Euler hydrodynamics for attractive particle systems in random environment

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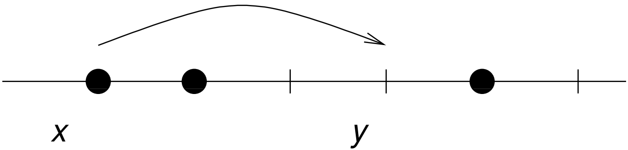
Outline

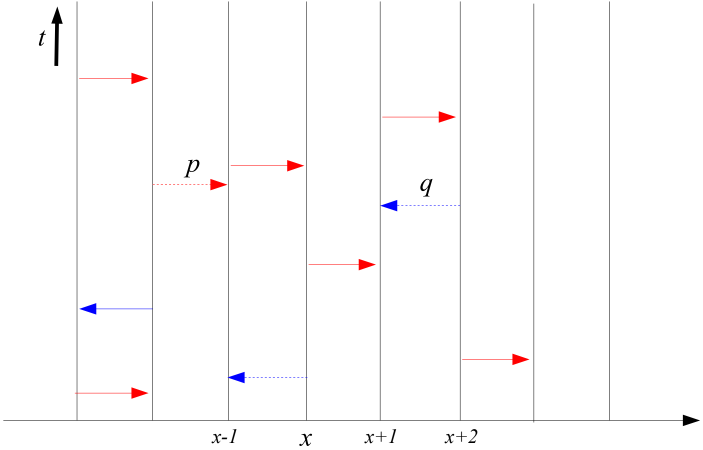
Basic example : The ASEP (without disorder)



- $K = 1$, for $z \in \mathbb{Z}$, $\eta(z) = 0$ or 1.
- From each site x, choice of y with $p(y - x)$.
- According to (independent) exponential clocks, jump from x to y if possible (*exclusion rule*).

Graphical construction [Harris]

$\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_H$ for initial configurations and Poisson processes.
(below for ASEP) $\rightsquigarrow$ basic coupling.



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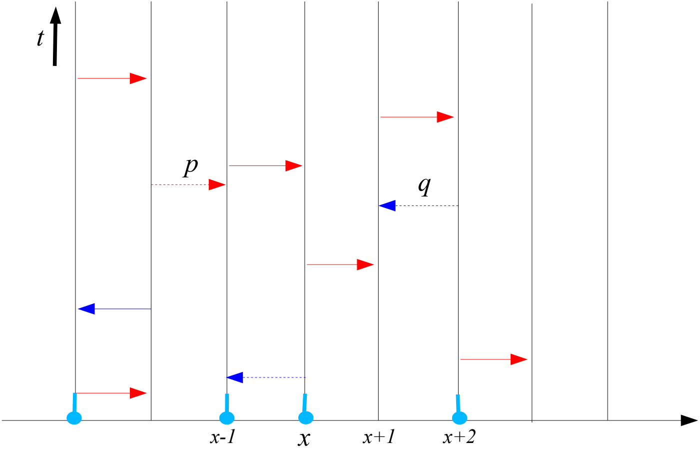

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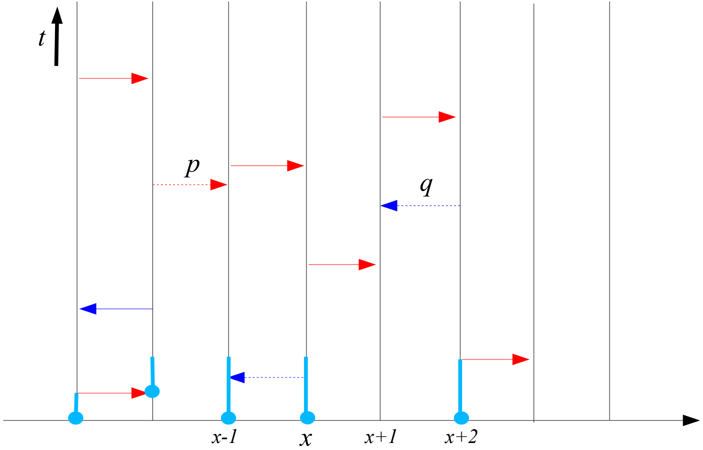

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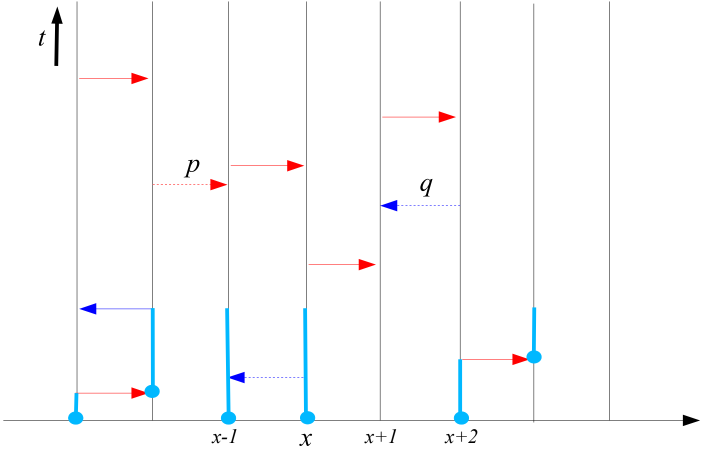

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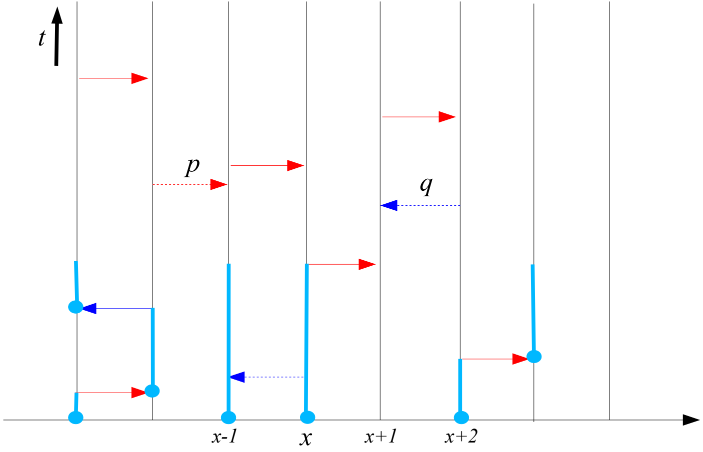

