# Uniqueness of Lagrangian trajectories for suitable weak solutions of the 3D Navier–Stokes equations

James C. Robinson

Department of Mathematics
University of Warwick

(joint work with Masoumeh Dashti & Witold Sadowski)

Warwick, June 8th 2009

# Lagrangian trajectories

We want to investigate uniqueness of solutions of the ODE for Lagrangian 'particle' trajectories

$$\mathrm{d}\xi/\mathrm{d}t = u(\xi, t) \qquad \xi(0) = a \in \Omega,$$

where $u$ is a weak solution of the Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \nabla \cdot u = 0 \qquad u(x, 0) = u_0(x)$$

for $x \in \Omega$ (periodic or Dirichlet boundary conditions).

- $u_0 \in H^{1/2}$: a unique $u \in L^\infty(0, T; H^{1/2}) \cap L^2(0, T; H^{3/2})$ for some $T > 0$ (Fujita & Kato, 1964), and every initial condition $a \in \Omega$ gives rise to a unique solution $\xi_a(t)$ on $[0, T)$ (Chemin & Lerner, 1995; Dashti & JCR, 2009);
- continue $u$ as a *suitable weak solution* for $t \geq T$; then almost every initial condition $a \in \Omega$ gives rise to a unique solution $\xi_a(t)$ for all $t \geq 0$ (Sadowski & JCR, 2009).

# Everywhere uniqueness for flows in $\mathbb{R}^3$

Take two solutions $X(t)$ and $Y(t)$ of $\mathrm{d}X/\mathrm{d}t = u(X,t)$ with the same initial condition.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|X-Y| \le |u(X,t)-u(Y,t)|.$$

Since (Zuazua, 2002)

$$|f(X)-f(Y)| \le c\|f\|_{H^{5/2}}|X-Y|(-\log|X-Y|)^{1/2}$$

we have, with $W = X - Y$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|W| \le \|u(t)\|_{H^{5/2}}|W|(-\log|W|)^{1/2}.$$

Integrate from $s$ to $t$:

$$-(-\log|W(t)|)^{1/2} \le -(-\log|W(s)|)^{1/2} + c\int_s^t \|u(r)\|_{H^2}\,\mathrm{d}r.$$

If $u \in L^1(0,T;H^{5/2})$ then we could simply let $s \downarrow 0$ to obtain uniqueness. But this is not true, even for the heat equation, when $u_0 \in H^{1/2}$.

**Theorem (Dashti & JCR)**

*Suppose that* $\Omega \subset \mathbb{R}^d$, *and for some* $p > 1$

$$u \in L^p(0,T;H^{(d/2)-1}) \qquad \text{and} \qquad \sqrt{t}u \in L^2(0,T;H^{(d/2)+1}).$$

*Then for every* $a \in \Omega$ *there exists a unique solution of* $\mathrm{d}X/\mathrm{d}t = u(X,t)$ *for* $t \in [0,T)$.

**Theorem (Dashti & JCR)**

*Suppose that* $\Omega \subset \mathbb{R}^3$,

$$u \in L^2(0,T;H^{1/2}) \qquad \textit{and} \qquad \sqrt{t}u \in L^2(0,T;H^{5/2}).$$

*Then for every* $a \in \Omega$ *there exists a unique solution of* $\mathrm{d}X/\mathrm{d}t = u(X,t)$ *for* $t \in [0,T)$.

**Theorem (Dashti & JCR)**

*Suppose that* $\Omega \subset \mathbb{R}^3$,

$$u \in L^2(0,T;H^{1/2}) \qquad \textit{and} \qquad \sqrt{t}u \in L^2(0,T;H^{5/2}).$$

*Then for every* $a \in \Omega$ *there exists a unique solution of* $\mathrm{d}X/\mathrm{d}t = u(X,t)$ *for* $t \in [0,T)$.

