [x_i]_{i=1,\ldots,N_s} \in \Re^{N_s \times 1}$, $\mathbf{F} = [F_i]_{i=1,\ldots,N_s} \in \Re^{N_s \times 1}$, and $\mathbf{G}(\mathbf{x}(t), t) \in \Re^{N_s \times N_w}$ are the state vector, state function, and diffusion matrix respectively. $\mathbf{w}(t) \in \Re^{N_w \times 1}$ is the vector of the zero-mean Gaussian process with an autocorrelation of $E[\mathbf{w}(t)\,\mathbf{w}(\tau)^{\mathrm{T}}] = \mathbf{Q}(t)\,\delta(t-\tau)$. The measurement $\mathbf{y}(t_k)$ taken at discrete time instants t_k is defined as

$$\mathbf{y}(t_k) = \mathbf{h}(\mathbf{x}(t_k), t_k) + \boldsymbol{\nu}(t_k) \quad k = 1, 2, \ldots \tag{2}$$

where $\mathbf{h}(\mathbf{x}(t_k), t_k) \in \Re^{N_y \times 1}$ is a measurement function (either linear or nonlinear). The measurement noise $\boldsymbol{\nu}(t_k)$ is assumed to be a Gaussian white noise with a covariance matrix of $\mathbf{R}$ and independent of $\mathbf{x}(0)$, $\mathbf{w}(t)$, and $\mathbf{h}$.

The basic procedure of the nonlinear filtering technique is illustrated in Fig. 1. If the process described by the SDE is a Markovian diffusion process, the probability density function characterizing this process between measurements ($t_k < t < t_{k+1}$) is governed by the Fokker–Planck Equation (FPE) (Fokker, 1940; Jazwinski, 1966, 1970; Planck, 1917) as

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N_s} \frac{\partial [pF_i]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{N_s}\sum_{j=1}^{N_s} \frac{\partial^2 \left[p\left(\mathbf{GQG}^{\mathrm{T}}\right)_{ij}\right]}{\partial x_i \partial x_j} \tag{3}$$

where $p = p(\mathbf{x}(t)|\,\mathbf{Y}(t_k))$ is the state conditional PDF and the measurement observation history is defined as $\mathbf{Y}(t) \triangleq \{\mathbf{y}_k,\ t_k \leq t\}$. The first term on the right hand side (RHS) of the FPE is the drift term whereas the second one is the diffusion term. The process becomes a deterministic one if the diffusion term is neglected.

Once the PDF function is found from Eq. (3), the measurement $\mathbf{y}(t_{k+1})$ made at the time instant t_{k+1} and the Bayes' formula are used together to update the conditional PDF $p\left(\mathbf{x}(t_{k+1})\,\middle|\mathbf{y}_{k+1}\right)$ as

$$p(\mathbf{x}(t_{k+1})|\,\mathbf{Y}(t_{k+1})) = \frac{p(\mathbf{x}(t_{k+1})|\,\mathbf{Y}(t_k))\,p(\mathbf{y}(t_{k+1})|\,\mathbf{x}(t_{k+1}))}{\int p(\boldsymbol{\xi}(t_{k+1})|\,\mathbf{Y}(t_k))\,p(\mathbf{y}(t_{k+1})|\,\boldsymbol{\xi}(t_{k+1}))\,\mathrm{d}\boldsymbol{\xi}}. \tag{4}$$

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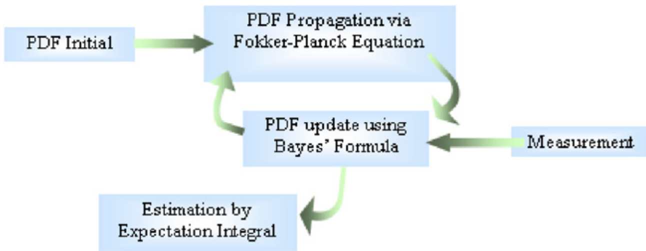

Fig. 1. FPE based nonlinear/linear filtering.

Hence, using the updated conditional PDF, the minimum mean-square estimate (MMSE) of any state variables or functions of state variables $\phi(\mathbf{x})$ can be obtained.

Since nonlinearities can be associated with either the process model or the measurement model, or both, the advantage of the nonlinear filtering technique as compared with the widely-used Kalman Filter (KF) or Extended Kalman Filter (EKF) methods is: no linearization is required. In results, the propagation of the state and error covariance matrix can be more accurate and the measurement update can be less frequent.

However, the central issue associated with the FPE and Bayes' based nonlinear filtering technique is the high computational cost, which explains the reason of why this type of nonlinear filtering method has not been used until recent years (Challa & Bar-Shalom, 2000; Kanchanavally, Zhang, Ordonez, & Layne, 2004; Kastella, 2000; Kastella & Kreucher, 2005; Tang & Ozguner, 2003).

Since it is difficult to obtain the exact solution of the FPE with the exception of some special cases, numerical approximations, such as the finite difference method (Spencer & Bergman, 1993; Zhou & Chirikjian, 2003), path integral method (Naess & Johnson, 1992), and cell-mapping method (Sun & Hsu, 1990), are typically used to evaluate the FPE between measurements. These methods are developed for systems with low dimensions and may not be appropriate for real-time applications. In Daum's paper (2005), the characteristics of nonlinear filtering with a numerical approximation of the FPE are discussed. This work points out that the accuracy of the state estimation by this algorithm is optimal if designed carefully but the computational cost will be prohibitive for high dimensional problems. It was also mentioned that the high computational cost may be mitigated with adaptive grids such as the ones been used by Challa and Bar-Shalom (2000), Musick, Greenswald, Kreucher, and Kastella (2001) and Yoon and Xu (2007). However, as shown in these papers, even with adaptive grids, the computational cost is still high even for low-dimension problems and the accuracy is very sensitive to the moving domain selection.

