THE UNIVERSITY *of* EDINBURGH

# Fixation and coarsening in Moran-type processes

Richard A Blythe

Some introductory material

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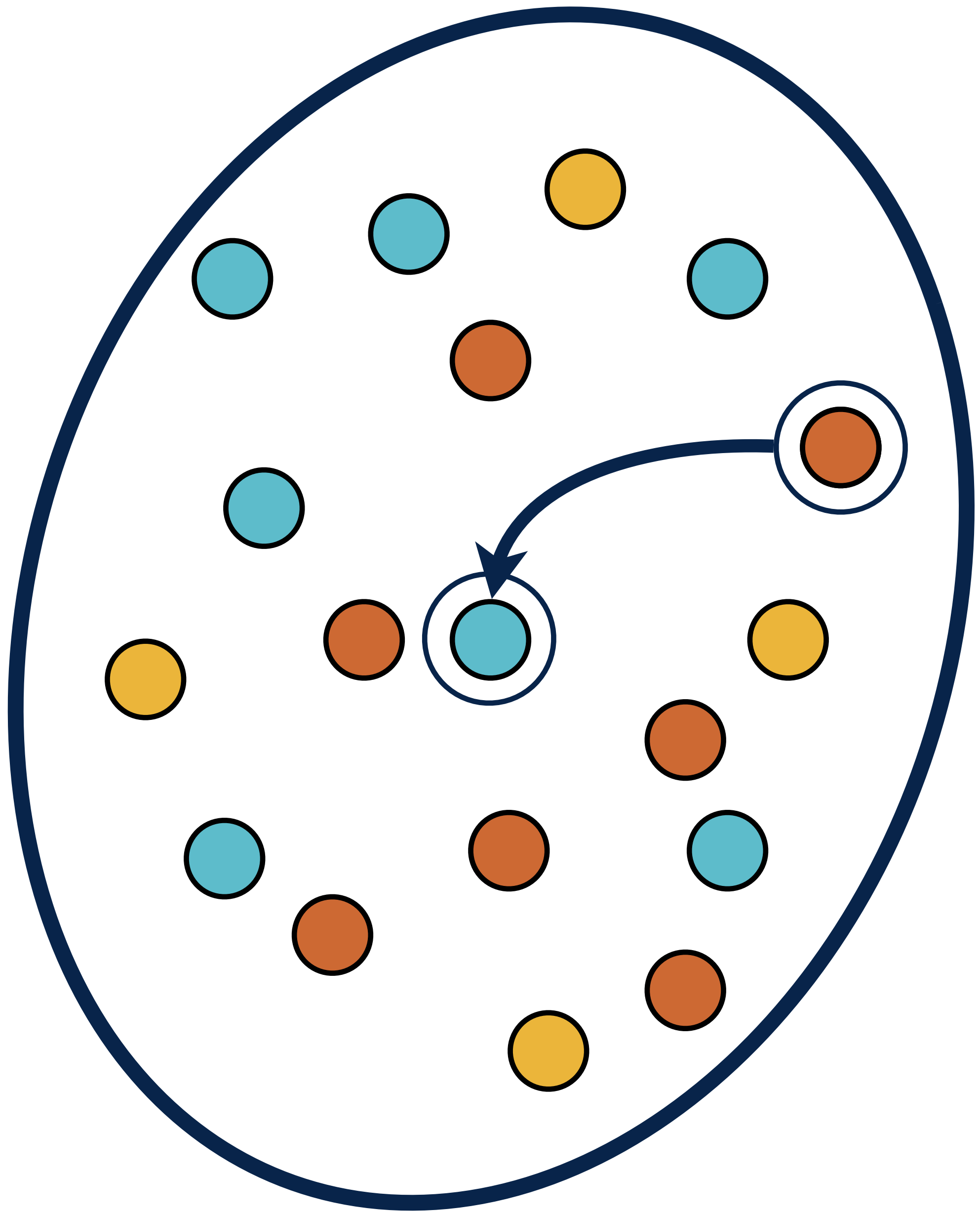
D I Russell and R A Blythe (2011) *Physical Review Letters* **106** 165702

# Moran-type process



**S** species, **N** individuals

Time **t**: $\mathbf{n_\alpha}$ individuals of species $\boldsymbol{\alpha}$

Choose an individual to **die** (species $\boldsymbol{\alpha}$)

Replace with a **copy** of a randomly-chosen **parent** (species $\boldsymbol{\beta}$)

→ State at **t+1**

$$(\cdots, n_\alpha, \cdots, n_\beta, \cdots) \to (\cdots, n_\alpha - 1, \cdots, n_\beta + 1, \cdots)$$

with probability $\frac{n_\alpha}{N}\frac{n_\beta}{N}$

“Moran model” – Moran (1958) *Proc Camb Phil Soc* **54** 60; also “mean-field voter model”

# Motivation: Hubbell's neutral model

Hubbell (2001)
*The unified neutral theory of biodiversity and biogeography*

birth death

# Fate of the population

$$(\cdots, n_\alpha, \cdots, n_\beta, \cdots) \to (\cdots, n_\alpha - 1, \cdots, n_\beta + 1, \cdots)$$

with probability $\frac{n_\alpha}{N}\frac{n_\beta}{N}$

Markov chain on a finite state space

States of **fixation** $(0, 0, \ldots, 0, N, 0, \ldots, 0)$ are absorbing

All other states are transient

$$n_\alpha(t+1) = \begin{cases} n_\alpha(t) + 1 & \text{w.p. } \frac{n_\alpha}{N}\left(\sum_{\beta\neq\alpha}\frac{n_\beta}{N}\right) \\ n_\alpha(t) - 1 & \text{w.p. } \left(\sum_{\beta\neq\alpha}\frac{n_\beta}{N}\right)\frac{n_\alpha}{N} \end{cases}$$