Note that under the assumptions of the theorem, for any $0 < r < 1$

$$\begin{aligned}\int_0^T \|u(t)\|_{H^{5/2}}^r \,\mathrm{d}t &\le \int_0^T t^{-r/2} t^{r/2} \|u(t)\|_{H^{5/2}} \,\mathrm{d}r \\ &\le \left(\int_0^T t^{-r/(2-r)} \,\mathrm{d}t\right)^{1-(r/2)} \left(\int_0^T t\|u(t)\|_{H^{5/2}}^2 \,\mathrm{d}t\right)^{r/2},\end{aligned}$$

i.e. $u \in L^r(0,T;H^{5/2})$.

$$-(-\log|W(t)|)^{1/2} \le -(-\log|W(s)|)^{1/2} + c\int_s^t \|u(r)\|_{H^{5/2}}\,\mathrm{d}r.$$

From $\mathrm{d}|W|/\mathrm{d}t \le 2\|u(t)\|_\infty$ we have

$$\begin{aligned}
|W(t)| &\le 2\int_0^t \|u(r)\|_\infty\,\mathrm{d}r \le c\int_0^t \|u(r)\|_{H^{1/2}}^{1/2}\|u(r)\|_{H^{5/2}}^{1/2}\,\mathrm{d}r \\
&= c\int_0^t r^{-1/4}\|u(r)\|_{H^{1/2}}^{1/2} r^{1/4}\|u(r)\|_{H^{5/2}}^{1/2}\,\mathrm{d}r \\
&\le c\left(\int_0^t r^{-1/2}\,\mathrm{d}r\right)^{1/2}\left(\int_0^t \|u(r)\|_{H^{1/2}}^2\,\mathrm{d}r\right)^{1/4}\left(\int_0^t r\|u(r)\|_{H^{5/2}}^2\,\mathrm{d}r\right)^{1/4} \\
&\le ct^{1/4}\|u\|_{L^2(0,t;H^{1/2})}^{1/2}\|\sqrt{t}u\|_{L^2(0,t;H^{5/2})}^{1/4}.
\end{aligned}$$

So

$$(-\log|W(s)|)^{1/2} \ge \alpha(-\log s)^{1/2}$$

for some fixed $\alpha > 0$ for all $s \le s_0$, $s_0$ small.

$$-(-\log|W(t)|)^{1/2} \leq -\alpha(-\log s)^{1/2} + c\int_s^t \|u(r)\|_{H^{5/2}}\,\mathrm{d}r.$$

$$\begin{aligned}
\int_s^t \|u(r)\|_{H^{5/2}}\,\mathrm{d}r &= \int_s^t r^{-1/2} r^{1/2}\|u(r)\|_{H^{5/2}}\,\mathrm{d}r \\
&\leq \left(\int_s^t r^{-1}\,\mathrm{d}r\right)^{1/2}\left(\int_s^t r\|u(r)\|_{H^{5/2}}^2\,\mathrm{d}r\right)^{1/2} \\
&= (\log t - \log s)^{1/2}\left(\int_0^t r\|u(r)\|_{H^{5/2}}^2\,\mathrm{d}r\right)^{1/2}.
\end{aligned}$$

For $t$ sufficiently small, $t < t^*$,

$$\int_0^t r\|u(r)\|_{H^{5/2}}^2\,\mathrm{d}r < \alpha^2/4:$$

$$-(-\log|W(t)|)^{1/2} \leq -\alpha(-\log s)^{1/2} + \frac{\alpha}{2}(\log t - \log s)^{1/2}.$$

Now let $s \to 0$ to deduce that $|W(t)| = 0$.

# Checking conditions for 3D Navier–Stokes

Consider here periodic BCs for simplicity. Take the inner product of the equation with $t\Delta u$ in $H^{1/2}$, and then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|u\|_{3/2}^2\right) - \frac{1}{2}\|u\|_{3/2}^2 + \nu t\|u\|_{5/2}^2 \leq t((u\cdot\nabla)u, \Delta u))_{1/2}.$$

Estimating the RHS using standard inequalities yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|u\|_{3/2}^2\right) - \|u\|_{3/2}^2 + \nu t\|u\|_{5/2}^2 \leq ct\|u\|_{3/2}^4.$$

Multiplying by $E(t) = \exp(-c\int_0^t \|u(s)\|_{3/2}^2\,\mathrm{d}s)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|u(t)\|_{3/2}^2 E(t)\right) + \nu t E(t)\|u\|_{5/2}^2 \leq E(t)\|u\|_{3/2}^2.$$

Integrating between $0$ and $t$ and multiplying by $E(t)^{-1}$ gives

$$\nu\int_0^t s\|u(s)\|_{5/2}^2\,\mathrm{d}s \leq \left(\int_0^t \|u(s)\|_{3/2}^2\,\mathrm{d}s\right)\exp\left(c\int_0^t \|u(s)\|_{3/2}^2\,\mathrm{d}s\right) < \infty.$$

# 3D: Lagrangian mapping of Foias-Guillopé-Temam (1985)

For any weak solution $u$, for every $a \in \Omega$ there exists a continuous function $\xi : [0,T] \to \bar{\Omega}$ satisfying

$$\xi(t) = a + \int_0^t u(\xi(s),s)\,\mathrm{d}s. \qquad (*)$$

Uses the fact that $u \in L^1(0,T;L^\infty(\Omega))$ (FGT, 1982) and that $u$ is regular on a collection of open intervals whose union has full measure (Leray, 1934).