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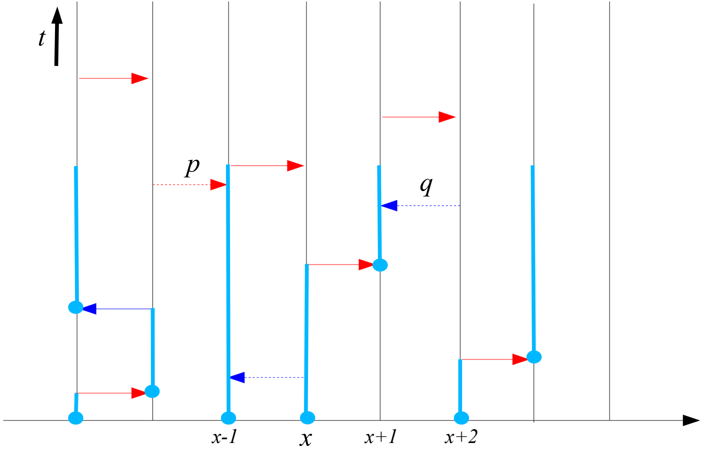

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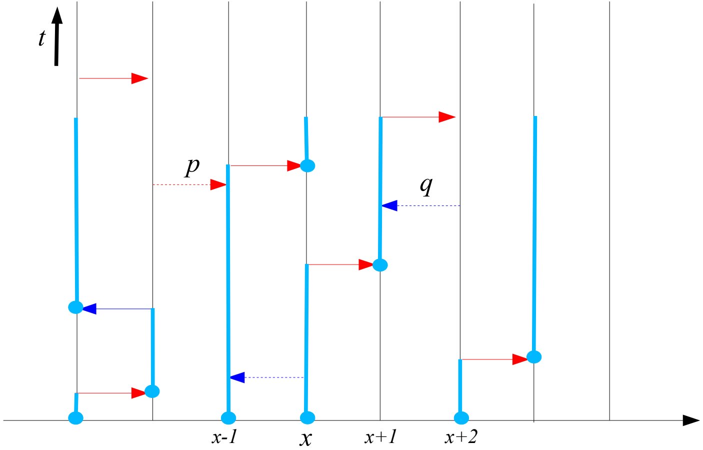

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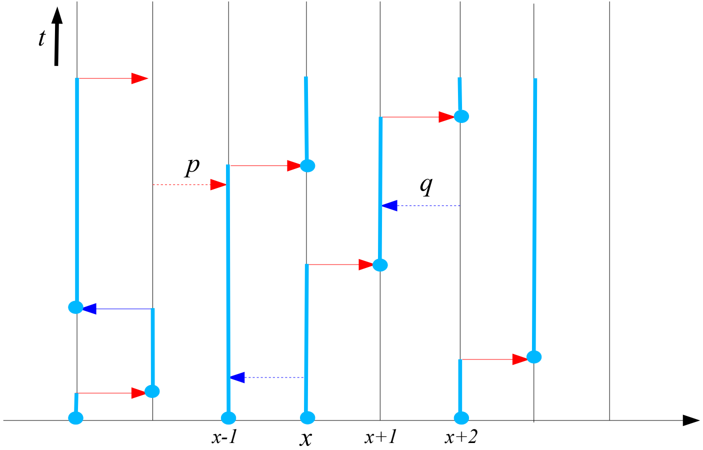

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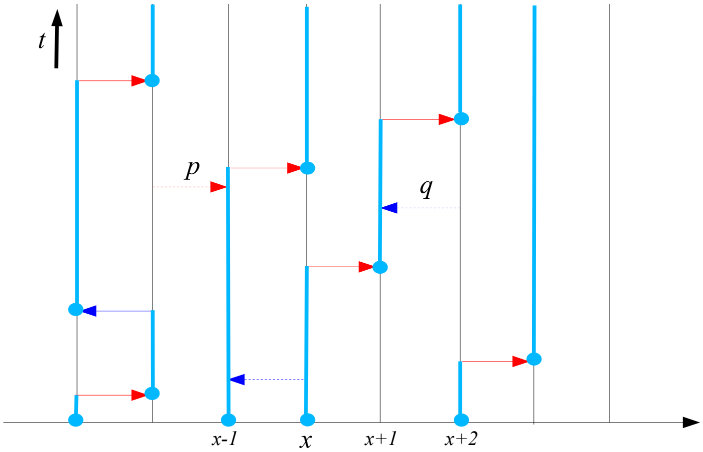

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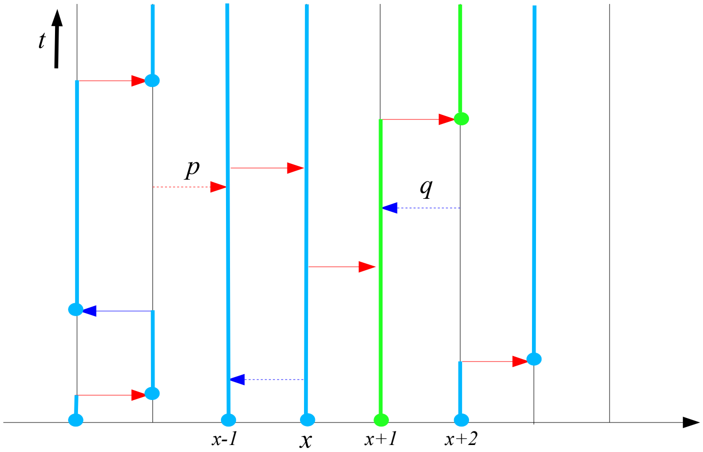


Attractive Systems

The Markov process $(\eta_t)_{t\geq 0}$ with generator L and semigroup $S(t)$ is attractive :

- *partial order* on $\mathbf{X}$: $\eta \leq \xi \Leftrightarrow \forall x \in \mathbb{Z},\ \eta(x) \leq \xi(x)$.
  Extended to probabilities on $\mathbf{X}$: $\mu_1 \leq \mu_2$
- *basic coupling* :
  $(\eta_t, \xi_t)_{t\geq 0}$ on $\mathbf{X} \times \mathbf{X}$; η_t and ξ_t obey the same clocks.
  $\eta_0 \leq \xi_0$ a.s. $\Rightarrow \eta_t \leq \xi_t$ a.s. $\forall t > 0$.

The Misanthrope type model (without disorder)

State space $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$, $\eta \in \mathbf{X}$, $0 \leq \eta(x) \leq K, \forall x \in \mathbb{Z}$

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}} p(y-x)b(\eta(x),\eta(y))\left[f\left(\eta^{x,y}\right) - f(\eta)\right]$$

- *(A1)* Irreducibility : $\forall z \in \mathbb{Z}$, $\sum_{n\in\mathbb{N}}[p^{*n}(z) + p^{*n}(-z)] > 0$;
- *(A2)* finite mean : $\sum_{z\in\mathbb{Z}} |z|\, p(z) < +\infty$;
- *(A3)* K-exclusion rule : $b(0,.) = 0$, $b(.,K) = 0$;
- *(A4)* non-degeneracy : $b(1, K-1) > 0$;
- *(A5)* attractiveness : $b(i,j)$ nondecreasing in i, nonincreasing in j.