On the other hand, in order to solve the FPE efficiently, an approximation approach, called the moment or the Gaussian approximation method, was introduced (Kushner, 1967; Kushner & Budhiraja, 2000). In this method, the PDF is assumed to have a particular form and then associated parameters are calculated. In this paper, the direct quadrature method of moments (DQMOM), along with Bayesian update of the conditional state PDF, is formulated for solving the FPE based nonlinear filtering problems. This approach involves representation of the state conditional PDF in terms of a finite summation of Dirac delta functions, whose weights and locations (abscissas) are determined based on constraints due to evolution of moments and modified using Bayes' rule for measurement update. Using a small number of scalars (in the Dirac delta function), the method is able to efficiently and accurately model stochastic processes described by the multivariable FPE through a set of ordinary differential equations (ODEs).

This paper is organized as follows: The Section 2 begins with descriptions of the DQMOM in solving the FPE. Then the procedure to obtain estimates through the Bayes' formula using weights and abscissas is discussed. Following this, the overall structure of the algorithm is outlined and two numerical examples are shown. Conclusions are summarized in the final section.

2. Direct quadrature method of moments (DQMOM)

To clarify the derivation, the FPE of the state conditional PDF (Eq. (3)) between measurements is rewritten here

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N_s} \frac{\partial [pF_i]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{N_s}\sum_{j=1}^{N_s} \frac{\partial^2 \left[p\left(\mathbf{GQG}^{\mathrm{T}}\right)_{ij}\right]}{\partial x_i \partial x_j},$$
$$t_k < t < t_{k+1}. \tag{5}$$

The DQMOM method, originally investigated by Marchisio and Fox for the population balance problem (Marchisio & Fox, 2005), is illustrated in terms of the nonlinear filtering. In DQMOM, the state conditional PDF is written as a summation of a multi-dimensional Dirac delta function

$$p\left(\mathbf{x}(t)\,|\,\mathbf{Y}(t_k)\right) = \sum_{\alpha=1}^{N} w_\alpha(t, \mathbf{Y}(t_k)) \prod_{j=1}^{N_s} \delta[x_j - \langle x_j\rangle_\alpha (t, \mathbf{Y}(t_k))] \tag{6}$$

where N is the number of nodes, $w_\alpha = w_\alpha(t, \mathbf{Y}(t_k))$, $\alpha = 1, \ldots, N$ is the corresponding weight for node α, and $\langle x_j\rangle_\alpha = \langle x_j\rangle_\alpha (t_k, \mathbf{Y}(t_k))$, $\alpha = 1, \ldots, N$; $j = 1, \ldots, N_s$ is the property vector of node α (called "abscissas"). The weights and abscissas will be computed next.

Substituting Eq. (6) into Eq. (5), then the left-hand side (LHS) of Eq. (5) becomes

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left\{\sum_{\alpha=1}^{N} w_\alpha \prod_{j=1}^{N_s} \delta[x_j - \langle x_j\rangle_\alpha]\right\} \\
&= \sum_{\alpha=1}^{N}\left(\frac{\partial w_\alpha}{\partial t}\right)\prod_{j=1}^{N_s} \delta[x_j - \langle x_j\rangle_\alpha] \\
&\quad - \sum_{\alpha=1}^{N} w_\alpha \sum_{j=1}^{N_s} \prod_{k=1,k\neq j}^{N_s} \delta[x_k - \langle x_k\rangle_\alpha] \frac{\partial \delta_{j\alpha}}{\partial \langle x_j\rangle_\alpha} \frac{\partial \langle x_j\rangle_\alpha}{\partial t} \\
&= \sum_{\alpha=1}^{N}\prod_{j=1}^{N_s} \delta_{j\alpha}\left(\frac{\partial w_\alpha}{\partial t}\right) - \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s} w_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{\partial \langle x_j\rangle_\alpha}{\partial t}
\end{aligned} \tag{7}$$

where $\delta_{j\alpha} \triangleq \delta[x_j - \langle x_j\rangle_\alpha]$ and $\delta'_{j\alpha} \triangleq \partial \delta_{j\alpha}/\partial \langle x_j\rangle_\alpha$.

If the weighted abscissas $\varsigma_{j\alpha} \triangleq w_\alpha \langle x_j\rangle_\alpha$ are introduced, after some manipulations, Eq. (7) can be rewritten as

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha}\left(\frac{\partial w_\alpha}{\partial t}\right) + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{\partial w_\alpha}{\partial t}\right] \\
&\quad - \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\frac{\partial \varsigma_{j\alpha}}{\partial t}.
\end{aligned} \tag{8}$$

Notice that w_α, $\varsigma_{j\alpha}$, and $\delta_{j\alpha}$ are functions of time, thus the partial derivatives of the functions can be written as total derivatives.

