Couplings and attractive particle systems

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Outline

Introduction

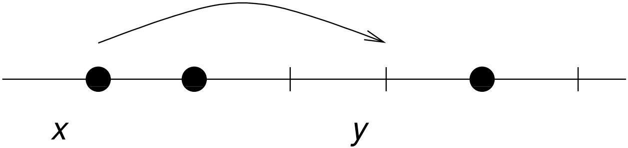
The multiple particle jump model

- Coupling for the multiple particle jump model
- Invariant measures

Exclusion process with speed change

Basic example : The simple exclusion process (SEP)

[Liggett]

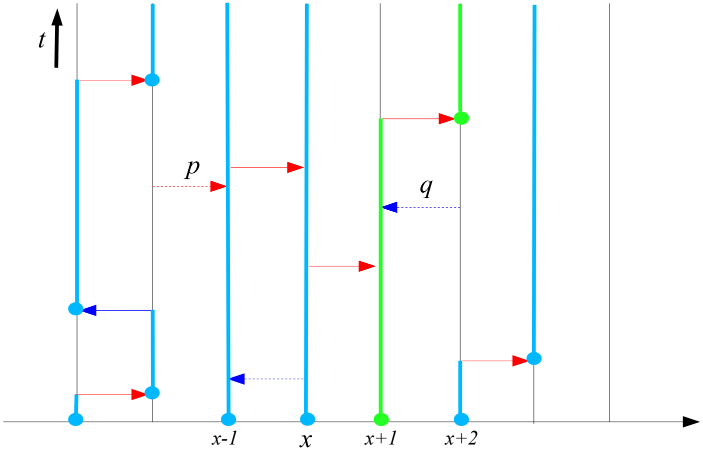


conservative dynamics :

- At most one particle per site : for $z \in \mathbb{Z}$, $\eta(z) = 0$ or 1.
- From each site x, choice of y with $p(y - x)$.
- According to an exponential clock, jump from x to y if possible (*exclusion rule*).

Graphical construction [Harris]

$\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_H$ for initial configurations and Poisson processes.



$\rightsquigarrow$ basic coupling : same clocks, different initial conditions.

Basic coupling preserves partial order : $\rightsquigarrow$ SEP is attractive

- $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho,\ \rho \in [0,1]\}$, Bernoulli product measures
- Euler hydrodynamics for ASEP (asymmetric SEP) :

$$\partial_t u + \partial_x[\gamma u(1-u)] = 0$$

$\gamma = \sum_z z p(z)$; $G(u) = u(1-u)$ concave flux function

Extension to other conservative dynamics ?

- Multiple particle jumps : $K \geq 1$ particles per site, $k \geq 1$ particles attempt to jump from x to y.
- Exclusion with speed change : at most 1 particle per site ; the rate to jump from x to y depends also on the occupation on sites $z \notin \{x, y\}$

Same questions : attractiveness, $(\mathcal{I} \cap \mathcal{S})_e$, Euler hydrodynamics ?

Is condensation compatible with attractiveness ?

Attractiveness [Liggett]

$(\eta_t)_{t\geq 0}$: an interacting particle system of state space $\Omega = X^S$, with $X \subset \mathbb{Z}$, $S = \mathbb{Z}^d$. It is a Markov process with generator L and semi-group $T(t)$.

- Partial order on Ω :
  $\forall \xi, \zeta \in \Omega, \quad \xi \leq \zeta \iff (\forall x \in S, \quad \xi(x) \leq \zeta(x))$
  $f : W \to \mathbb{R}$ is monotone if :
  $\forall l, m \in W,\ l \leq m \Longrightarrow f(l) \leq f(m)$
  $\mathcal{M} = \{$ bounded, monotone, continuous functions on $\Omega\}$
- Stochastic order on probability measures $\mathcal{P}(\Omega)$:
  $\forall \nu, \nu' \in \mathcal{P}(\Omega), \quad \nu \leq \nu' \iff (\forall f \in \mathcal{M}, \nu(f) \leq \nu'(f))$.
- $(\eta_t)_{t\geq 0}$ is attractive if the following equivalent conditions are satisfied.
  (a) $f \in \mathcal{M}$ implies $T(t)f \in \mathcal{M}$ for all $t \geq 0$.
  (b) For $\nu, \nu' \in \mathcal{P}(\Omega)$, $\nu \leq \nu'$ implies $\nu T(t) \leq \nu' T(t)$ for all $t \geq 0$.