Some Classical Examples :

- Simple Exclusion : $K = 1$, $b(1,0) = 1$ [Liggett]
- T.A. K-exclusion : $p(1) = 1$, $b(i,j) = \mathbf{I}_{\{i>0,j<K\}}$ [Seppäläinen]
- Misanthropes : + algebraic relations on b's [Cocozza]

The model in random environment

Disorder $\alpha = (\alpha(x), x \in \mathbb{Z}) \in \mathbf{A} = (c, 1/c)^{\mathbb{Z}}$, for $c \in (0,1)$.
The dist. Q of α on $\mathbf{A}$ is ergodic w.r.t. τ_x (on $\mathbb{Z}$).
For $\alpha \in \mathbf{A}$, quenched process $(\eta_t)_{t \geq 0}$ on $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$:

$$L_\alpha f(\eta) = \sum_{x,y \in \mathbb{Z}} \alpha(x) p(y-x) b(\eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta) \right] \quad (1)$$

Our method is robust w.r.t. the model and disorder (e.g. no restriction to site or bond disorder).
We detail the misanthropes' process with site disorder, then explain how to deal with other models.

Graphical construction

Let $\mathcal{V} = \mathbb{Z} \times [0,1]$, $(\Omega, \mathcal{F}, \mathbb{P})$ proba. space of locally finite point measures $\omega(dt, dx, dv)$ on $\mathbb{R}^+ \times \mathbb{Z} \times \mathcal{V}$, where $\mathcal{F}$ is generated by the mappings $\omega \mapsto \omega(S)$ for Borel sets S of $\mathbb{R}^+ \times \mathbb{Z} \times \mathcal{V}$, and $\mathbb{P}$ makes ω a Poisson process with intensity

$$M(dt, dx, dv) = \lambda_{\mathbb{R}^+}(dt)\lambda_{\mathbb{Z}}(dx)m(dv)$$

where λ denotes either the Lebesgue or the counting measure. $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.
For the case (1) we take $\mathcal{V} := \mathbb{Z} \times [0,1]$, $v = (z, u) \in \mathcal{V}$,

$$m(dv) = c^{-1}||b||_\infty p(dz)\lambda_{[0,1]}(du) \qquad (2)$$

(A2) $\Rightarrow$ for $\mathbb{P}$-a.e. ω, $\exists$ a unique mapping

$$(\alpha, \eta_0, t) \in \mathbf{A} \times \mathbf{X} \times \mathbb{R}^+ \mapsto \eta_t = \eta_t(\alpha, \eta_0, \omega) \in \mathbf{X} \qquad (3)$$

satisfying : *(a)* $t \mapsto \eta_t(\alpha, \eta_0, \omega)$ is right-continuous ; *(b)* $\eta_0(\alpha, \eta_0, \omega) = \eta_0$; *(c)* the particle configuration is updated at points $(t, x, v) \in \omega$ (and only at such points - where $(t, x, v) \in \omega$ means $\omega\{(t, x, v)\} = 1$) according to the rule

$$\eta_t(\alpha, \eta_0, \omega) = \mathcal{T}^{\alpha,x,v}\eta_{t-}(\alpha, \eta_0, \omega) \qquad (4)$$

where, for $v = (z, u) \in \mathcal{V}$,

$$\mathcal{T}^{\alpha,x,v}\eta = \begin{cases} \eta^{x,x+z} & \text{if} \quad u < \alpha(x)\dfrac{b(\eta(x), \eta(x+z))}{c^{-1}||b||_\infty} \\ \eta & \quad\text{otherwise} \end{cases} \qquad (5)$$

Shift commutation property

$$\mathcal{T}^{\tau_x\alpha,y,v}\tau_x = \tau_x\mathcal{T}^{\alpha,y+x,v} \qquad (6)$$

where τ_x on the r.h.s. acts only on η.

Attractiveness

By *(A5)*,

$$\mathcal{T}^{\alpha,x,v} : \mathbf{X} \to \mathbf{X} \text{ is nondecreasing} \tag{7}$$

Hence,

$$(\alpha, \eta_0, t) \mapsto \eta_t(\alpha, \eta_0, \omega) \text{ is nondecreasing w.r.t. } \eta_0 \tag{8}$$

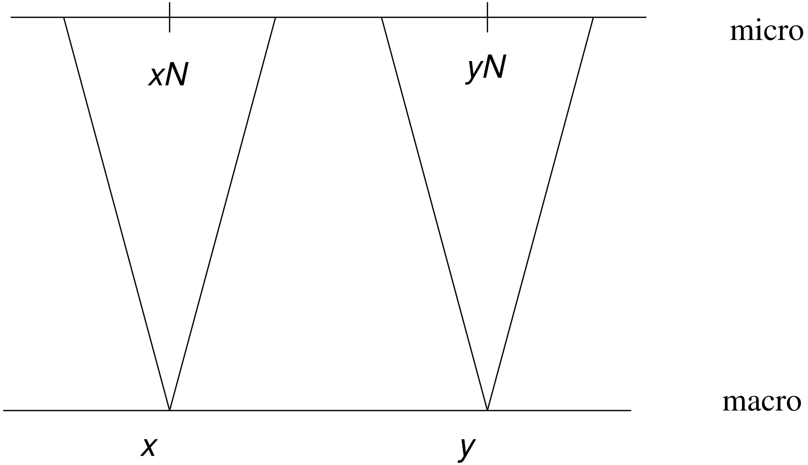
Thus for any $t \in \mathbb{R}^+$ and continuous function f on $\mathbf{X}$, $\mathbb{E}[f(\eta_t(\alpha, \eta_0, \omega))] = S_\alpha(t) f(\eta_0)$, where S_α denotes the semigroup generated by L_α. From (8), for $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$,

$$\mu_1 \leq \mu_2 \Rightarrow \forall t \in \mathbb{R}^+,\ \mu_1 S_\alpha(t) \leq \mu_2 S_\alpha(t) \tag{9}$$

Property (9) is usually called *attractiveness*. Condition (7) implies the stronger complete monotonicity property : existence of a monotone Markov coupling for an arbitrary number of processes with generator (1).