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha}\left(\frac{\mathrm{d} w_\alpha}{\mathrm{d}t}\right) + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{\mathrm{d} w_\alpha}{\mathrm{d}t}\right] \\
&\quad - \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\frac{\mathrm{d} \varsigma_{j\alpha}}{\mathrm{d}t}.
\end{aligned} \tag{9}$$

With the definitions

$$\mathrm{d}w_\alpha/\mathrm{d}t \triangleq a_\alpha, \quad \alpha = 1, \ldots, N \tag{10}$$

and

$$d\varsigma_{j\alpha}/dt \triangleq b_{j\alpha}, \quad j = 1, \ldots, N_s; \alpha = 1, \ldots, N. \tag{11}$$

Eq. (9) (LHS of Eq. (5)) can be further simplified as

$$\frac{\partial p}{\partial t} = \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha} + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\right]a_\alpha - \sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha}. \tag{12}$$

The right-hand side (RHS) of Eq. (5) is now given by the expression

$$S_{\mathbf{x}}(\mathbf{x}) = -\sum_{i=1}^{N_s}\frac{\partial pF_i}{\partial x_i} + \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{\partial^2[1/2p(GQG^{\mathrm{T}})_{ij}]}{\partial x_i\partial x_j}. \tag{13}$$

The FPE can be written in terms of the multi-variable Dirac delta function as

$$\sum_{\alpha=1}^{N}\left(\prod_{j=1}^{N_s}\delta_{j\alpha}\right)a_\alpha + \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}a_\alpha - \sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha} = S_{\mathbf{x}}(\mathbf{x}). \tag{14}$$

There are in total $N(1+N_s)$ parameters (in Eq. (14)) which need to be found in order to construct the conditional PDF $p(\mathbf{x}(t)|\mathbf{Y}(t))$: a_α and $b_{j\alpha}, j = 1, \ldots, N_s, \alpha = 1, \ldots, N$. The DQMOM method applies an independent set of user-defined moment constraints to construct $N(1+N_s)$ ordinary differential equations (ODEs).

Given the following three Dirac delta function properties

$$\int_{-\infty}^{+\infty} x^k\delta(x-\langle x\rangle_\alpha)dx = \langle x\rangle_\alpha^k \tag{15}$$

$$\int_{-\infty}^{+\infty} x^k\delta'(x-\langle x\rangle_\alpha)dx = -k\langle x\rangle_\alpha^{k-1} \tag{16}$$

and

$$\int_{-\infty}^{+\infty} x^k\delta''(x-\langle x\rangle_\alpha)dx = k(k-1)\langle x\rangle_\alpha^{k-2}. \tag{17}$$

The $k_1, k_2, \ldots, k_{N_s}$ moments of Eq. (14) can be written as followed

$$\begin{aligned}&\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\prod_{j=1}^{N_s}\delta_{j\alpha}a_\alpha\right)\prod_{l=1}^{N_s}dx_l \\ &+\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha\delta_{k\alpha}\delta'_{j\alpha}a_\alpha\right)\prod_{l=1}^{N_s}dx_l \\ &-\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha}\right)\prod_{l=1}^{N_s}dx_l \\ &=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}[S_{\mathbf{x}}(\mathbf{x})]\prod_{l=1}^{N_s}dx_l.\end{aligned} \tag{18}$$

After rearranging and simplifying Eq. (18), the $N(1+N_s)$ unknown parameters can be found in

$$\sum_{\alpha=1}^{N}\left[\left(1-\sum_{j=1}^{N_s}k_j\right)\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right]a_\alpha + \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}k_j\langle x_j\rangle_\alpha^{k_j-1}\prod_{k=1,k\neq j}^{N_s}\langle x_k\rangle_\alpha^{k_k}b_{j\alpha} = \bar{S}_{k_1,\ldots,k_{N_s}} \tag{19}$$

where $\bar{S}_{k_1,\ldots,k_{N_s}} \triangleq \bar{S}^1_{k_1,\ldots,k_{N_s}} + \bar{S}^2_{k_1,\ldots,k_{N_s}}$. The detailed derivation and expressions of $\bar{S}^1_{k_1,\ldots,k_{N_s}}$ and $\bar{S}^2_{k_1,\ldots,k_{N_s}}$ can be found in Appendix.

For example, if the number of states is $N_s = 2$ and the number of nodes used in the multi-dimensional Dirac delta function is $N = 2$, there will be $N(1+N_s) = 6$ unknown parameters in Eq. (19). In order to solve these six DAEs, the following six moment constraints

$$(k_1, k_2) = (0,0), (1,0), (0,1), (2,0), (1,1), (0,2) \tag{20}$$

can be applied such that there are enough equations for solving a_α, $\alpha = 1, 2$ and $b_{j\alpha}$, $j = 1, 2$; $\alpha = 1, 2$ explicitly.

In general, it may be expected that the accuracy of estimation and the computational cost will be higher when the number of nodes increases. The selected moment constraint $k_1, k_2, \ldots, k_{N_s}$ will guarantee the PDF approximated by the Eq. (6) has an exact value for this moment of the PDF. For a typical estimation problem, the accuracy of the first moment (e.g. minimum mean-square estimate (MMSE) estimates of any state variables or functions of state variables $\phi(\mathbf{x})$ can be obtained) is automatically guaranteed.