The multiple particle jump model

$S = \mathbb{Z}^d$; either $X \subset \mathbb{Z}$ or $X = \mathbb{N}$.

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega = X^S$ with infinitesimal generator :

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} \sum_{\alpha,\beta \in X} \chi_{x,y}^{\alpha,\beta}(\eta) \sum_{k \in \mathbb{N}} \Gamma_{\alpha,\beta}^{k}(y-x)(f(S_{x,y}^{k}\eta) - f(\eta))$$

$$\chi_{x,y}^{\alpha,\beta}(\eta) = \begin{cases} 1 & \text{if } \eta(x) = \alpha \text{ and } \eta(y) = \beta \\ 0 & \text{otherwise} \end{cases}$$

$$(S_{x,y}^{k}\eta)(z) = \begin{cases} \eta(x) - k & \text{if } z = x \text{ and } \eta(x) - k \in X, \eta(y) + k \in X \\ \eta(y) + k & \text{if } z = y \text{ and } \eta(x) - k \in X, \eta(y) + k \in X \\ \eta(z) & \text{otherwise} \end{cases}$$

- assumption : For all $\forall z \in S$, $\alpha, \beta \in X$, $\sum_{k \in \mathbb{N}} \Gamma_{\alpha,\beta}^{k}(z) < \infty$
- This particle system is conservative :
  $\eta(x) + \eta(y)$ is a *conserved quantity* in the transition.

Some Classical Examples with $k = 1$:

- SEP : $\Gamma^1_{1,0}(y-x) = p(y-x) \times 1$ [Liggett]
- zero range process (ZRP) : $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)g(\alpha)$ [Andjel]
- misanthropes process (MP) : $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)b(\alpha,\beta)$ [Cocozza]
  misanthropes $\leftrightarrow$ attractiveness
  philanthropes $\leftrightarrow$ condensation

A two species exclusion model with charge conservation [Collet et al.]

- $\Omega = \{-1, 0, 1\}^{\mathbb{Z}}$; $\eta(x) = \pm 1$ means a particle with a positive (negative) charge on x ; $\eta(x) = 0$ if x empty.
- A transition between neighboring sites x, y is allowed if $\eta(x) + \eta(y)$ conserved : There are possibly ten different transition rates, with $k \in \{1, 2\}$: $\Gamma^1_{0,-1}(r)$, $\Gamma^2_{1,-1}(r)$, $\Gamma^1_{1,-1}(r)$, $\Gamma^1_{0,0}(r)$, $\Gamma^1_{1,0}(r)$, $r = \pm 1$
- Under the condition

$$\frac{\Gamma^1_{0,0}(-1)\Gamma^1_{1,-1}(1) - \Gamma^1_{0,0}(1)\Gamma^1_{1,-1}(-1)}{\Gamma^1_{0,0}(1) + \Gamma^1_{0,0}(-1)} =$$

$$\sum_{r=\pm 1} r\big(\Gamma^2_{1,-1}(r) + \Gamma^1_{1,-1}(r) - \Gamma^1_{1,0}(r) - \Gamma^1_{0,-1}(r)\big)$$

the process has a one-parameter family of stationary product measures $\{\mu_\rho,\ \rho \in [-1, 1]\}$.

A stick process

[Seppäläinen], [Ferrari & Martin]

- $\Omega \subset X^S$ with $X = \mathbb{N}, S = \mathbb{Z}$:

$$\Omega = \{\eta \in \mathbb{N}^{\mathbb{Z}};\ \lim_{n \to -\infty} n^{-2} \sum_{x=n}^{-1} \eta(x) = \lim_{n \to +\infty} n^{-2} \sum_{x=1}^{n} \eta(x) = 0\}$$

- Transition rates are, with $p(1) + p(-1) = 1$,

$$\begin{cases} \Gamma^k_{\alpha,\beta}(r) = p(r)\mathbf{1}_{\{k \le \alpha\}} & \text{for } r = \pm 1, \\ \Gamma^k_{\alpha,\beta}(r) = 0 & \text{otherwise} \end{cases}$$

- In bijection with the generalized discrete HAD process seen from a tagged particle, when holes (resp. positions of particles) of the HAD are turned into particles (resp. sites) for the stick model.
- The geometric product measures $\{\mu_\rho,\ \rho \in [0, +\infty)\}$ with parameter $\rho(1+\rho)^{-1}$ are invariant for the generalized stick process (ρ is the average particles' density per site).