Hydrodynamic scaling

$N \in \mathbb{N}$: scaling parameter for the hydrodynamic limit (*i.e.* inverse of the macroscopic distance between two consecutive sites, and time rescaling).



Hydrodynamic limit Theorem

The empirical measure of configuration η viewed on scale N is

$$\pi^N(\eta)(dx) = N^{-1}\sum_{y\in\mathbb{Z}}\eta(y)\delta_{y/N}(dx) \in \mathcal{M}^+(\mathbb{R}) \tag{10}$$

(positive locally finite measures ; topology of vague cv : for continuous test functions with compact support).

Theorem

Assume $p(.)$ has finite third moment. Let Q be an ergodic proba. dist. on $\mathbf{A}$. Then there exists a Lipschitz continuous function G^Q on $[0,K]$ (depending only on $p(.)$, $b(.,.)$ and Q) such that :
Let $(\eta_0^N,\ N\in\mathbb{N})$ be a sequence of $\mathbf{X}$-valued r.v. on a proba. space $(\Omega_0,\mathcal{F}_0,\mathbb{P}_0)$ such that

$$\lim_{N\to\infty}\pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-a.s.} \tag{11}$$

for some measurable $[0,K]$-valued profile $u_0(.)$.

Then for Q-a.e. $\alpha \in \mathbf{A}$, the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}(\alpha, \eta_0^N(\omega_0), \omega))(dx) = u(., t)dx \qquad (12)$$

holds uniformly on all bounded time intervals, where $(x,t) \mapsto u(x,t)$ denotes the unique entropy solution with initial condition u_0 to the conservation law

$$\partial_t u + \partial_x[G^Q(u)] = 0 \qquad (13)$$

We have now to precise what we mean by

- having a strong density profile and hydrodynamic limit ;
- what is an entropy solution of (13).

Hydrodynamics vs a.s. hydrodynamics

The sequence $(\eta^N)_N$ has

- *(weak) density profile* $u(.)$ if : $\forall \varepsilon > 0, \psi$,

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{R}} \psi(x)\pi^N(\eta^N)(dx) - \int_{\mathbb{R}} \psi(x)u(x)dx\right| > \varepsilon\right) = 0$$

- *strong density profile* $u(.)$ if : $\forall \psi$,

$$\mathbb{P}\left(\lim_{N\to\infty} \int_{\mathbb{R}} \psi(x)\pi^N(\eta^N)(dx) = \int_{\mathbb{R}} \psi(x)u(x)dx\right) = 1,$$

The sequence $(\eta^N_t,\ t \geq 0)_N$ has

- *hydrodynamic limit* (resp. *a.s.* hydrodynamic limit) $u(.,.)$ if : $\forall t \geq 0$, $(\eta^N_{Nt})_N$ has weak (resp. strong) density profile $u(.,t)$.

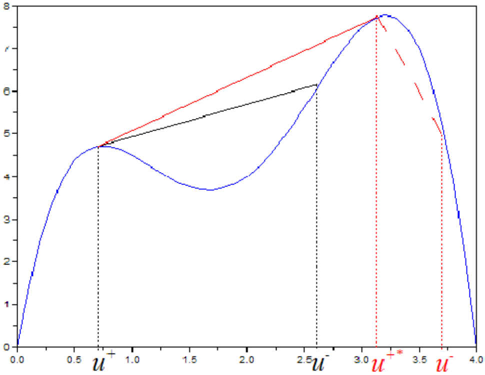
Entropy solution

$u = u(.,.) : \mathbb{R} \times \mathbb{R}^{+*} \to \mathbb{R}$ has *locally bounded space variation* if : $\forall J \subset \mathbb{R}^{+*}, I = [a, b] \subset \mathbb{R}$,

$$\sup_{t \in J} \sup_{x_0 = a < x_1 \cdots < x_n = b} \sum_{i=0}^{n-1} |u(x_{i+1}, t) - u(x_i, t)| < +\infty$$

Let u be a weak solution to (13) with locally bounded space variation. Then u is an entropy solution to (13) iff, for a.e. $t > 0$, all discontinuities of $u(., t)$ are entropy shocks.

Oleĭnik's entropy condition



A discontinuity (u^-, u^+), with $u^\pm := u(x \pm 0, t)$, is an entropy shock, iff :

> The chord of the graph of G between u^- and u^+ lies above (below) the graph if $u^+ < u^-$ $(u^- < u^+)$.

The macroscopic flux function G^Q

the microscopic flux through site 0 is

$$\begin{aligned} j(\alpha,\eta) &= j^+(\alpha,\eta) - j^-(\alpha,\eta) \\ j^+(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\, y\leq 0<y+z} \alpha(y)p(z)b(\eta(y),\eta(y+z)) \\ j^-(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\, y+z\leq 0<y} \alpha(y)p(z)b(\eta(y),\eta(y+z)) \end{aligned} \tag{14}$$

We will see below that :
$\exists\mathcal{R}^Q \subset [0,K]$ closed, $\exists\widetilde{\mathbf{A}}^Q \subset \mathbf{A}$ with $Q(\widetilde{\mathbf{A}}^Q) = 1$ (both depending also on $p(.)$ and $b(.,.)$), and $\exists$ a family of proba. measures $(\nu_\alpha^{Q,\rho} :\ \alpha \in \widetilde{\mathbf{A}}^Q, \rho \in \mathcal{R}^Q)$ on $\mathbf{X}$, such that

$\forall \rho \in \mathcal{R}^Q$:

- *(B1)* $\forall \alpha \in \widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$ is an invariant measure for L_α.
- *(B2)* $\forall \alpha \in \widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$-a.s.,

$$\lim_{l\to\infty} (2l+1)^{-1} \sum_{x\in\mathbb{Z}:\,|x|\leq l} \eta(x) = \rho$$

- *(B3)*

$$G_\alpha^Q(\rho) := \int j(\alpha,\eta)\nu_\alpha^{Q,\rho}(d\eta) \tag{15}$$

  does not depend on $\alpha \in \widetilde{\mathbf{A}}^Q$.

Hence we define $G^Q(\rho)$ as (15) for $\rho \in \mathcal{R}^Q$ and extend it by linear interpolation on the complement of $\mathcal{R}^Q$, which is a finite or countably infinite union of disjoint open intervals.
Lipschitz constant V of G^Q :

$$V = 2c^{-1}||b||_\infty \sum_{z\in\mathbb{Z}} |z|p(z) \tag{16}$$

Remark. $\nu_\alpha^{Q,\rho}$ not explicit $\Longrightarrow G^Q(\rho)$ not explicit.
$\rightsquigarrow$ influence of disorder not visible in $G^Q(\rho)$.