For simplicity, Eq. (19) can be rewritten in a matrix form as

$$\mathbf{A}\boldsymbol{\mu} = \mathbf{s} \tag{21}$$

where the unknown parameters are

$$\boldsymbol{\mu} \triangleq [a_1, a_2, \ldots, a_N, b_{11}, b_{12}, \ldots b_{1N}, \ldots, b_{N_s1}, b_{N_s2}, \ldots, b_{N_sN}]^{\mathrm{T}} \in \mathfrak{R}^{N(1+N_s)\times 1} \tag{22}$$

and matrix $\mathbf{A}$ can be derived from Eq. (19) as a nonlinear function of the abscissas. The moment constraints are

$$\mathbf{s} = [\bar{S}_{0,\ldots,0}, \bar{S}_{1,\ldots,0}, \ldots]^{\mathrm{T}} \in \mathfrak{R}^{N(N_s+1)\times 1}. \tag{23}$$

As compared with the widely used finite difference methods (Challa & Bar-Shalom, 2000; Daum, 2005; Naess & Johnson, 1992; Yoon & Xu, 2007), with the help of the DQMOM scheme, the partial differential equation is reduced to a set of ordinary differential equations and the computational cost is expected to be reduced.

3. Bayes' formula and nonlinear estimation by DQMOM

Once the weights and abscissas in the "predictor" PDF are found through the DQMOM Eq. (21) propagation, the "updated" conditional PDF can be found using the new measurement $\mathbf{y}(t_{k+1})$ made at the time instant t_{k+1} and the Bayes' formula (Eq. (4)). Substituting Eq. (6) into Eq. (4), the DQMOM based Bayes' equation can be derived as

$$\begin{aligned}&p(\mathbf{x}_{k+1}|\mathbf{Y}(t_{k+1})) \\ &= \frac{p(\mathbf{y}(t_{k+1})|\mathbf{x}(t_{k+1}))\sum_{\alpha=1}^{N}w_\alpha(t_{k+1},\mathbf{Y}(t_k))\prod_{j=1}^{N_s}\delta[x_j-\langle x_j\rangle_\alpha(t_{k+1},\mathbf{Y}_k(t_k))]}{\int p(\mathbf{y}(t_{k+1})|\boldsymbol{\xi}(t_{k+1}))\sum_{\alpha=1}^{N}w_\alpha(t_{k+1},\mathbf{Y}(t_k))\prod_{j=1}^{N_s}\delta[\xi_j-\langle x_j\rangle_\alpha(t_{k+1},\mathbf{Y}(t_k))]d\boldsymbol{\xi}} \\ &= \frac{\sum_{\alpha=1}^{N}w_\alpha(t_{k+1},\mathbf{Y}(t_k))p(\mathbf{y}(t_{k+1})|\mathbf{x}(t_{k+1}))\prod_{j=1}^{N_s}\delta[x_j-\langle x_j\rangle_\alpha(t_{k+1},\mathbf{Y}(t_k))]}{\sum_{\alpha=1}^{N}w_\alpha(t_{k+1},\mathbf{Y}(t_k))\,p\left(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha\right)}.\end{aligned} \tag{24}$$

The new weights in the update step (i.e. after accounting for measurements at $t = t_{k+1}$) for the DQMOM are obtained by renormalizing the old weights as

$$w_\alpha(t_{k+1},\mathbf{Y}(t_{k+1})) = \frac{w_\alpha(t_{k+1},\mathbf{Y}(t_k))p\left(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha\right)}{\sum_{\alpha=1}^{N}w_\alpha(t_{k+1},\mathbf{Y}(t_k))p\left(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha\right)} \tag{25}$$

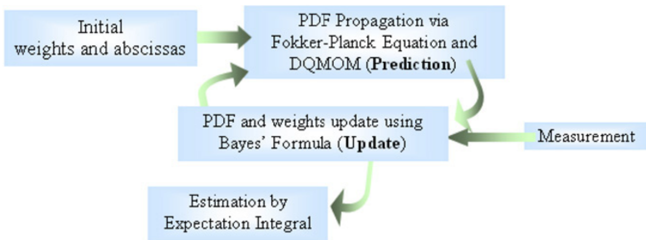

Fig. 2. DQMOM used in the nonlinear/linear filtering.

while the abscissas are unchanged as

$$x_\alpha(t_{k+1}, \mathbf{Y}(t_{k+1})) = x_\alpha(t_{k+1}, \mathbf{Y}(t_k)), \quad \alpha = 1, \dots, N_s. \tag{26}$$

Hence, using the computed state conditional PDF value of $p\left(\mathbf{x}\left(t\right)|\mathbf{Y}\left(t\right)\right)$, $t = t_1, t_2, \dots$, the MMSE estimates of the state variables or functions of state variables $\phi(\mathbf{x}, t)$ at each time step can be obtained as

$$\begin{aligned}\hat{\phi}(\mathbf{x}(t)) &= E[\mathbf{x}(t)\,|\mathbf{Y}(t)] \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi(\mathbf{x}(t)) \sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t)) \\ &\quad \times \prod_{j=1}^{N_s} \delta[x_j - \langle x_j \rangle_\alpha (\mathbf{Y}(t))] \mathrm{d}x_1 \cdots \mathrm{d}x_{N_s} \\ &= \sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t)) \phi(\langle x_1 \rangle_\alpha, \dots, \langle x_{N_s} \rangle_\alpha)\end{aligned} \tag{27}$$