$$\langle n_\alpha(t+1)\rangle = n_\alpha(t)$$

$$\lim_{t\to\infty}\langle n_\alpha(t)\rangle = n_\alpha(0)$$

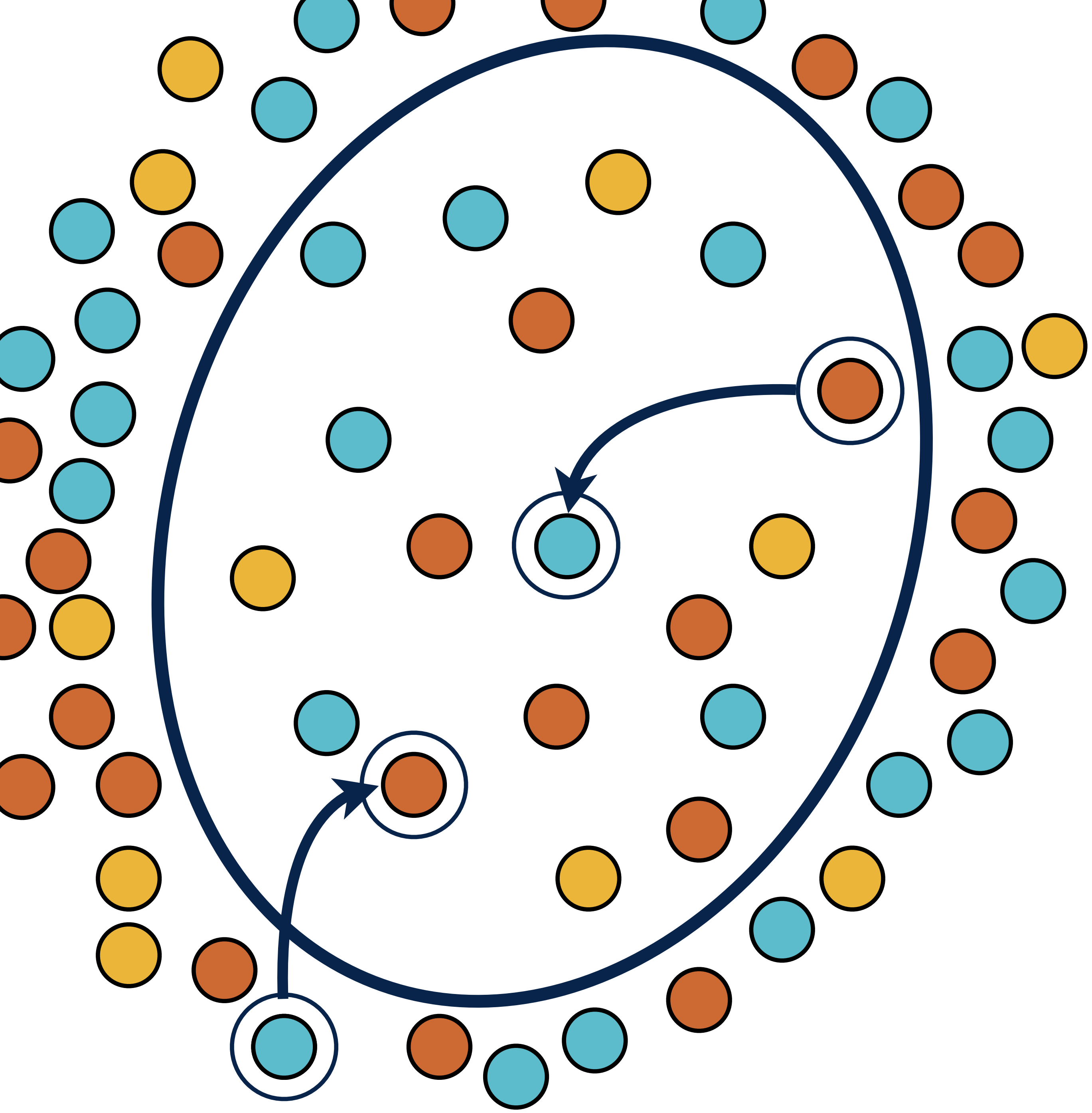
Fixation of species $\alpha$ occurs with probability $\frac{n_\alpha(0)}{N}$

# Contact with a metacommunity

Hubbell (2001)
*The unified neutral theory of biodiversity and biogeography*

birth death migration

# Birth, Death & Migration



with probability **1-m** :
do the same as before

with probability **m** :
copy from a (fixed) environment

**Absorbing states are eliminated**

$$(\cdots, n_\alpha, \cdots, n_\beta, \cdots) \to (\cdots, n_\alpha - 1, \cdots, n_\beta + 1, \cdots)$$

with probability $\quad (1-m)\frac{n_\alpha}{N}\frac{n_\beta}{N} + m\frac{n_\alpha}{N}\overline{x_\beta}$

Wright (1931) *Genetics* **16** 97; see also Crow and Kimura (1970) *An introduction to population genetics theory*

# Birth, Death & Migration

$$(\cdots, n_\alpha, \cdots, n_\beta, \cdots) \to (\cdots, n_\alpha - 1, \cdots, n_\beta + 1, \cdots)$$

c.f. inclusion process $p(\alpha,\beta) n_\alpha \left(n_\beta + \frac{m}{2}\right)$

with probability $\quad (1-m)\frac{n_\alpha}{N}\frac{n_\beta}{N} + m\frac{n_\alpha}{N}\overline{x_\beta}$

Continuum limit $\quad n_\alpha = N x_\alpha \quad m = \frac{\theta}{N} \quad \delta t = \frac{1}{N^2} \quad N \to \infty$

With just two species

$$\frac{\langle x(t+\delta t) - x(t)\rangle}{\delta t} \to \theta\left[\bar{x} - x\right] \qquad \frac{\langle [x(t+\delta t) - x(t)]^2\rangle}{\delta t} \to 2x(1-x)$$