Examples without disorder

Invariant measures

- Simple exclusion processes : [Liggett]
  $\mathcal{R} = [0,1]$, $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho,\ \rho \in [0,1]\}$, Bernoulli product.
- T.A. K-Exclusion : [BGRS2]
  Known : 0 and K are limit points of $\mathcal{R}$ and
  $\mathcal{R} \cap [\frac{1}{3}, K - \frac{1}{3}] \neq \emptyset$.

Macroscopic flux

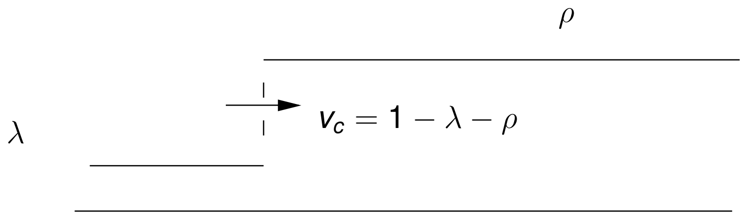
- Exclusion Process : $G(u) = \gamma u(1-u)$, $u \in [0,1]$
- T.A. K-Exclusion : $G(u) = G(K-u)$, $u \in [0,K]$.

Hydrodynamics : The TASEP

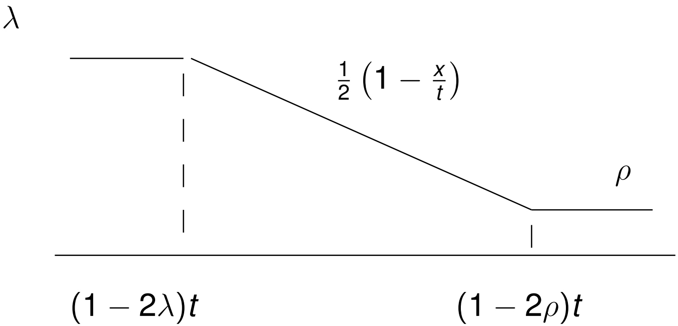
$$\partial_t u + \partial_x [u(1-u)] = 0$$

$G(u) = u(1-u)$ concave flux function
Riemann problem : Initial condition (λ, ρ)

- case $\lambda < \rho$: shock



- case $\lambda > \rho$: rarefaction fan



Previous results (in random environment)

under attractiveness ;
i. i.d. site disorder ; quenched hydrodynamic limit ;

- [Benjamini, Ferrari & Landim] ('96) : Asymmetric ZRP on $\mathbb{Z}^d$ (an extension of [Evans] ('96)).
- [Seppäläinen] ('99) : TA K-exclusion on $\mathbb{Z}$.

Our results : ergodic disorder ; no restriction to site disorder.

Our method

- Since the quenched process lacks translation invariance, an essential ingredient in proving hydrodynamics, we study a joint disorder-particle process with its invariant measures.
- then follow the scheme of proof of a.s. hydrodynamics in [BGRS3] (without disorder) :
- hydrodynamics for the Riemann problem

$$R_{\lambda,\rho}(x,0) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x\geq 0\}} \tag{17}$$

- it implies the result for general u_0 by an approximation scheme.

The disorder-particle process

$$Lf(\alpha,\eta)=\sum_{x,y\in\mathbb{Z}}\alpha(x)p(y-x)b(\eta(x),\eta(y))\left[f\left(\alpha,\eta^{x,y}\right)-f(\alpha,\eta)\right] \tag{18}$$

$(S(t), t\in\mathbb{R}^+)$ is the semigroup generated by L. Given $\alpha_0=\alpha$, this means that $\alpha_t=\alpha$ for all $t\geq 0$, while $(\eta_t)_{t\geq 0}$ is a Markov process with generator L_α given by (1).
L is translation invariant :

$$\tau_x L = L\tau_x \tag{19}$$

where τ_x acts jointly on (α,η). This is equivalent to a commutation property for the quenched dynamics :

$$L_\alpha\tau_x=\tau_x L_{\tau_x\alpha} \tag{20}$$

where the first τ_x on the r.h.s. acts only on η.

A conditional stochastic order

Define

$$\begin{aligned} \overline{\mathcal{O}} &= \overline{\mathcal{O}}_+ \cup \overline{\mathcal{O}}_- \\ \overline{\mathcal{O}}_+ &= \{(\alpha,\eta,\xi)\in \mathbf{A}\times\mathbf{X}^2 : \eta\leq\xi\} \\ \overline{\mathcal{O}}_- &= \{(\alpha,\eta,\xi)\in \mathbf{A}\times\mathbf{X}^2 : \xi\leq\eta\} \end{aligned} \tag{21}$$

Lemma

For $\mu^1=\mu^1(d\alpha,d\eta)$, $\mu^2=\mu^2(d\alpha,d\eta)$ proba. measures on $\mathbf{A}\times\mathbf{X}$, the following properties (denoted by $\mu^1\ll\mu^2$) are equivalent :
(i) $\forall f$ bounded measurable local function on $\mathbf{A}\times\mathbf{X}$, such that $f(\alpha,.)$ is nondecreasing for all $\alpha\in\mathbf{A}$, we have $\int f\,d\mu^1\leq\int f\,d\mu^2$.
(ii) μ^1 and μ^2 have a common α-marginal Q, and $\mu^1(d\eta|\alpha)\leq\mu^2(d\eta|\alpha)$ for Q-a.e. $\alpha\in\mathbf{A}$.
(iii) $\exists\overline{\mu}(d\alpha,d\eta,d\xi)$ coupling measure supported on $\overline{\mathcal{O}}_+$, under which $(\alpha,\eta)\sim\mu^1$ and $(\alpha,\xi)\sim\mu^2$.

Invariant Measures

$\mathcal{I}_L$, $\mathcal{S}$ and $\mathcal{S}^{\mathbf{A}}$: the sets of proba. measures respectively invariant for L, shift-invariant on $\mathbf{A} \times \mathbf{X}$ and shift-invariant on $\mathbf{A}$.