4. Algorithm outline

In DQMOM implementation, both sides of Eq. (21) are standard and can be coded in a general form for any nonlinear filtering problems. This section describes the basic procedures of applying the DQMOM in the nonlinear filtering which involves the following two steps: (1) prediction and (2) update. In the prediction stage (as shown in Fig. 2), Eqs. (10), (11) and (19) will be used together to propagate the weights $w_\alpha = w_\alpha(\mathbf{Y})$ and abscissas $\langle x_j \rangle_\alpha$, $\alpha = 1, \dots, N$; $j = 1, \dots, N_s$. During update, the new weights will be calculated using the Bayes' equation Eq. (25) and the measurement PDF

$$p\left(\mathbf{y}_{k+1} \middle| \mathbf{x}_{k+1}\right) = \frac{1}{(2\pi)^{m/2} |\mathbf{R}|^{1/2}} e^{\left[-\frac{1}{2}[\mathbf{Y}(t_{k+1}) - \mathbf{h}(\mathbf{x}_{k+1})]^{\mathrm{T}} \mathbf{R}^{-1} [\mathbf{Y}(t_{k+1}) - \mathbf{h}(\mathbf{x}_{k+1})]\right]} \tag{28}$$

where $\mathbf{Y}(t_{k+1})$ is the current measurement and $\mathbf{h}(\mathbf{x}_{k+1})$ is the predicted measurement function of the state estimated (first moment of the PDF). The initial value of abscissas and weights (initialization of the DQMOM filter) can be generated randomly, where (1) the mean value of the abscissas generated equals the mean value of the states and (2) the norm of the randomly generated weights is one.

The states or function of the states estimation can be obtained through Eq. (27).

5. Numerical examples

In this section, two numerical examples are used to demonstrate the capabilities of the proposed method.

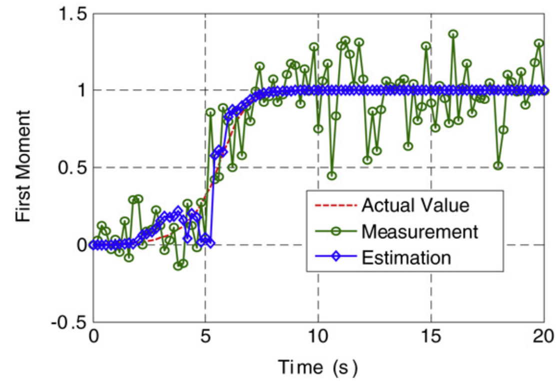

Fig. 3. First moment with Bayes' update.

5.1. Example 1: A nonlinear process

In this subsection, the performance of the proposed nonlinear filter is demonstrated through a nonlinear process governed by the following nonlinear stochastic ordinary differential equation

$$\mathrm{d}x = (x - x^3)\mathrm{d}t + \sigma \mathrm{d}w \tag{29}$$

where σ is the standard deviation and w is a Wiener process. The corresponding FPE is derived as

$$\frac{\partial p}{\partial t} = -\frac{\partial[(x - x^3)]p}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}. \tag{30}$$

The simple measurement model is used

$$y = x + v \tag{31}$$

where $v \sim (0, R)$. To show the effectiveness, the performance of the proposed nonlinear filtering technique is compared with the widely used EKF and the finite difference method (FDM) based nonlinear filtering techniques. In the simulation, the random process is assumed to have a zero mean and a standard deviation of $\sigma = 0.02$. The standard deviation of the measurement noise is assumed to be 0.2 (i.e. $R = 0.2^2$). Both the SDE and the FPE are propagated with a step size of 0.01 s through the DQMOM, whereas the measurements (in the EKF, FDM, and DQMOM methods) are updated at different sampling rates to show the effectiveness of the proposed nonlinear filtering technique. In this example, three nodes are used corresponding to a total of six unknowns (i.e. three weights and three abscissas) to characterize the conditional PDF. These six unknowns are determined using constraints due to the evolution of integer moments of type $\langle x^k \rangle$, where $k = 0, 1, \dots 5$.

To achieve a similar estimation precision, in the FDM, the spatial difference is chosen to be 0.01 and the time difference is selected to be 5×10^{-4} s. The integration domain is $[-1.5, 1.5]$.

Fig. 3 shows a case where weights and abscissas are propagated through the DQMOM with a measurement update. In the case without a measurement update, the first moment of the PDF (estimate of the mean value of the state variable) will stay at zero, which equals the analytical solution. When the Bayes' formula (in the DQMOM framework, Eq. (25)) is used to update the weights, the first moment of the DQMOM solution (estimation) follows the actual state value, as one should expect. Fig. 3 suggests that the measurement update was implemented correctly.

A set of one hundred Monte Carlo runs have been done for all three methods. All the codes are written in Matlab and run in a Sony VAIO laptop (Intel Core CPU with 2.16 GHz, 1.99 GB and 998 MHz of RAM). The Runge–Kutta 4th order method is used for integration.

As shown in Figs. 4 and 5, when the measurement update rate is set at every 0.2 and 0.7 s, the estimation errors from all

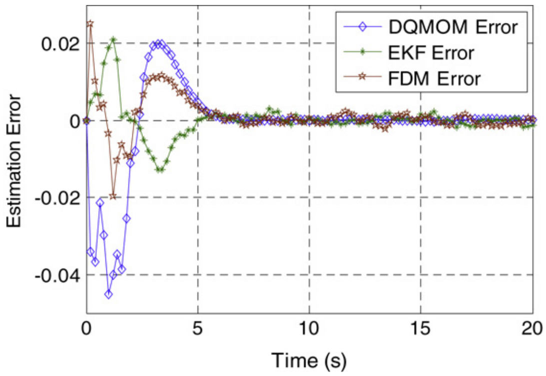

Fig. 4. Update rate at 0.2 s.