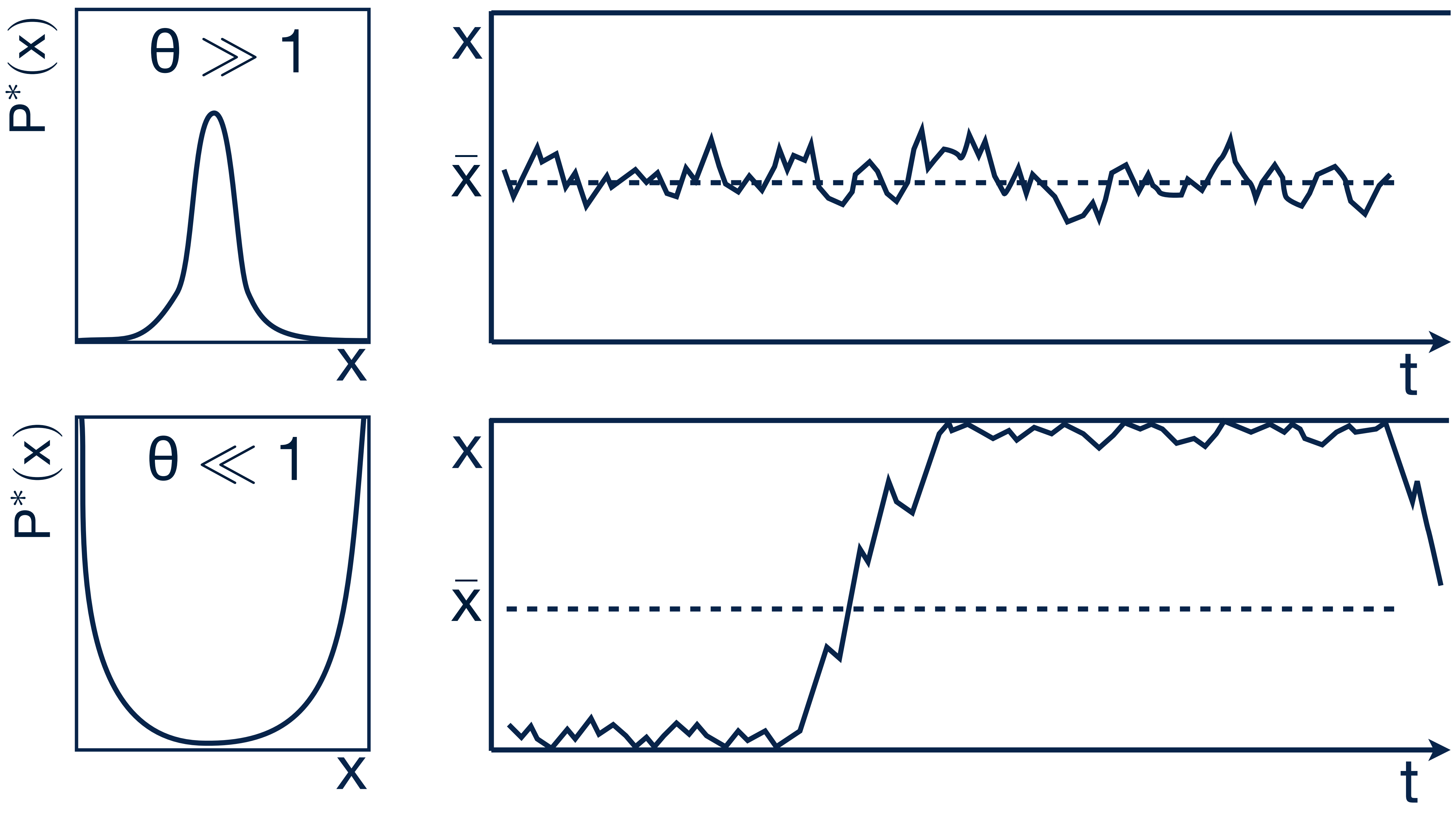
$$\frac{\partial}{\partial t}P(x,t) = -\theta\frac{\partial}{\partial x}(\bar{x} - x)P(x,t) + \frac{\partial^2}{\partial x^2}x(1-x)P(x,t)$$

$$\dot{x}(t) = \theta(\bar{x} - x) + \sqrt{x(1-x)}\,\eta(t) \qquad \text{(Itō)}$$

Wright (1931) *Genetics* **16** 97; see also Crow and Kimura (1970) *An introduction to population genetics theory*

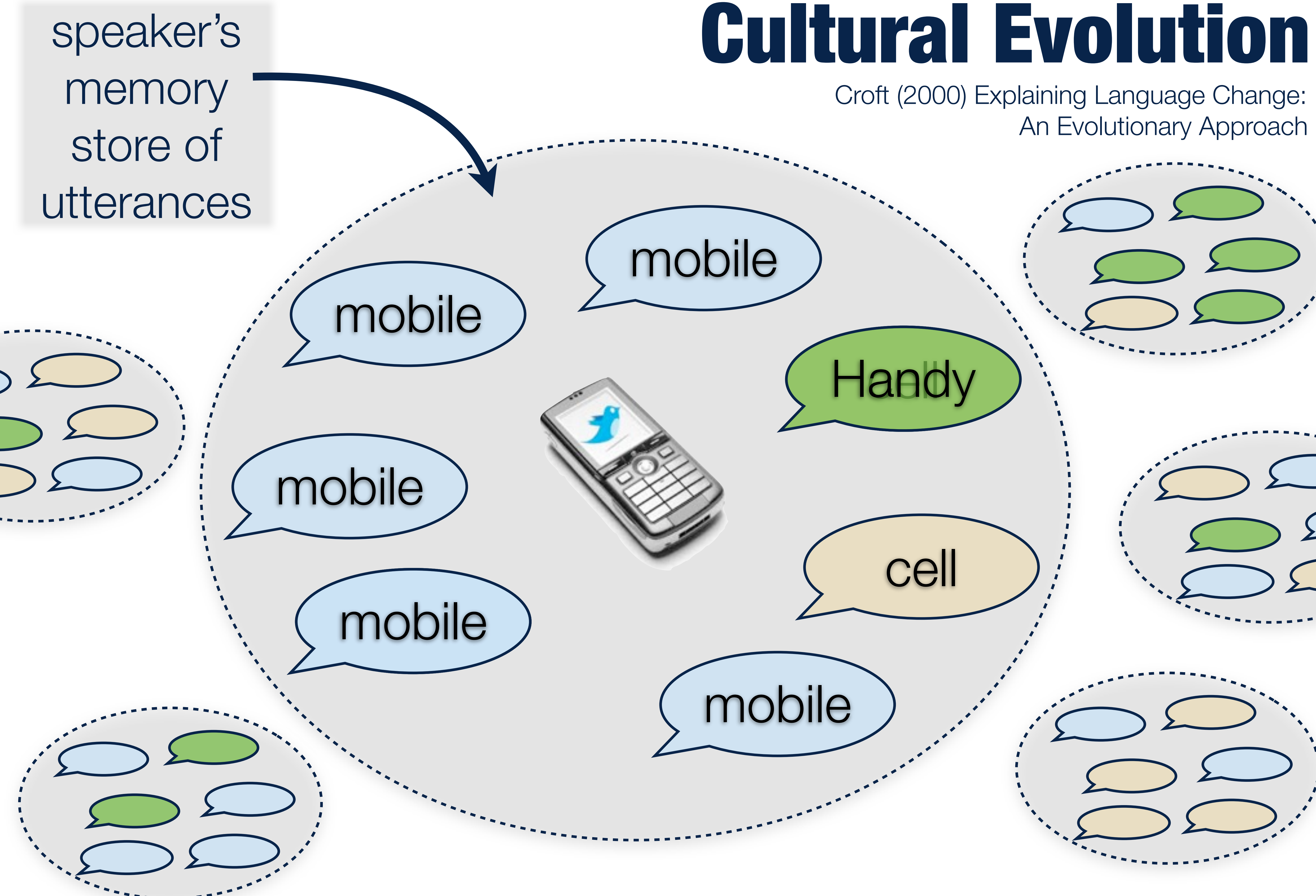
$$\frac{\partial}{\partial t}P(x,t) = -\theta\frac{\partial}{\partial x}(\bar{x}-x)P(x,t) + \frac{\partial^2}{\partial x^2}x(1-x)P(x,t)$$