Coupling for the multiple particle jump model

Strategy : Six main steps :

1. Obtain necessary conditions on the transition rates for attractiveness. [Massey]
2. Construct a Markovian coupling.
3. Show that it is increasing under conditions from Step 1 (thus proving their sufficiency).
4. Modify it so that discrepancies between unordered configurations do not increase.
5. Derive irreducibility conditions for the coupled process, under which discrepancies of opposite sign have a positive probability to coalesce.
6. Applications :
   - Determination of extremal invariant and translation invariant probability measures of the initial process.
   - Derivation of hydrodynamics under Euler scaling for conservative processes.

Systems with denumerable state space

[Massey] [Kamae et al.],[Kamae & Krengel]

Definition

W : a set endowed with a partial order relation.

$$V \subset W \text{ is increasing if : } \forall l \in V, m \in W, \quad l \leq m \Longrightarrow m \in V$$

$$V \subset W \text{ is decreasing if : } \forall l \in V, m \in W, \quad l \geq m \Longrightarrow m \in V$$

$$f : W \to \mathbb{R} \text{ is monotone if : } \forall l, m \in W, \quad l \leq m \Longrightarrow f(l) \leq f(m)$$

Examples. For $l \in W$, $I_l = \{m \in W : l \leq m\}$ is increasing ; $D_l = \{m \in W : l \geq m\}$ is decreasing.

Remark. $\forall V \subset W$,
V is increasing $\iff$ $W \setminus V$ is decreasing $\iff$ $\mathbf{1}_V$ is monotone

Necessary conditions

$(\xi, \zeta) \in \Omega \times \Omega$, $\xi \leq \zeta$; $V \subset \Omega$ increasing cylinder set.

$$\text{If } \xi \in V \text{ or } \zeta \notin V, \quad \mathbf{1}_V(\xi) = \mathbf{1}_V(\zeta)$$

By attractiveness, since V increasing,

$$\forall t \geq 0,\ (T(t)\mathbf{1}_V)(\xi) \leq (T(t)\mathbf{1}_V)(\zeta)$$

Combining both, one get

$$\begin{aligned} t^{-1}[(T(t)\mathbf{1}_V)(\xi) - \mathbf{1}_V(\xi)] &\leq t^{-1}[(T(t)\mathbf{1}_V)(\zeta) - \mathbf{1}_V(\zeta)] \\ \text{thus when } t \to 0, \qquad (\mathcal{L}\mathbf{1}_V)(\xi) &\leq (\mathcal{L}\mathbf{1}_V)(\zeta) \end{aligned}$$

This gives two sets of inequalities for the rates

$$\zeta \notin V \Longrightarrow \quad \sum_{\eta \in V} \Gamma(\xi, \eta) \leq \sum_{\eta \in V} \Gamma(\zeta, \eta)$$

$$\xi \in V \Longrightarrow \quad \sum_{\eta \in V^c} \Gamma(\xi, \eta) \geq \sum_{\eta \in V^c} \Gamma(\zeta, \eta)$$

Attractiveness conditions

Theorem

$(\eta_t)_{t\geq 0}$ *is attractive iff :*

$\forall(\alpha,\beta,\gamma,\delta)\in X^4$ *with* $(\alpha,\beta)\leq(\gamma,\delta)$*,* $\forall(x,y)\in S^2$*,*

$$\begin{aligned}
\forall l \geq 0, \sum_{k'>\delta-\beta+l} \Gamma^{k'}_{\alpha,\beta}(y-x) &\leq \sum_{l'>l} \Gamma^{l'}_{\gamma,\delta}(y-x)\\
\forall k \geq 0, \quad \sum_{k'>k} \Gamma^{k'}_{\alpha,\beta}(y-x) &\geq \sum_{l'>\gamma-\alpha+k} \Gamma^{l'}_{\gamma,\delta}(y-x)
\end{aligned}$$