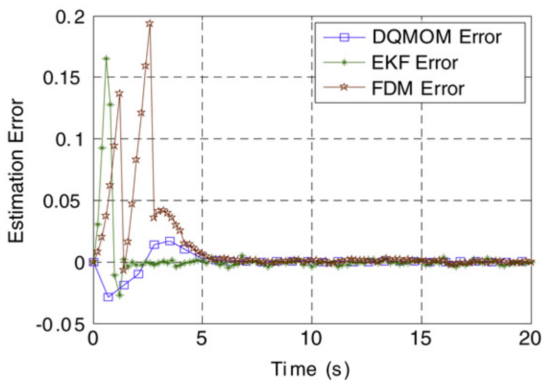

Fig. 5. Update rate at 0.7 s.

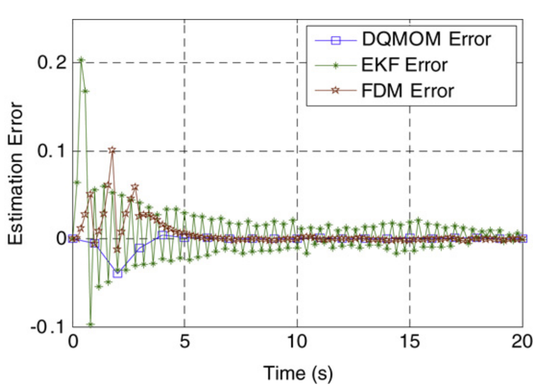

Fig. 6. Update rate at 1.0 s.

these filters have a comparable precision on the order of 10^{-3} in the stationary state (also shown in Table 1). However, as the update delay increases as shown in Fig. 6 (1.0 s) and Fig. 7 (1.5 s), the estimation errors due to the EKF increase dramatically (from 0.004 to 0.4) whereas those of the FDM and DQMOM methods are consistent and still kept in the order of 10^{-3}. The decrease in accuracy observed in EKF can be attributed to errors due to linearization of nonlinear dynamics.

To achieve the same precision, for all cases, the FDM needs approximately 14 s and the time spent is almost constant for the range of update rates considered (see Table 2). The time taken in the DQMOM approach is much less than that of the FDM and as the update frequency decreases, the speed of the DQMOM increases without compromising in the estimation precision (as shown in Table 2).

We observe that for a given accuracy, the estimation due to the DQMOM approach is significantly faster (by an order of magnitude) than the FDM for the low update rate (e.g. 1.5 s).

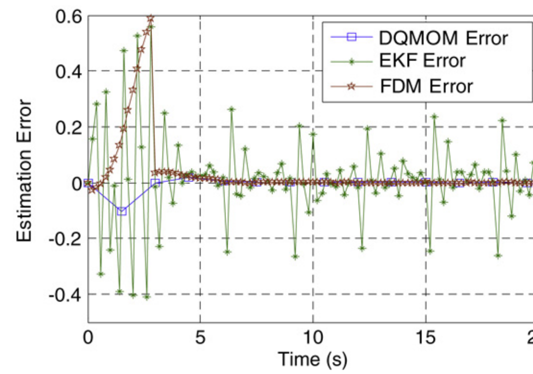

Fig. 7. Update rate at 1.5 s.

Table 1
Bounds of the estimation error.

Update delay (s)	EKF	FDM	DQMOM
0.2	0.003	0.003	0.001
0.7	0.004	0.003	0.001
1.0	0.04	0.002	0.002
1.5	0.4	0.003	0.002

Table 2
Computational cost (in seconds).

Update delays (s)	EKF	FDM	DQMOM
0.2	0.1959	14.31	6.336
0.7	0.1952	13.93	2.371
1.0	0.1956	13.91	1.733
1.5	0.1989	13.91	1.100

5.2. Example 2: Univariate nonstationary growth model

To further test the proposed DQMOM approach, the method is applied in a continuous time version of a nonstationary problem, modified from the discrete-time univariate nonstationary growth model (UNGM) (Kitagawa, 1987; Kotecha & Djuric, 2003; Wu, Hu, Wu, & Hu, 2005), where both the state dynamics and measurement models are nonlinear. The process equation of the UNGM is

$$x_n = \alpha x_{n-1} + \beta \frac{x_{n-1}}{1 + x_{n-1}^2} + \gamma \cos\left[1.2(n-1)\right] + w_n,$$
$$n = 1, 2, \ldots \tag{32}$$

and the measurement model is

$$z = x^2/20 + v. \tag{33}$$

A continuous time version of the process equation is derived as

$$\dot{x} = \alpha^* x + \beta^* \frac{x}{1 + x^2} + \gamma^* \cos\left[\frac{1.2(t - t_0)}{\Delta t}\right] + w,$$
$$t \geq t_0 = 1 \tag{34}$$

where $\alpha^* = (\alpha - 1)/\Delta t$, $\beta^* = \beta/\Delta t$, and $\gamma^* = \gamma/\Delta t$ are based upon the first order Euler scheme. $\alpha = 0.5$, $\beta = 10$, and $\gamma = 8$ are used in the simulation. The time-step used in the conversion from the discrete time model (Kitagawa, 1987) to the continuous time model is $\Delta t = 0.1$ s. The process noise w is t-distributed with 10 degrees of freedom and $v \sim N(0, 0.01)$. A step size of 0.1 s is applied in the propagation of the corresponding SDE and FPE equations, whereas the measurements are updated at different sampling rates to show the consistence in estimation precision of the state.