$$P^*(x) \propto x^{\theta\bar{x}-1}(1-x)^{\theta(1-\bar{x})-1}$$

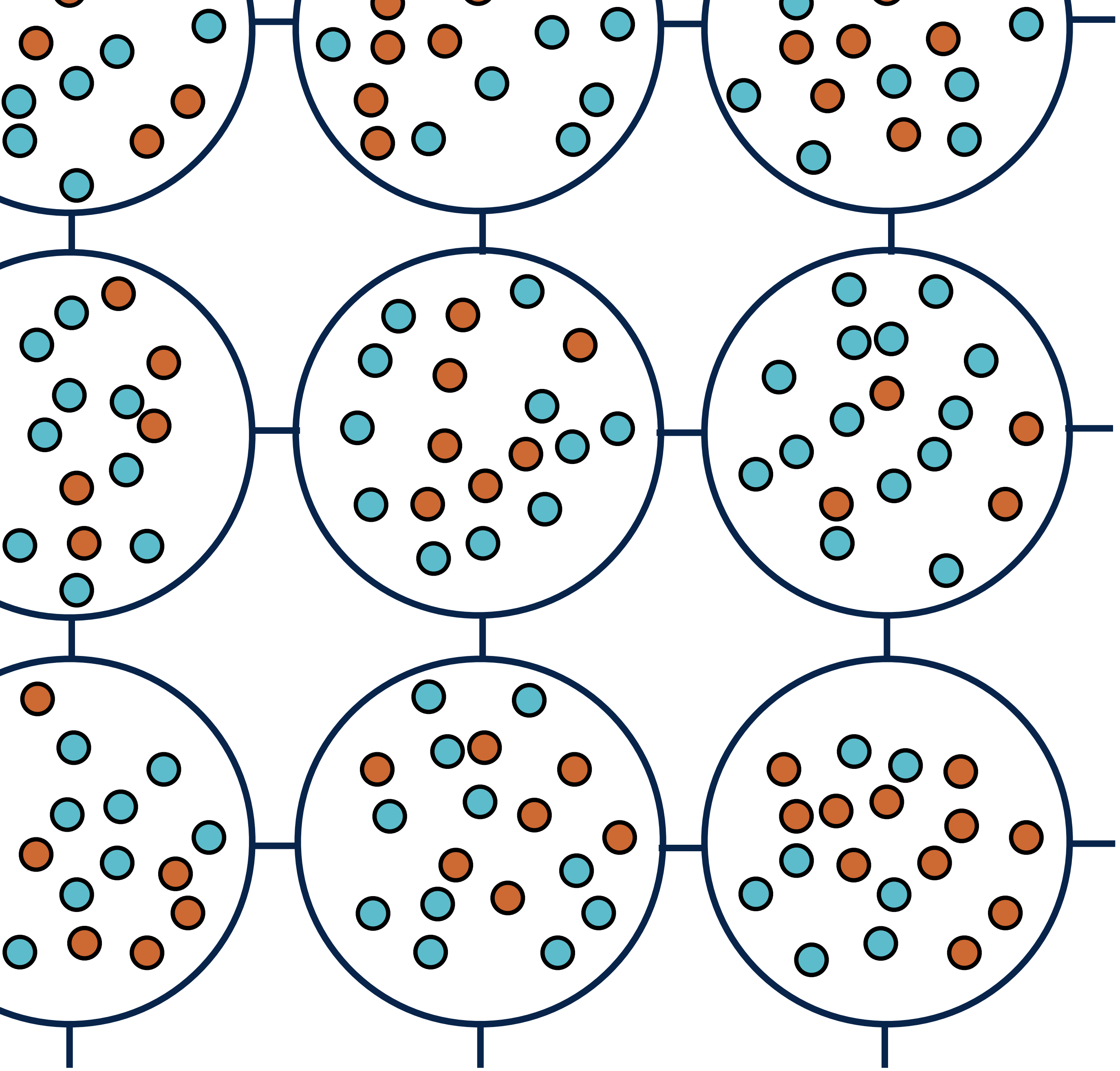


# Cultural Evolution

Croft (2000) Explaining Language Change:
An Evolutionary Approach




**birth** **death** **migration**



pick a site **i** at random

choose an **individual** to die

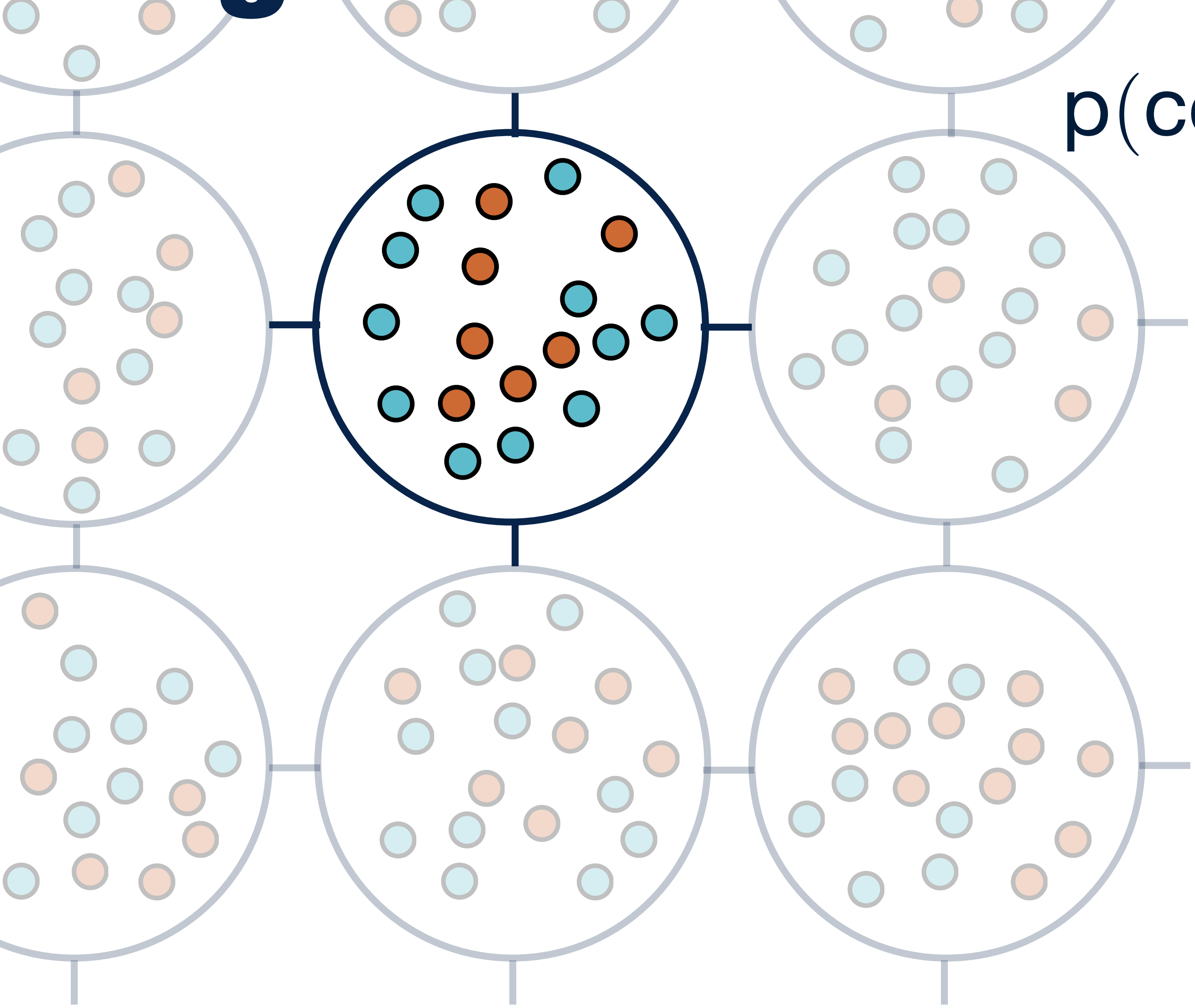
with probability $\frac{h}{N}$ choose **parent** from **neighbouring** site