Proposition

For every $Q \in \mathcal{S}_e^{\mathbf{A}}$, there exists a closed subset $\mathcal{R}^Q$ of $[0, K]$ containing 0 and K, such that

$$(\mathcal{I}_L \cap \mathcal{S})_e = \left\{\nu^{Q,\rho},\ Q \in \mathcal{S}_e^{\mathbf{A}},\ \rho \in \mathcal{R}^Q\right\}$$

where index e denotes the set of extremal elements, and $(\nu^{Q,\rho} :\ \rho \in \mathcal{R}^Q)$ is a family of shift-invariant measures on $\mathbf{A} \times \mathbf{X}$, weakly continuous with respect to ρ, such that

$$\int \eta(0)\nu^{Q,\rho}(d\alpha, d\eta) = \rho$$

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:|x|\le l}\eta(x) = \rho, \quad \nu^{Q,\rho}-\text{a.s.}$$

$$\rho\le\rho' \Rightarrow \nu^{Q,\rho} \ll \nu^{Q,\rho'}$$

For $\rho = 0 \in \mathcal{R}^Q$ (resp. $\rho = K \in \mathcal{R}^Q$) we get the invariant dist. $\delta_0^{\otimes\mathbb{Z}}$ (resp. $\delta_K^{\otimes\mathbb{Z}}$), the deterministic dist. of the configuration with no particles (resp. with maximum number of particles K everywhere).

Corollary

(i) The family $\nu_\alpha^{Q,\rho}(.) := \nu^{Q,\rho}(.|\alpha)$ *on* $\mathbf{X}$ *satisfies properties (B1)–(B3) on page 27 ;*
(ii) for $\rho \in \mathcal{R}^Q$, $G^Q(\rho) = \int j(\alpha,\eta)\nu^{Q,\rho}(d\alpha, d\eta)$.

Remark

By (ii), and shift-invariance of $\nu^{Q,\rho}(d\alpha, d\eta)$,

$$G^Q(\rho) = \int j(\alpha,\eta)\nu^{Q,\rho}(d\alpha, d\eta) = \int \widetilde{\jmath}(\alpha,\eta)\nu^{Q,\rho}(d\alpha, d\eta) \quad (22)$$

$\forall \rho \in \mathcal{R}^Q$, *where*

$$\widetilde{\jmath}(\alpha,\eta) := \alpha(0)\sum_{z\in\mathbb{Z}} zp(z)b(\eta(0),\eta(z)) \quad (23)$$

$\rightsquigarrow$ *alternatively take* $\widetilde{\jmath}(\alpha,\eta)$ *as microscopic flux function.*

The properties used to prove Theorem 1

- The set of environments is a proba. space $(\mathbf{A}, \mathcal{F}_{\mathbf{A}}, Q)$, for $\mathbf{A}$ a compact metric space. On $\mathbf{A}$ we have a group of space shifts $(\tau_x : x \in \mathbb{Z})$, w.r.t. which Q is ergodic. $\forall \alpha \in \mathbf{A}$, L_α is the generator of a Feller process on $\mathbf{X}$ (by *(A3)*) satisfying the commutation property (20). It is equivalent to L translation-invariant on $\mathbf{A} \times \mathbf{X}$.
- For L_α, $\exists$ (by *(A2)*) a graphical construction on a space-time Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$ such that L_α is given by some mapping $\mathcal{T}^{\alpha,z,v}$ satisfying the shift commutation and strong attractiveness properties (6) and (7).
- Irreducibility and non-degeneracy *(A1), (A4)* (combined with attractiveness *(A5)*) ($\rightsquigarrow$ for Proposition 3.1).

In our examples, particle jumps : $\mathcal{T}^{\alpha,z,v}\eta = \eta^{x(\alpha,z,v),y(\alpha,z,v)}$. So

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} c_\alpha(x,y,\eta)\left[f\left(\eta^{x,y}\right) - f(\eta)\right] \tag{24}$$

$$c_\alpha(x,y,\eta) = \sum_{z\in\mathbb{Z}} m\left(\{v\in\mathcal{V}:\ \mathcal{T}^{\alpha,z,v}\eta = \eta^{x,y}\}\right) \tag{25}$$

shift-commutation property (6) $\Longrightarrow$

$$c_\alpha(x,y,\eta) = c_{\tau_x\alpha}(0,y-x,\tau_x\eta) \tag{26}$$

(is equivalent to (20) for (24)). Micro. fluxes (14), (23) write

$$\begin{aligned} j^+(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\ y\leq 0<y+z} c_\alpha(\eta(y),\eta(y+z)) \\ j^-(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\ y+z\leq 0<y} c_\alpha(\eta(y),\eta(y+z)) \\ \widetilde{\jmath}(\alpha,\eta) &= \sum_{z\in\mathbb{Z}} z c_\alpha(0,z,\eta) \end{aligned} \tag{27}$$

Generalized misanthropes' process

$c \in (0,1)$, and $p(.)$, $P(.)$ proba. dist. on $\mathbb{Z}$ satisfying *(A1)*, resp. *(A2)*. **A** : functions $B : \mathbb{Z}^2 \times \{0,\ldots,K\}^2 \to \mathbb{R}^+$ such that $\forall (x,z) \in \mathbb{Z}^2$, $B(x,z,.,.)$ satisfies *(A3)–(A5)* and

$$B(x,z,1,K-1) \geq cp(z) \qquad (28)$$

$$B(x,z,K,0) \leq c^{-1}P(z) \qquad (29)$$

The shift operator τ_y on **A** : $(\tau_y B)(x,z,n,m) = B(x+y,z,n,m)$.

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} B(x,y-x,\eta(x),\eta(y))\left[f\left(\eta^{x,y}\right) - f(\eta)\right] \qquad (30)$$

where the dist. Q of $B(.,.,.,.)$ is ergodic w.r.t. τ_y.

For $v = (z, u)$, set $m(dv) = c^{-1}P(dz)\lambda_{[0,1]}(du)$ in (2), and replace (5) with

$$\mathcal{T}^{\alpha,x,v}\eta = \begin{cases} \eta^{x,x+z} & \text{if} \quad u < \dfrac{B(x,z,\eta(x),\eta(x+z))}{c^{-1}P(z)} \\ \eta & \text{otherwise} \end{cases} \tag{31}$$

microscopic flux (27) writes

$$\widetilde{\jmath}(\alpha,\eta) = \sum_{z\in\mathbb{Z}} zB(0,z,\eta(0),\eta(z))$$

