Condensation in Totally Asymmetric Inclusion Process

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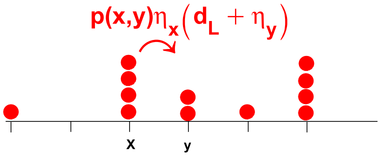
Outline

1. Totally Asymmetric Inclusion Process (TASIP)
2. Condensation in TASIP Model
3. Dynamics of Condensation
   - Stationary Regime
   - Saturation Regime
   - Coarsening Regime
   - Nucleation Regime

Totally Asymmetric Inclusion Process



Lattice : $\Lambda_L = \{1, 2, 3, ..., L\}$ with periodic boundary condition
State space : $\mathbf{X} = \{0, 1, 2, ...\}^{\Lambda_L}$

Configuration : $\boldsymbol{\eta} = (\eta_x)_{x \in \Lambda_L}$. Conserved particles: $\sum_{x \in \Lambda_L} \eta_x = \rho_L L = N$

$$\textbf{Generator} : \mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda} p(x,y)\eta_x(d_L + \eta_y)(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}))$$

where $p(x,y) = \begin{cases} 1, & \text{if } y = x+1 \\ 0, & \text{otherwise} \end{cases}$ (nearest neighbor jump).

Stationary product measure [Grosskinsky, Redig, Vafayi, 2011]

Condensation in TASIP model

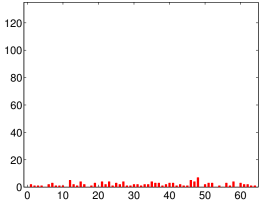
Condensation : all particles accumulate on a single site.

In the limit $d_L \to 0$, a condensation phenomenon occurs in inclusion process :

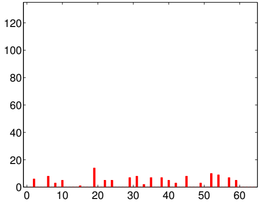
- L and N are both fixed. [Grosskinsky, Redig, Vafayi. 2011]
- in the limit $L \to \infty$, $N \to \infty$ s.t. $\frac{N}{L} \to \rho > 0$. [Chleboun. 2011]
- in the limit L fixed, and $N \to \infty$. [Grosskinsky, Redig, Vafayi. 2012]

MOVIE $[L = 64,\ \rho_L = 2,\ d_L = \frac{1}{L^2}]$

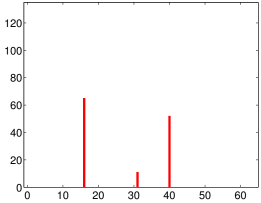
Dynamics of Condensation



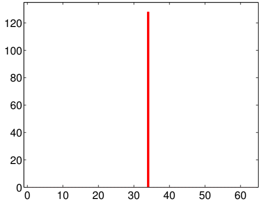
Nucleation →



Coarsening →



Saturation →

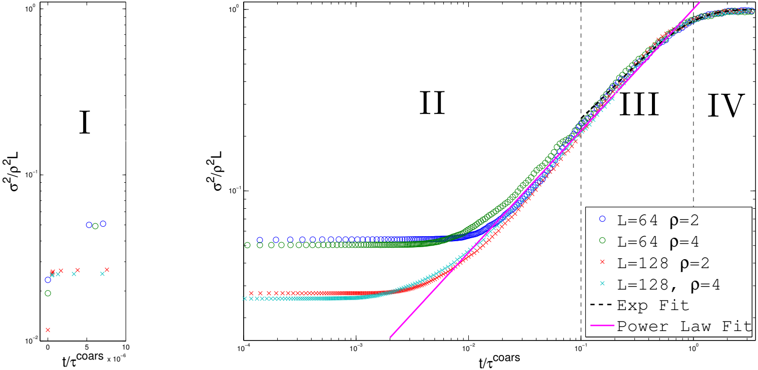


Stationary →

Condensation in TASIP model

Observable: $\sigma^2(t) = \mathbb{E}[\boldsymbol{\eta}^2] = \frac{1}{L}\sum_{x\in\Lambda_L}\eta_x^2$