otherwise, choose **parent** from **same** site

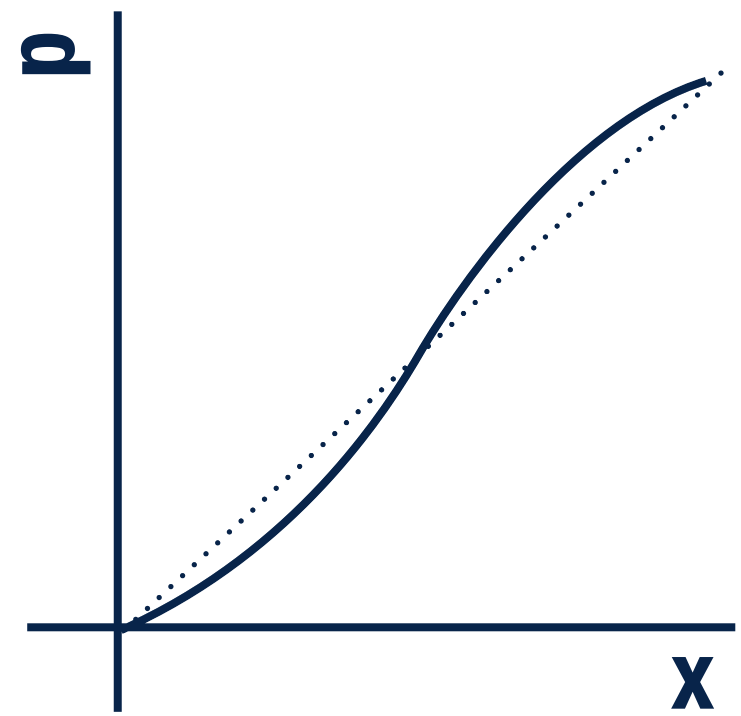
$$\frac{\partial}{\partial t}P(\{x\},t) = -h\sum_{i,j}\frac{\partial}{\partial x_i}(x_j - x_i)P(\{x\},t) + \sum_i \frac{\partial^2}{\partial x_i^2}x_i(1-x_i)P(\{x\},t)$$

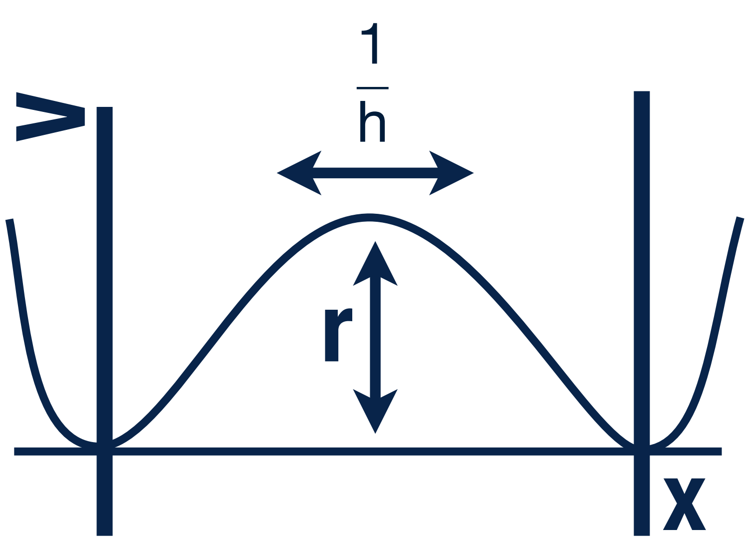
$$\dot{x}_i(t) = h\sum_j (x_j - x_i) + \sqrt{x_i(1-x_i)}\eta_i(t)$$

# Regularisation of variation



$$p(\text{copy}\ \textcolor{red}{\bullet}) = x + \frac{a}{N}x(1-x)(2x-1)$$



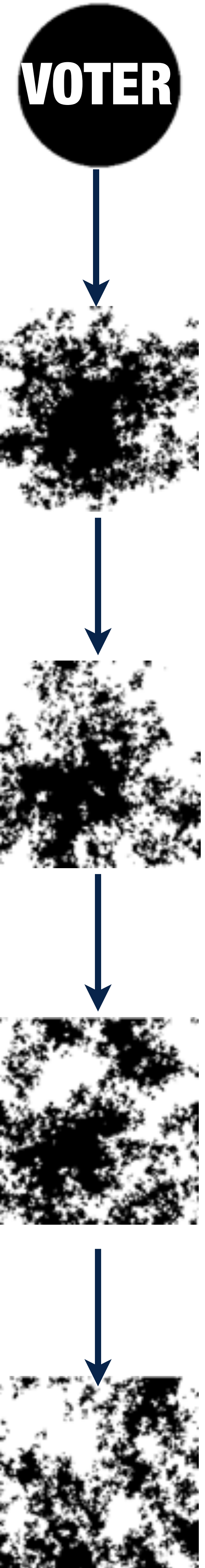




$$a = hr$$

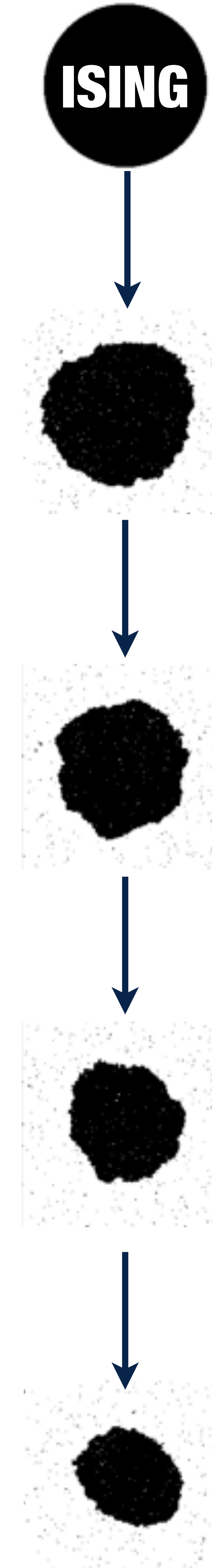
$$\dot{x}_i(t) = h\left[r\frac{\partial}{\partial x_i}x_i^2(1-x_i)^2 + \sum_j (x_j - x_i)\right] + \sqrt{x_i(1-x_i)}\,\eta_i(t)$$