Lipschitz constant $V = 2c^{-1}\sum_{z\in\mathbb{Z}}|z|P(z)$ for G^Q

Examples

- The basic model (1) : $B(x,z,n,m)=\alpha(x)p(z)b(n,m)$, for $p(.)$ a proba. dist. on $\mathbb{Z}$ satisfying *(A1)–(A2)*, $\alpha(.)$ an ergodic $(c,1/c)$-valued random field, $b(.,.)$ a function satisfying *(A3)–(A5)*. (28)–(29) hold with $P(.)=p(.)$.
- bond-disorder version of (1) :
  $B(x,z,n,m)=\alpha(x,x+z)b(n,m)$ for a positive random field $\alpha=(\alpha(x,y):\ x,y\in\mathbb{Z})$ on $\mathbb{Z}^2$, bounded away from 0, ergodic w.r.t. space shift $\tau_z\alpha=\alpha(.+z,.+z)$. Sufficient assumptions replacing *(A1)*, *(A2)* are

$$c\,p(y-x)\leq\alpha(x,y)\leq c^{-1}P(y-x) \tag{32}$$

  for $c>0$, and proba. dist. $p(.)$, $P(.)$ on $\mathbb{Z}$, satisfying *(A1)* resp. *(A2)*.
- Switch between two rate functions according to environment : $(\alpha(x),\ x\in\mathbb{Z})$ is an ergodic $\{0,1\}$-valued field, $p(.)$ satisfies *(A1)*, and b_0, b_1 *(A3)–(A5)*, $B(x,z,n,m)=p(z)[(1-\alpha(x))b_0(n,m)+\alpha(x)b_1(n,m)]$.

Generalized k-step K-exclusion process

The k-step exclusion process

$k \in \mathbb{N}$, $p(.)$ a jump kernel on $\mathbb{Z}$ satisfying *(A1), (A2)*.
A particle at x performs a random walk with $p(.)$ and jumps to the first vacant site it finds along this walk, unless it returns to x or does not find an empty site within k steps, in which case it stays at x.

Generalization without disorder : the (q, β)-k step K-exclusion

$K \geq 1$, $c \in (0,1)$; $\mathcal{D}$ is the set of functions $\beta = (\beta^1, \ldots, \beta^k) : \mathbb{Z}^k \to (0,1]^k$ s.t.

$$\beta^1(.) \in [c, 1] \tag{33}$$

$$\beta^i(.) \geq \beta^{i+1}(.), \forall i \in \{1, \ldots, k-1\} \tag{34}$$

q is a proba. dist. on $\mathbb{Z}^k$, and $\beta \in \mathcal{D}$.
A particle at x picks a q-dist. random vector $\underline{Z} = (Z_1, \ldots, Z_k)$, and jumps to the first site $x + Z_i$ ($i \in \{1, \ldots, k\}$) with strictly less than K particles along the path $(x + Z_1, \ldots, x + Z_k)$, if such a site exists, with rate $\beta^i(\underline{Z})$. Otherwise, it stays at x.

This extends k-step exclusion in different directions (apart from $K \geq 1$) :

- The random path followed by the particle need not be a Markov process.
- The dist. q is not necessarily supported on paths absorbed at 0.
- Different rates can be assigned to jumps according to the number of steps, and the collection of these rates may depend on the path realization.

Generalization with disorder

The environment is a field $\alpha = ((q_x, \beta_x) : x \in \mathbb{Z}) \in \mathbf{A} := (\mathcal{P}(\mathbb{Z}^k) \times \mathcal{D})^{\mathbb{Z}}$. For a given α, the dist. of the path $\underline{Z}$ picked by a particle at x is q_x, and the rate at which it jumps to $x + Z_i$ is $\beta^i_x(\underline{Z})$.
The corresponding generator is given by (24) with $c_\alpha = \sum_{i=1}^k c^i_\alpha$, where (with the convention that an empty product is equal to 1)

$$c^i_\alpha(x, y, \eta) = \mathbf{1}_{\{\eta(x)>0, \eta(y)<K\}} \int [\beta^i_x(\underline{z}) \mathbf{1}_{\{x+z_i=y\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{\eta(x+z_j)=K\}}] dq_x(\underline{z})$$

dist. Q on $\mathbf{A}$ is ergodic w.r.t. space shift τ_y, where $\tau_y \alpha = ((q_{x+y}, \beta_{x+y}) : x \in \mathbb{Z})$.

replace *(A1)–(A2)* by

$$\inf_{x\in\mathbb{Z}} q_x^1(.) \geq cp(.) \tag{35}$$

$$\sup_{i=1,\dots,k}\sup_{x\in\mathbb{Z}} q_x^i(.) \leq c^{-1}P(.) \tag{36}$$

for $c>0$, q_x^i : i-th marginal of q_x, $p(.)$, resp. $P(.)$ proba. dist. satisfying *(A1)*, resp. *(A2)*.
For $(x,\underline{z},\eta)\in\mathbb{Z}\times\mathbb{Z}^k\times\mathbf{X}$, $\beta\in\mathcal{D}$ and $u\in[0,1]$,

$$\begin{aligned}
N(x,\underline{z},\eta) &= \inf\{i\in\{1,\dots,k\}:\ \eta(x+z_i)<K\}\ \text{with}\inf\emptyset=+\infty\\
Y(x,\underline{z},\eta) &= \begin{cases} x+z_{N(x,\underline{z},\eta)} & \text{if}\quad N(x,\underline{z},\eta)<+\infty\\ x & \text{if}\quad N(x,\underline{z},\eta)=+\infty\end{cases}\\
\mathcal{T}_0{}^{x,\underline{z},\beta,u}\eta &= \begin{cases}\eta^{x,Y(x,\underline{z},\eta)} & \text{if}\quad \eta(x)>0 \text{ and } u<\beta^{N(x,\underline{z},\eta)}(\underline{z})\\ \eta & \quad\text{otherwise}\end{cases}
\end{aligned}$$

(where the definition of $\beta^{+\infty}(\underline{z})$ has no importance).

$$c_\alpha(x,y,\eta) = \mathbf{1}_{\{\eta(x)>0\}}\mathbb{E}_{q_0}\left[\beta_0^{N(x,\underline{Z},\eta)}\mathbf{1}_{\{Y(x,\underline{Z},\eta)=y\}}\right] \quad (37)$$

$$\widetilde{\mathcal{J}}(\alpha,\eta) = \mathbf{1}_{\{\eta(0)>0\}}\mathbb{E}_{q_0}\left[\beta_0^{N(0,\underline{Z},\eta)}Y(0,\underline{Z},\eta)\right] \quad (38)$$

where expectation is w.r.t. $\underline{Z}$.
for G^Q Lipschitz constant $V = 2k^2c^{-1}\sum_{z\in\mathbb{Z}}|z|P(z)$. Let $\mathcal{V} = [0,1]\times[0,1]$, $m = \lambda_{[0,1]}\otimes\lambda_{[0,1]}$. $\forall$ proba. dist. q on $\mathbb{Z}^k$, $\exists$ a mapping $F_q : [0,1]\to\mathbb{Z}^k$ s.t. $F_q(V_1)$ has distribution q if V_1 is uniformly distributed on $[0,1]$. Then $\mathcal{T}$ in (4) is defined by (with $v = (v_1, v_2)$ and $\alpha = ((q_x,\beta_x) : x\in\mathbb{Z}))$

$$\mathcal{T}^{\alpha,x,v}\eta = \mathcal{T}_0^{x,F_{q_x}(v_1),\beta_x(F_{q_x}(v_1)),v_2}\eta \quad (39)$$

Strong attractiveness by

Lemma

$\forall(x,\underline{z},u)\in\mathbb{Z}\times\mathbb{Z}^k\times[0,1]$, $\mathcal{T}_0^{x,\underline{z},\beta,u}$ is an increasing mapping from $\mathbf{X}$ to $\mathbf{X}$.