A Markovian increasing coupling

Proposition

$\overline{\mathcal{L}}$ defined on $\Omega \times \Omega$ is the generator of a Markovian coupling

$$\begin{aligned}
\overline{\mathcal{L}} f(\xi,\zeta) &= \sum_{x,y\in S}\sum_{\alpha,\beta\in X}\sum_{\gamma,\delta\in X} \chi_{x,y}^{\alpha,\beta}(\xi)\chi_{x,y}^{\gamma,\delta}(\zeta) \times \\
&\qquad \sum_{(k,l)\neq(0,0)} G_{\alpha,\beta;\,\gamma,\delta}^{k;\,l}(y-x)\big(f(S_{x,y}^k\xi, S_{x,y}^l\zeta) - f(\xi,\zeta)\big)
\end{aligned}$$

$$\begin{aligned}
\textit{with} \qquad & \forall(\alpha,\beta,\gamma,\delta)\in X^4, \forall k>0, \forall l>0, \\
G_{\alpha,\beta;\,\gamma,\delta}^{k;\,l} &= \Big(\Gamma_{\alpha,\beta}^k - \Gamma_{\alpha,\beta}^k \wedge \big(\Sigma_{\gamma,\delta}^l - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^l\big)\Big) \\
&\qquad \wedge \Big(\Gamma_{\gamma,\delta}^l - \Gamma_{\gamma,\delta}^l \wedge \big(\Sigma_{\alpha,\beta}^k - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^l\big)\Big) \\
G_{\alpha,\beta;\,\gamma,\delta}^{0;\,l} &= \Gamma_{\gamma,\delta}^l - \Gamma_{\gamma,\delta}^l \wedge \big(\Sigma_{\alpha,\beta}^0 - \Sigma_{\alpha,\beta}^0 \wedge \Sigma_{\gamma,\delta}^l\big) \\
G_{\alpha,\beta;\,\gamma,\delta}^{k;\,0} &= \Gamma_{\alpha,\beta}^k - \Gamma_{\alpha,\beta}^k \wedge \big(\Sigma_{\gamma,\delta}^0 - \Sigma_{\alpha,\beta}^k \wedge \Sigma_{\gamma,\delta}^0\big)
\end{aligned}$$

where $\Sigma_{\alpha,\beta}^k = \sum_{k'>k} \Gamma_{\alpha,\beta}^{k'}$.

Remarks

- same departure and arrival sites ;
- At most one non zero rate for fixed value of $k+l$.
- diagonal rates : $G^{k;\,k}_{\alpha,\beta;\,\alpha,\beta}=\Gamma^{k}_{\alpha,\beta}$
- when only $k=1$, reduces to basic coupling.

Proposition

Under "Attractiveness Conditions", this Markovian coupling is increasing.

Evolution of discrepancies

Fix $x, y \in S$; $\overline{\mathcal{L}}_{x,y}$ is the generator of jumps between x and y :
$\overline{\mathcal{L}} = \sum_{x',y'} \overline{\mathcal{L}}_{x',y'}$.
For $(\xi, \zeta) \in \Omega \times \Omega$, a measure of positive discrepancies :

$$\begin{aligned} f^+_{x,y}(\xi,\zeta) = f^+_{y,x}(\xi,\zeta) &:= [\xi(x) - \zeta(x)]^+ + [\xi(y) - \zeta(y)]^+ \\ \Delta(\xi(x), \xi(y), \zeta(x), \zeta(y), k, l) &:= f^+_{x,y}(S^k_{x,y}\xi, S^l_{x,y}\zeta) - f^+_{x,y}(\xi,\zeta) \end{aligned}$$