$V(x_i)$

inspired by experimental psychology research performed by Hudson Kam and Newport (2005) and others




# Coarsening behaviour

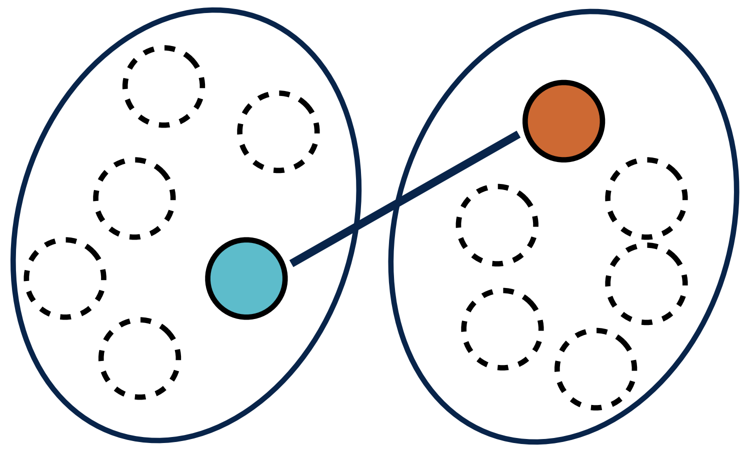


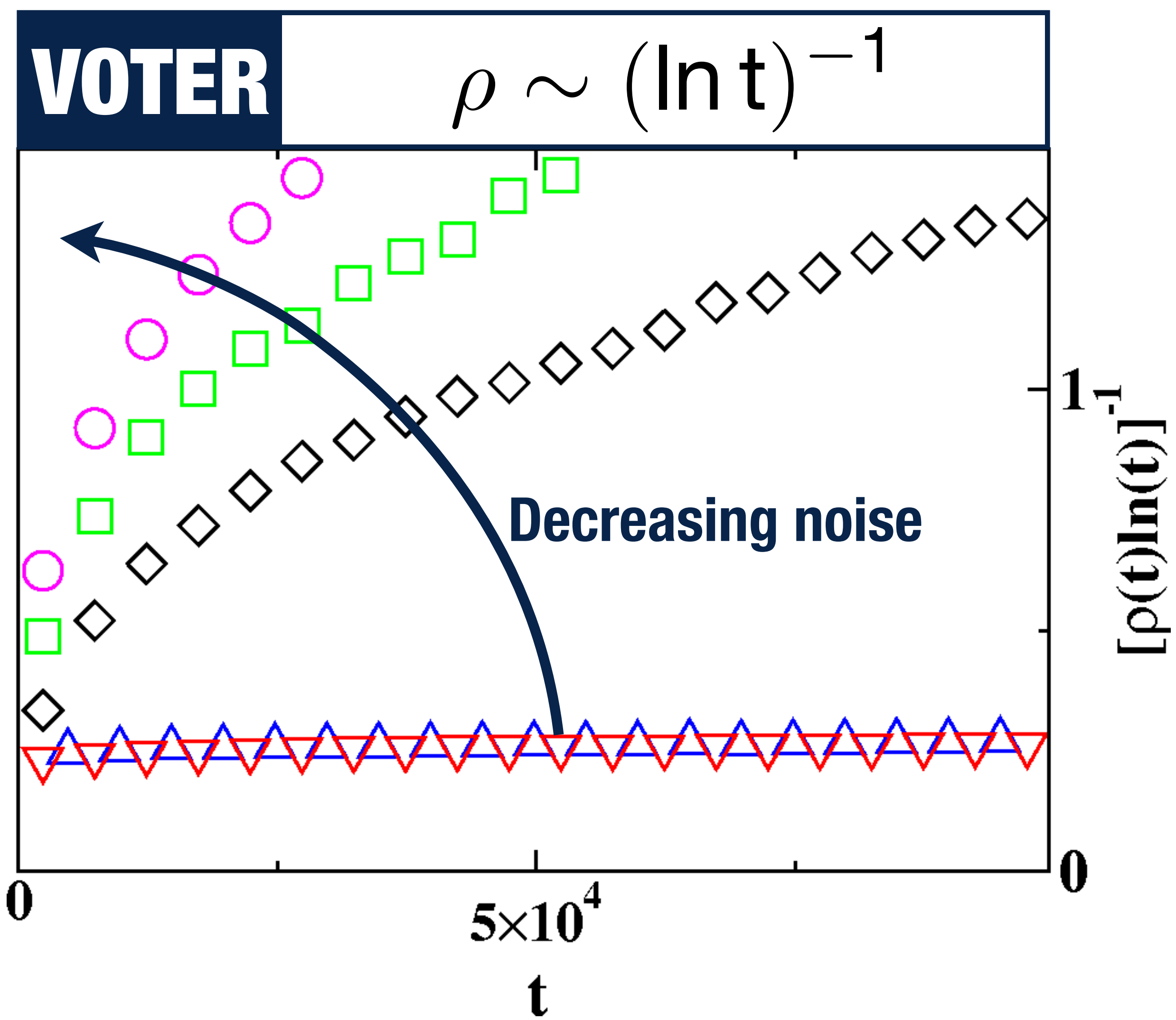

$$\dot{x}_i(t) = h\left[r\frac{\partial}{\partial x_i}x_i^2(1-x_i)^2 + \sum_j (x_j - x_i)\right] + \sqrt{x_i(1-x_i)}\eta_i(t)$$