Examples

Let $c \in (0,1)$.

- A disordered k-step exclusion with jump kernel r :
  $K = 1$, $(\alpha_x : x \in \mathbb{Z})$ is an ergodic $[c, 1/c]$-valued random field, and $r(.)$ is a proba. measure on $\mathbb{Z}$ satisfying *(A1)*, *(A2)*.
  Multiply the rate of any jump from x by α_x.
  $q_x = q^k_{RW}(r)$, and $\beta_x(\underline{z}) = (\alpha_x, \ldots, \alpha_x)$ for every $\underline{z} \in \mathbb{Z}^k$.
- $(\gamma_x, \iota_x)_{x\in\mathbb{Z}}$ is an ergodic $[c, 1]^{2k}$-valued random field, where $\gamma_x = (\gamma^n_x,\ 1 \leq n \leq k)$ and $\iota_x = (\iota^n_x,\ 1 \leq n \leq k)$.
  $q_x = \frac{1}{2}\delta_{(1,2,\ldots,k)} + \frac{1}{2}\delta_{(-1,-2,\ldots,-k)}$
  $\beta^i_x(1, 2, \ldots, k) = \gamma^i_x$, $\beta^i_x(-1, -2, \ldots, -k) = \iota^i_x$

K-exclusion process with speed change and traffic flow model

$\mathcal{K} := \{-k, \ldots, k\} \setminus \{0\}$; $\alpha = ((\upsilon(x), \beta_x^1) : x \in \mathbb{Z})$ is an ergodic $[0, +\infty)^{2k} \times (0, +\infty)$-valued field, with $\upsilon(x) = (\upsilon_z(x) : z \in \mathcal{K})$.

$$\begin{aligned} \Theta(x,\eta) &:= \{y \in \mathbb{Z} : y - x \in \mathcal{K}, \eta(y) < K\} \\ Z(\alpha,x,\eta) &:= \sum_{z \in \Theta(x,\eta)} \upsilon_{z-x}(x) \end{aligned}$$

In configuration η, if $Z(\alpha,x,\eta) > 0$, a particle at x picks a site y at random in $\Theta(x,\eta)$ with proba. $Z(\alpha,x,\eta)^{-1}\upsilon_{y-x}(x)$, and jumps to this site at rate β_x^1. If $Z(\alpha,x,\eta) = 0$, nothing happens. For instance, if $\upsilon_z(x) \equiv 1$, the particle uniformly chooses a site with strictly less than K particles. Generator is given by (24), with

$$c_\alpha(x,y,\eta) = \mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{Z(\alpha,x,\eta)>0\}}\mathbf{1}_{\Theta(x,\eta)}(y)Z(\alpha,x,\eta)^{-1}\upsilon_{y-x}(x)$$

microscopic flux (23) writes

$$\mathfrak{J}(\alpha,\eta) = \beta_0^1\mathbf{1}_{\{\eta(0)>0\}}Z(\alpha,0,\eta)^{-1}\sum_{z\in\mathcal{K}} z\upsilon_z(0)\mathbf{1}_{\{\eta(z)<K\}}$$

Totally asymmetric case with $K = 1$: $\upsilon_z(x) = 0$ for $z < 0$: It is a traffic-flow model with maximum overtaking distance k.

True also for Example 2 above in the totally asymmetric setting $\iota_x^i = 0, 1 \leq i \leq k$. But there an overtaking car has only one choice for its new position.

This dynamics is a $2k$-step model, thus strongly attractive :

$\beta_x = (\beta_x^1, \ldots, \beta_x^1)$, $q_x := q(\upsilon(x))$, where $q(\upsilon_z : z \in \mathcal{K})$ is the dist. of a random self-avoiding path $(Z_1, \ldots, Z_{2k})$ in $\mathcal{K}$ such that

$$\begin{aligned} \mathbf{P}(Z_1 = y) &= \frac{\upsilon_y}{\sum_{z \in \mathcal{K}} \upsilon_z} \\ \mathbf{P}\left(Z_i = y | Z_1, \ldots, Z_{i-1}\right) &= \frac{\upsilon_y}{\sum_{z \in \mathcal{K} \setminus \{Z_1, \ldots, Z_{i-1}\}} \upsilon_z} \quad \text{for } 2 \leq i \leq 2k \end{aligned}$$

Lemma

Assume $(Z_1, \ldots, Z_{2k}) \sim q(\upsilon_z : z \in \mathcal{K})$*. Let* Θ *be a nonempty subset of* $\{z \in \mathcal{K} : \upsilon_z \neq 0\}$*,* $\tau := \inf\{i \in \{1, \ldots, 2k\} : Z_i \in \Theta\}$*, and* $Y = Z_\tau$*. Then*

$$\mathbf{P}\left(Y = y\right) = \mathbf{1}_\Theta(y) \frac{\upsilon_y}{\sum_{y' \in \Theta} \upsilon_{y'}}$$

Other examples

- *k*-step misanthrope process

$$c^i_\alpha(x,y,\eta) = \mathbf{1}_{\{\eta(y)<K\}} \int [b^i_x(\underline{z},\eta(x),\eta(x+z_i)) \times \mathbf{1}_{\{x+z_i=y\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{\eta(x+z_j)=K\}}]\, dq_x(\underline{z})$$

$$\forall j = 2,\ldots,k, \quad b^j(.,K,0) \le b^{j-1}(.,1,K-1) \qquad (40)$$

- Generalized *k*-step misanthrope process
  A particle at x picks a q-dist. random vector $\underline{Z} = (Z_1, ..., Z_k)$ and moves to the first site $x + Z_i$ $(i \le k)$ that has strictly less particles than x and which carries the minimal number of particles among the sites of the random path $x + Z_1, ..., x + Z_k$, if such a site exists, with rate $b(\underline{Z}, \eta(x), \eta(x + Z_i))$; otherwise it stays at x.

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Thank you for your Attention