In this example, two nodes are selected. Therefore, there is a total of four unknowns (i.e. two weights and two abscissas) to characterize the conditional PDF and $k = 0, 1, \ldots, 7$.

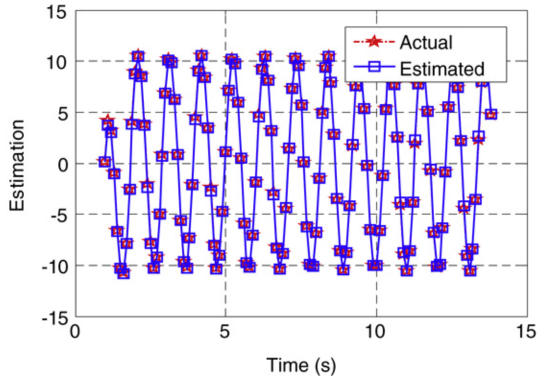

Fig. 8a. Estimated state history using the DQMOM approach (update rate at 0.1 s).

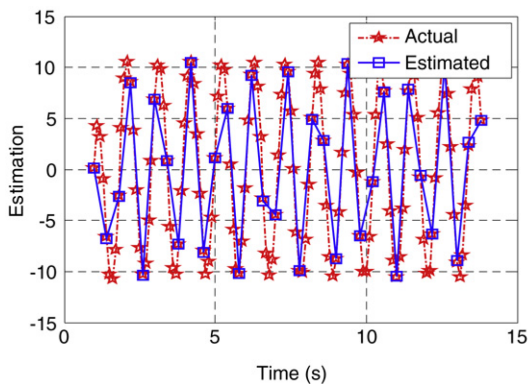

Fig. 8b. Estimated state history using the DQMOM approach (update rate at 0.4 s).

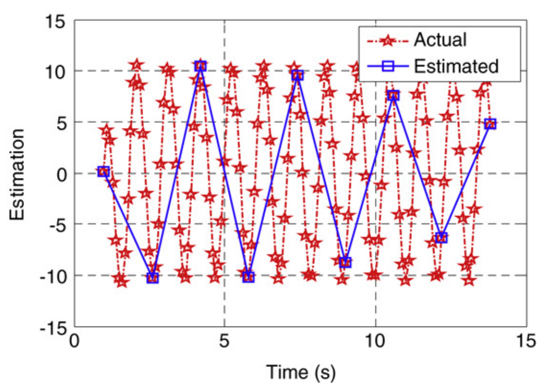

Fig. 8c. Estimated state history using the DQMOM approach (update rate at 1.6 s).

A set of fifty Monte Carlo runs have been used for testing the algorithm. The measurement update rates are set at every 0.1, 0.2, 0.4, 0.8, and 1.6 s respectively. As shown in Figs. 8a–8c, the estimated state tracks the actual state history for all the cases under different update rates.

The mean square error of the estimation procedure based on DQMOM has a consistent precision in the range of 0.218–0.260 for update rates between 0.1s and 1.6 s (shown in Table 3). With reference to the signal amplitude, which is around 10 according to the simulation results as shown in Figs. 8a–8c, the percentage of the state estimation error is only about 2.5%. As expected, we find that the computational cost of estimation based on DQMOM (shown in Table 3), also decreases monotonically as the measurement updates become less frequent.

Simulation results for this numerical example also lead us to conclusions that are similar to those observed from the analysis of numerical example 1. While, the error in estimation based on EKF type approaches increases with a decrease in update rate, owing to the increase in errors due to linearization of dynamics, the DQMOM approach can give better estimation performance than EKF especially at low update rates. Besides, the computational cost of estimation based on the DQMOM approach was also found to be significantly lower than FDM for comparable levels of accuracy (as in numerical example 1).

Table 3
DQMOM for the UNGM.

Update delay (s)	MSE	Error percentage (%)	Computational cost (in seconds)
0.1	0.254	2.54	8.23
0.2	0.260	2.60	4.79
0.4	0.248	2.48	3.18
0.8	0.218	2.18	2.55
1.6	0.246	2.46	2.33

6. Concluding remarks

A new approach for solving the nonlinear filtering problem is proposed using the direct-quadrature method of moments coupled with Bayesian update of the state conditional PDF. In the DQMOM, the state conditional PDF is represented by a summation of the weighted product of multi-dimensional Dirac delta functions. The location of the quadrature abscissas and their weights in the probability space are obtained through a set of independent moment constraints. Based on this method, the Fokker–Planck equation (a partial differential equation) can be transformed into a set of differential algebraic equations in terms of Dirac delta functions. It is expected that the DQMOM approach could lead to a significant reduction in computational cost, compared to finite difference (and other equivalent) methods, especially for high-dimensional problems (as low-order moments are preserved in the DQMOM approach). As compared with EKF based methods, which are used widely in the nonlinear filtering problems, the nonlinear dynamics are not required to be linearized. The new approach for optimal estimation appears to be very promising in the field of nonlinear filtering theory, due to the improvements in accuracy and reduction in computational cost.