Theorem

$\forall (\xi, \zeta)$ *in* $\Omega \times \Omega$, $(x, y) \in S^2$, $\overline{\mathcal{L}}_{x,y} f^+_{x,y}(\xi,\zeta) \leq 0$.
For all allowed transitions between x, y*, if*
$(\alpha, \beta, \gamma, \delta) = (\xi(x), \xi(y), \zeta(x), \zeta(y))$,

$$\begin{cases} \Delta(\alpha,\beta,\gamma,\delta,k,l) = 0 & \textit{for } (\alpha,\beta),(\gamma,\delta) \textit{ ordered} \\ \Delta(\alpha,\beta,\gamma,\delta,k,l) = 0 & \textit{for } (\alpha,\beta),(\gamma,\delta) \textit{ not ordered} \\ & \textit{and } k - l \in \{0, (\alpha-\gamma)+(\delta-\beta)\} \\ \Delta(\alpha,\beta,\gamma,\delta,k,l) < 0 & \textit{otherwise} \end{cases}$$

Definition

$\overline{\mathcal{L}}$ allows exchange of discrepancies (E.D.) when $\exists(x,y), \exists(\alpha,\beta,\gamma,\delta)$ with $(\alpha-\gamma)(\beta-\delta)<0$, $\exists|k-l|>|\alpha-\gamma|\vee|\beta-\delta|$ for which $G^{k;\,l}_{\alpha,\beta;\,\gamma,\delta}>0$.

extreme case : $k-l=(\alpha-\gamma)+(\delta-\beta)$, total E.D. (otherwise partial E.D.)

- $\overline{\mathcal{L}}$ does not allow E.D. iff $\forall(x,y), \forall(\alpha,\beta,\gamma,\delta), \forall(k,l)$ with $G^{k;\,l}_{\alpha,\beta;\,\gamma,\delta}(y-x)>0$,

$$-\left([\alpha-\gamma]^+\vee[\delta-\beta]^+\right)\le l-k\le[\gamma-\alpha]^+\vee[\beta-\delta]^+$$

  This condition stronger than attractiveness conditions, but coincides with them when $(\alpha,\beta),(\gamma,\delta)$ are ordered.

Hydrodynamic limit Theorem

The empirical measure of configuration η viewed on scale N is

$$\pi^N(\eta)(dx) = N^{-1} \sum_{y \in \mathbb{Z}} \eta(y) \delta_{y/N}(dx) \in \mathcal{M}^+(\mathbb{R})$$

(positive locally finite measures ; topology of vague cv : for continuous test functions with compact support).

Theorem

Assume $p(.)$ has finite third moment. There exists a Lipschitz continuous function G on $[0, K]$ (depending only on $p(.)$, $b(.,.)$) such that :

Let $(\eta_0^N,\ N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued r.v. on a proba. space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that

$$\lim_{N \to \infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-a.s.}$$

for some measurable $[0, K]$-valued profile $u_0(.)$.

Then the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}(\alpha, \eta_0^N(\omega_0), \omega))(dx) = u(., t)dx$$

holds uniformly on all bounded time intervals, where $(x, t) \mapsto u(x, t)$ *denotes the unique entropy solution with initial condition* u_0 *to the conservation law*

$$\partial_t u + \partial_x[G(u)] = 0$$

The macroscopic flux function G

the microscopic flux through site 0 is

$$j_{0,1}(\eta) = \sum_k [\Gamma^k_{\eta(0),\eta(1)}(1) - \Gamma^k_{\eta(1),\eta(0)}(-1)]$$

We define :

$$G(\rho) = \mu_\rho[j_{0,1}(\eta)] \qquad \text{for } \rho \in \mathcal{R}$$

and G is interpolated linearly outside $\mathcal{R}$, so that G is Lipschitz continuous.

In the case $X = \{a, \cdots, b\}$, $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho,\ \rho \in \mathcal{R}\}$, where $\mathcal{R}$ is a closed subset of $[a, b]$ with $\{a, b\} \in \mathcal{R}$, and $\mu_\rho[\eta(0)] = \rho$.