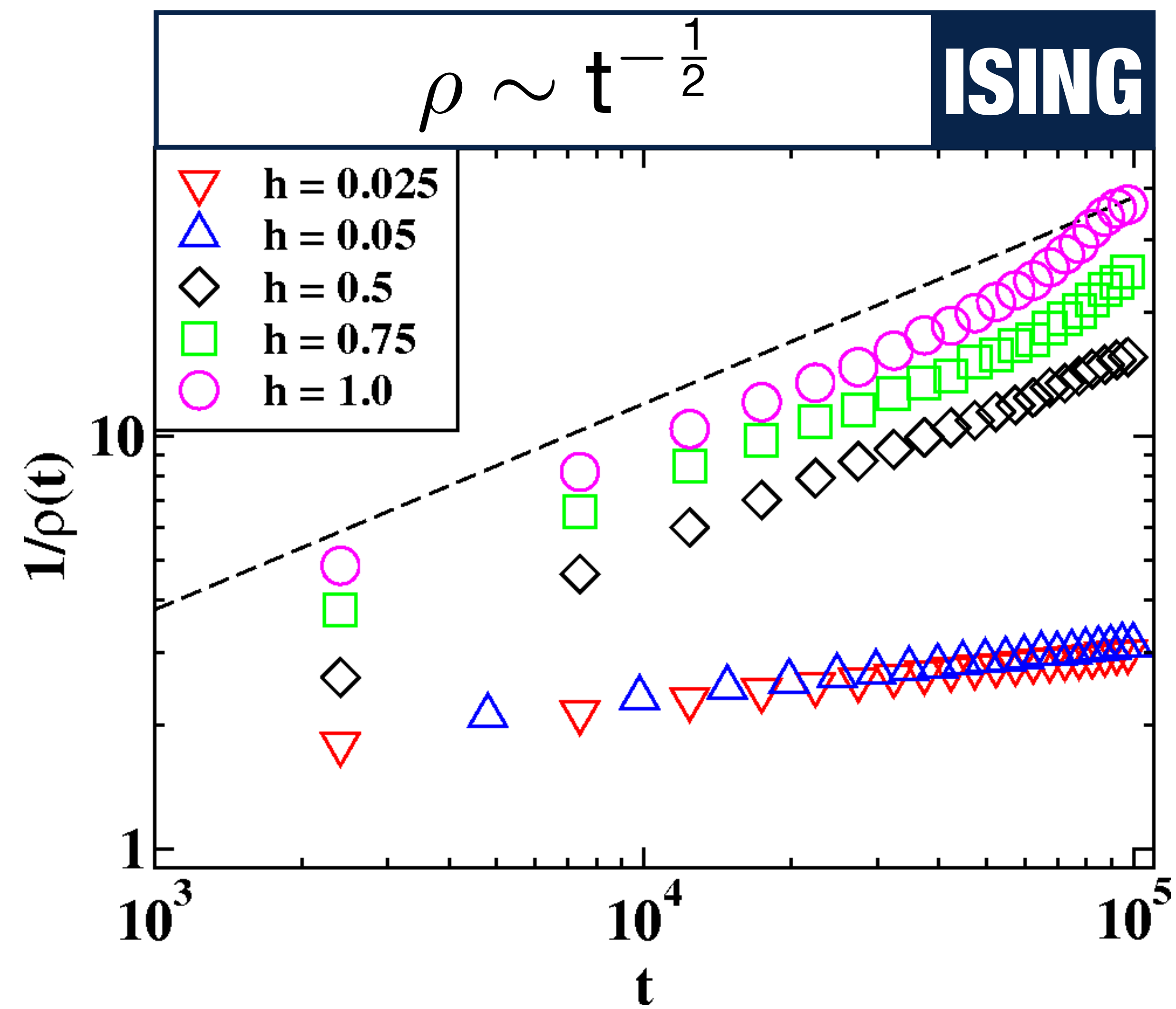
**r=0:** coarsens like the Moran or voter model