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Appendix

The first term in the LHS of Eq. (18) can be simplified as

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\prod_{j=1}^{N_s}\delta_{j\alpha}a_\alpha\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}\\
&=\sum_{\alpha=1}^{N}a_\alpha\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\delta_{1\alpha}\delta_{2\alpha}\cdots\delta_{N_s\alpha}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}\\
&=\sum_{\alpha=1}^{N}\left(\prod_{j=1}^{N_s}\langle x_j\rangle_\alpha^{k_j}\right)a_\alpha \qquad \text{(A.1)}
\end{aligned}$$

whereas the second term in the LHS of Eq. (18) is

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{N_s}x_m^{k_m}\left(\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha\,\delta_{k\alpha}\delta'_{j\alpha}a_\alpha\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{\alpha=1}^{N}a_\alpha\sum_{j=1}^{N_s}\int_{-\infty}^{+\infty}\left\{\prod_{m=1,m\neq j}^{N_s}x_m^{k_m}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\times\left[\int_{-\infty}^{+\infty}x_j^{k_j}\langle x_j\rangle_\alpha\,\delta'_{j\alpha}\mathrm{d}x_j\right]\prod_{l=1,l\neq j}^{N_s}\mathrm{d}x_l\right\}$$
$$=\sum_{\alpha=1}^{N}a_\alpha\sum_{j=1}^{N_s}\left[(-k_j)\langle x_j\rangle_\alpha^{k_j}\right]\int_{-\infty}^{+\infty}\left\{\prod_{m=1,m\neq j}^{N_s}x_m^{k_m}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\prod_{l=1,l\neq j}^{N_s}\mathrm{d}x_l\right\}$$
$$=-\sum_{\alpha=1}^{N}\left(\sum_{j=1}^{N_s}k_j\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)a_\alpha. \tag{A.2}$$

In the same way, the third term of the LHS in Eq. (18) is

$$-\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}b_{j\alpha}\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\delta'_{j\alpha}\left(\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}k_j\langle x_j\rangle_\alpha^{k_j-1}\prod_{k=1,k\neq j}^{N_s}\langle x_k\rangle_\alpha^{k_k}\,b_{j\alpha}. \tag{A.3}$$

The $k_1,\ldots,k_{N_s}$ moments of the RHS of Eq. (18) are derived to be

$$\bar{S}_{k_1,\ldots,k_{N_s}}\triangleq\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}S_{\mathbf{x}}(\mathbf{x})\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=-\sum_{i=1}^{n}\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\frac{\partial pF_i}{\partial x_i}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$+\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left[\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{1}{2}\frac{\partial^2 p(GQG^{\mathrm{T}})_{ij}}{\partial x_i\partial x_j}\right]\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\bar{S}^1_{k_1,\ldots,k_{N_s}}+\bar{S}^2_{k_1,\ldots,k_{N_s}} \tag{A.4}$$

where

$$\bar{S}^1_{k_1,\ldots,k_{N_s}}=-\sum_{i=1}^{N_s}\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\frac{\partial pF_i}{\partial x_i}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=-\sum_{i=1}^{N_s}\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\frac{\partial}{\partial x_i}\left(F_i(\mathbf{x})\sum_{\alpha=1}^{N}w_\alpha(t)\prod_{j=1}^{N_s}\delta_{j\alpha}\right)\times\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{i=1}^{N_s}\sum_{\alpha=1}^{N}k_iw_\alpha(t)\,\langle x_1\rangle_\alpha^{k_1}\cdots\langle x_{i-1}\rangle_\alpha^{k_{i-1}}\,\langle x_i\rangle_\alpha^{k_i-1}\times\langle x_{i+1}\rangle_\alpha^{k_{i+1}}\cdots\langle x_{N_s}\rangle_\alpha^{k_{N_s}}F_i(\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha). \tag{A.5}$$

When $i\neq j$,

$$\bar{S}^2_{k_1,\ldots,k_{N_s}}=\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left[\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{1}{2}\frac{\partial^2 p(GQG^{\mathrm{T}})_{ij}}{\partial x_i\partial x_j}\right]\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left\{\frac{\partial^2 p[D(\mathbf{x})]_{ij}}{\partial x_i\partial x_j}\right\}\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{\alpha=1}^{N}w_\alpha k_ik_j\left(\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)\Big/\langle x_i\rangle_\alpha\langle x_j\rangle_\alpha\times[D(\mathbf{x})]_{ij}\Big|_{\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha} \tag{A.6}$$

whereas when $i=j$,

$$\bar{S}^2_{k_1,\ldots,k_{N_s}}=\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\frac{\partial^2 f[D(\mathbf{x})]_{ii}}{\partial x_i^2}\mathrm{d}x_1\cdots\mathrm{d}x_{N_s}$$
$$=\sum_{\alpha=1}^{N}w_\alpha k_i(k_i-1)\left(\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)\Big/\langle x_i\rangle_\alpha^2\,[D(\mathbf{x})]_{ij}\Big|_{\langle x_1\rangle_\alpha,\ldots,\langle x_{N_s}\rangle_\alpha}. \tag{A.7}$$

Notice that $D(\mathbf{x})\triangleq(1/2)GQG^{\mathrm{T}}$. Thus, the $N(1+N_s)$ ODEs can be constructed using a set of independent moment constraints $k_1,\ldots,k_{N_s}$ as shown in Eq. (19).

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