Method

[Bahadoran et al., 1,2,3] Constructive method inspired by [Andjel & Vares] :

1. A variational formula for the entropy solution under Riemann initial condition : $u_0(x) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x \geq 0\}}$, $(\lambda, \rho \in \mathcal{R})$, then derive hydrodynamics under this profile.
2. Extend to any u_0 through an approximation scheme, which requires
   - a finite propagation property (at microscopic and macroscopic levels) ;
   - macroscopic stability [Bramson & Mountford], [MRS] $\rightsquigarrow$ problem with $k, l > 1$.

   control "distance" between particle system and entropy solution.

Proposition

When $S = \mathbb{Z}$*, with only nearest neighbor transitions, and* $\overline{\mathcal{L}}$ *does not allow E.D.,* $\forall(\xi_0, \zeta_0) \in \mathbf{X}^2$ *with* $\sum_{x\in\mathbb{Z}}[\xi_0(x) + \zeta_0(x)] < +\infty$*,* $\forall t > 0$*,*

$$S(\xi_t, \zeta_t) \leq S(\xi_0, \zeta_0)$$

where

$$S(\xi, \zeta) := \sup_{x\in\mathbb{Z}} \left| \sum_{y\leq x} [\xi(y) - \zeta(y)] \right|$$

Irreducibility conditions

(*a*) *For every pair of neighboring sites* (x, y) *and any couple of discrepancies of opposite sign, there is a finite path of coupled transitions on* (x, y) *along which* $f^{+}_{x,y}$ *decreases.*

(*b*) *For every* (x, y)*, let* $X_R(y - x) = X_L(x - y)$ *be the set of values* $\varepsilon \in X$ *such that for all discrepancies* (α, γ)*, there is a finite path of coupled transitions on* (x, y) *starting from* $(\alpha, \varepsilon, \gamma, \varepsilon)$ *which creates a discrepancy on* y*. Then for all* $(x, y) \in E_\Gamma$*, either* $X_R(y - x) = X$ *or* $X_L(y - x) = X$*.*

Irreducibility conditions

When (b) *is not satisfied, we replace* (b) *by* (b') *:*
There exists a set of oriented wedges W_Γ *such that :*

(b'_0) *S is W_Γ-connected*

(b'_1) *for all* $w = (x_0, x_1, x_2) \in W_\Gamma$*,*
$X_R(x_1 - x_0) \bigcup X_R(x_1 - x_2) = X$ *;*

(b'_2) *for all* $w = (x_0, x_1, x_2) \in W_\Gamma$*, all* $\varepsilon_1 \notin X_R(x_1 - x_0)$ *and all* $\varepsilon_2 \in X_R(x_2 - x_1)$*, there is a finite path of coupled transitions on* (x_1, x_2) *from* $(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2)$ *to* $(\varepsilon_3, \varepsilon_4, \varepsilon_3, \varepsilon_4)$ *such that* $\varepsilon_3 \in X_R(x_1 - x_0)$*.*

Invariant Measures

Theorem

If the process is attractive and satisfies Assumption (**IC**)*, then*
1) if the state space of $(\eta_t)_{t\geq 0}$ *is compact, that is* $X = \{a, \cdots, b\}$ *for* $(a, b) \in \mathbb{Z}^2$*, we have* $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho,\ \rho \in \mathcal{R}\}$*, where* $\mathcal{R}$ *is a closed subset of* $[a, b]$ *containing* $\{a, b\}$*, and* μ_ρ *is a translation invariant probability measure on* Ω *with* $\mu_\rho[\eta(0)] = \rho$ *; the measures* μ_ρ *are stochastically ordered, that is,* $\mu_\rho \leq \mu_{\rho'}$ *if* $\rho \leq \rho'$ *;*
2) if $(\eta_t)_{t\geq 0}$ *possesses a one parameter family* $\{\mu_\rho\}_\rho$ *of product invariant and translation invariant probability measures, we have* $(\mathcal{I} \cap \mathcal{S})_e = \{\mu_\rho\}_\rho$*.*

Conclusions

- Starting from basic coupling construction, we constructed increasing Markovian couplings for more general models under necessary and sufficient conditions for attractiveness.
- These constructions uncover new features of increasing couplings : crossing of discrepancies, irreducibility criteria for the coupling, ”non-locality” of coupled jumps, $\cdots$
- There is a number of applications to traffic model, stochastic lattice gas, possibly“attractive condensation”,$\cdots$