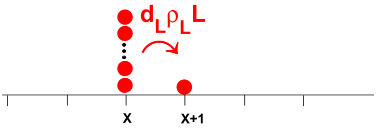
$\sigma^2(0) = \rho^2 + \rho - \frac{1}{L}\rho$ (converges to $\rho(1+\rho)$ as $L \to \infty$).
$\sigma^2(t) \xrightarrow{t\to\infty} \rho^2 L$



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations

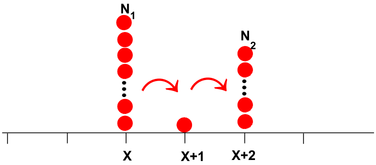
I : Nucleation Regime. II : Coarsening Regime.
III : Saturation Regime. IV : Stationary Regime.

Stationary Regime



- A particle jumps by **diffusion** with rate $d_L \rho_L L$.
- Other particles on site x follow immediately by **inclusion**.
- The single condensate moves ballistically with speed $d_L \rho_L L$.

Two condensates interaction



- N_1, N_2 are both in order L.
- Large condensate will **penetrate** small one without macroscopic change of number of particles.

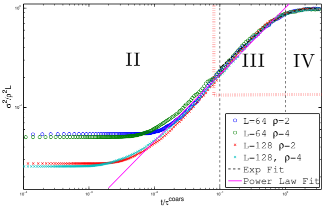
Exponential Approximation of $\sigma^2(t)$

$$\frac{d}{dt}\mathbb{E}\left[f(\boldsymbol{\eta}_t)\right] = \mathbb{E}[(\mathcal{L}_L f)(\boldsymbol{\eta}_t)] \text{ , take } f(\boldsymbol{\eta}) = \eta_z^2 \text{ , } z \in \Lambda_L .$$

Assumption : $\eta_{z-1} = \eta_{z+1} = \frac{\rho_L L - \eta_z}{L-1}$

$\Rightarrow \sigma^2(t) \simeq \rho_L^2 L \left(1 - e^{-\frac{2}{L}t}\right)$

Saturation Regime



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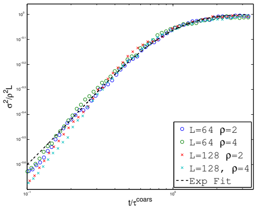


Exponential fit of saturation regime

Coarsening Regime

$n(t)$: **number** of piles. $m(t)$: **size** of a typical pile. $(m(t)n(t) = \rho_L L)$

$v \sim d_L m(t)$: **speed** of a pile. $s \sim \frac{L}{n(t)}$: average **distance** of two piles.

Differential equation

$$\frac{d}{dt}m(t) \sim \frac{v}{s} = \rho_L d_L \quad \text{, initial condition : } m(0) = \frac{\rho_L}{r}$$

r : average ratio of occupied sites after nucleation.

$\Rightarrow$

$$m(t) = C_1 \rho_L d_L t + \frac{\rho_L}{r}$$

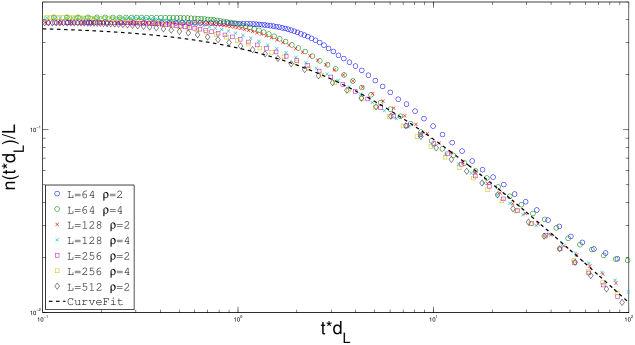
After time τ_L^{coars}, $m(t)$ will grow to size L.

$$m(\tau_L^{coars}) \sim L \quad \Rightarrow \quad \tau_L^{coars} \sim \frac{L}{d_L}$$

Coarsening Regime

Ratio of occupied sites:

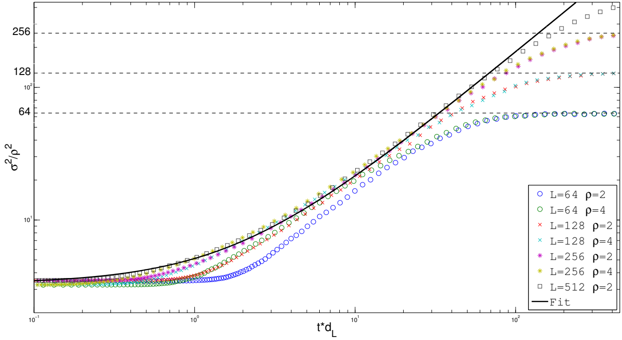
$$\frac{n(t)}{L} = \frac{1}{C_1 d_L t + 1/r} \ ,$$



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations. Fitting constant $C_1 \approx 0.8538$

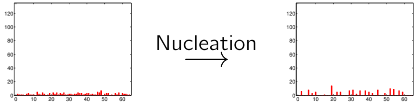
Second moment:

$$\sigma_L^2(t) \sim \frac{1}{L} n(t) m^2(t) \quad \Rightarrow \quad \frac{\sigma_L^2(t)}{\rho_L^2} = \tilde{C}\left(C_1 d_L t + \frac{1}{r}\right)$$

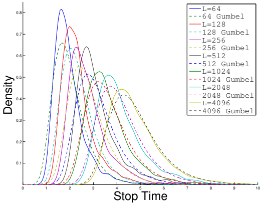


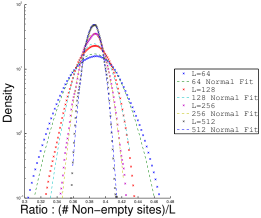
Simulation results with $d_L = 1/L^2$, averaged with 200 realizations. Fitting constant $C_1 \approx 0.5513$ and $\tilde{C} \approx 3.3070$

Nucleation Regime [in progress]



- Time scale of this regime is much smaller compared with coarsening regime.
- R: **ratio of occupied sites** when this regime ends. (appr. Normal distributed with mean r).
- T : **duration** of this regime. (appr. Gumbel distributed).





Toy model for R:

$\eta_x = 1, \forall x \in \Lambda_L$. Waiting time T_x, i.i.d. r.v.
Particles **merge** if they stay on the same site.

$$...11111111... \Rightarrow ...11111101...$$

Absorbing state:

$$...1010001\mathbf{0001}00...$$

Constructed by **blocks** (e.g.**0001**) of size X_n, where $2 \leq X_n \leq L$ and $\sum_n X_n = L$.

Block with '1' on fixed site x has k '0's $\iff T_{x-k} < T_{x-k+1} < ... < T_{x-1}$

T_x are i.i.d. $\implies$ uniform permutation.

Toy model for R:

$$\mathbb{P}[X_n - 1 \geq k] = \frac{1}{k!} \Rightarrow \mathbb{E}[X_n] = \sum_{k=1}^{L-1} \frac{1}{k!} + 1 \to e \ \text{, as } \ L \to \infty.$$

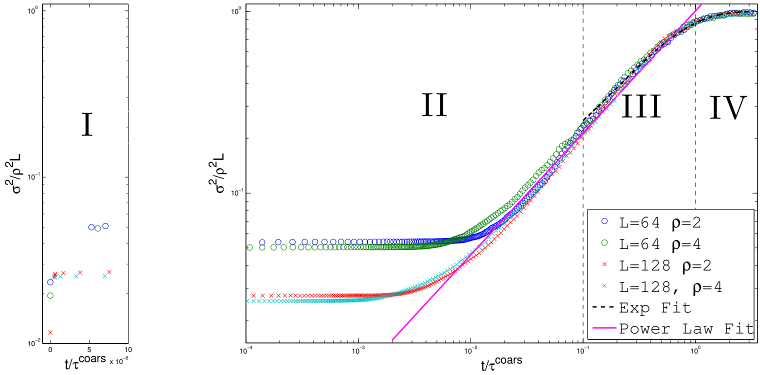
X_n forms a **renewal process** :

$$N(L) = \max \left\{ n \ : \ \sum_{i=1}^{n} X_i \leq L \right\},$$

where $N(L)$ is number of particles in n blocks.
From renewal theorem:

$$\frac{N(L)}{L} \to \frac{1}{\mathbb{E}[X_1]} = \frac{1}{e} \ , as \ \ L \to \infty \ \text{ almost surely } \ .$$

Review



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THANK YOU