**r>0, thermal noise:** coarsens like the Ising model

# Density of Interfaces

each  counts as one interface

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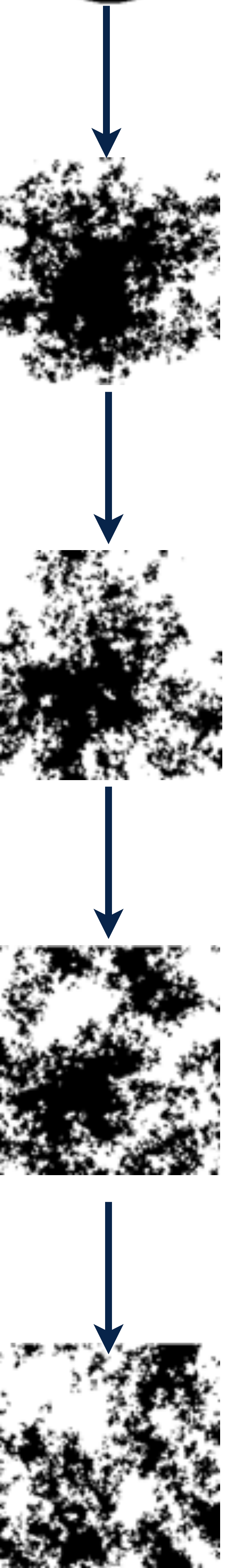
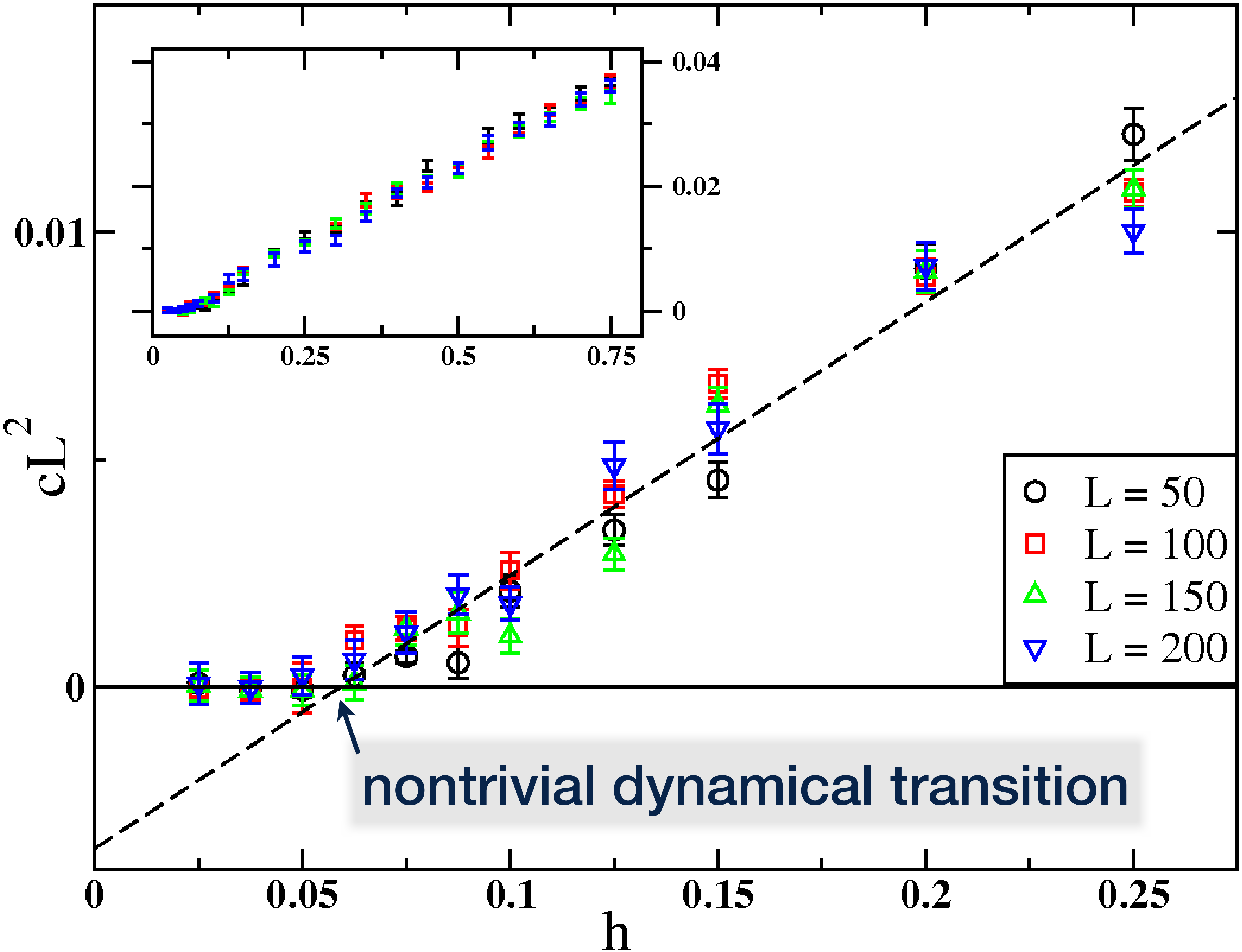
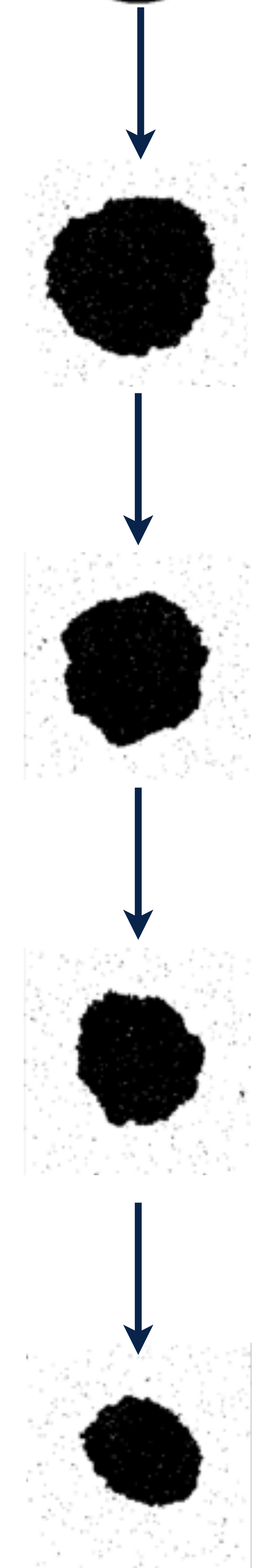
**VOTER** $c = 0$

$$A(t) = A(0) - ct$$

$c > 0$ **ISING**





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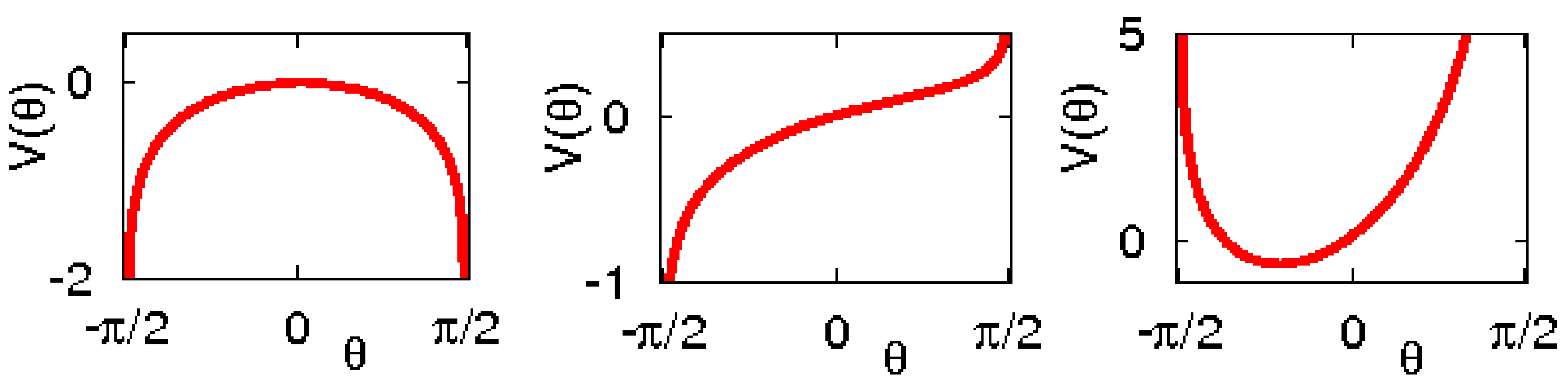
$$\dot{x}_i(t) = h\left[r\frac{\partial}{\partial x_i}x_i^2(1-x_i)^2 + \sum_j (x_j - x_i)\right] + \sqrt{x_i(1-x_i)}\,\eta_i(t)$$

$$x_i = \frac{1}{2}(1+\sin\theta_i) \qquad\qquad \dot{\theta}_i = -\frac{\partial}{\partial\theta_i}V_D(\theta_i) + \eta_i$$

for fixed neighbourhood




small **h** medium **h** big **h**

The diffusion of θ is **always** biased towards the absorbing boundaries when $\mathbf{h < h_c = 1/4z = 0.0625}$

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**Birth-death** dynamics in Moran processes leads to **fixation** of a single species (trivial condensate)

Low level of **migration** generates periods over which **one species** dominates (is this a condensate?)

**Athermal noise** near the boundaries mediates a **transition** between **Ising-like** (curvature-driven) and **Voter-like** (fluctuation-driven) coarsening

Traditionally, these **dynamical universality classes** are distinguished by a “conservation law” – in fact, this seems to be an **emergent property** of the dynamics

## Outstanding questions

temporal properties of the “condensate”, existence of the dynamical transition on networks, domain structure in fluctuation-driven coarsening on networks, empirical relevance, ...