Furthermore there exists at least one 'solution mapping' $\Phi : \Omega \times [0,T] \to \Omega$ such that

(i) $\xi_a(\cdot) = \Phi(a,\cdot)$ satisfies $(*)$,

(ii) $\xi_a(\cdot) \in W^{1,1}(0,T)$,

(iii) the mapping $a \mapsto \Phi(a,\cdot)$ belongs to $L^\infty(\Omega; C([0,T],\bar{\Omega}))$, and

(iv) $\Phi$ is volume-preserving: for any Borel set $B \subset \Omega$,

$$\mu[\Phi(\cdot,t)^{-1}(B)] = \mu(B),$$

where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^3$.

# Aside – generalised flows of Di Perna & Lions (1989)

With $u \in L^1(0,T;W^{1,1})$, for each $\theta_0 \in L^\infty(Q)$, there exists a unique 'renormalised solution' $\theta(t;\theta_0) \in L^\infty(\mathbb{R} \times Q)$ of $\theta_t + (u \cdot \nabla)\beta = 0$, i.e.

$$(\beta(\theta))_t + (u \cdot \nabla)\beta(\theta) = 0 \qquad \text{and} \qquad \beta(\theta(0)) = \beta(\theta_0)$$

for a large class of well-behaved functions $\beta$.

There is a one-to-one correspondence between such $\theta$s and generalised flows of $\dot{\xi} = u(\xi,t)$ (Lions, 1998): $\theta \mapsto \Phi$ via

$$[\Phi(t,a)]_i = [\theta(t;x_i)](a).$$

In particular this proves the uniqueness of the FGT solution mapping.

Unique flow $\not\Rightarrow$ almost everywhere uniqueness: for $\dot{x} = |x|^{1/2}$ there is a unique flow defined for all $t \in \mathbb{R}$, corresponding to the solution $x(t) = t|t|/4$, but uniqueness nowhere (Beck, 1973).

# Partial regularity – box-counting dimension of singular set

Take a suitable weak solution with $p \in L^{5/3}(\Omega \times (0,T))$ (Sohr-von Wahl, 1986).

**Theorem (Caffarelli, Kohn, & Nirenberg, 1982; Ladyzenskaya & Seregin, 1999)**

*There is an absolute constant $\alpha > 0$ such that if $(u,p)$ is a suitable weak solution, $r$ is sufficiently small that $Q_r(x,t) \subset \Omega \times (0,T)$, and*

$$r^{-2/3}\left(\int_{Q_r(x,t)} |u|^3\right)^{1/3} + r^{-4/3}\left(\int_{Q_r(x,t)} |p|^{3/2}\right)^{2/3} < \alpha,$$

*then $u(x,t)$ is Hölder continuous in a neighbourhood of $(x,t)$.*

In the statement of the theorem,

$$Q_r(x,t) = \{(y,s) \in \Omega \times (0,T) : \ |y-x| < r, \ |s-t| < r^2\}.$$

**Theorem (L & S)**

*Let $S$ be the 'singular set' of points for which $u(x,t)$ is not Hölder continuous in a neighbourhood of $(x,t)$. Then $P^1(S) = 0 \Rightarrow d_{\rm H}(S) \le 1$.*

The box-counting dimension of $X$, $d_{\text{box}}(X)$, is given by

$$\limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon}$$

where $N(X, \epsilon)$ is the minimum number of $\epsilon$-balls that cover $X$.

The box-counting dimension of $X$, $d_{\mathrm{box}}(X)$, is given by

$$\limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon}$$

where $N(X, \epsilon)$ is the maximum number of $\epsilon$-separated points in $X$.

The box-counting dimension of $X$, $d_{\text{box}}(X)$, is given by

$$\limsup_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon}$$

where $N(X,\epsilon)$ is the maximum number of $\epsilon$-separated points in $X$.

**Theorem (JCR & Sadowski)**

*If $S$ denotes the singular set of a suitable weak solution $(u,p)$ then $d_{\text{box}}(S\cap K)\le 5/3$ for any compact subset $K$ of $\Omega\times(0,T)$.*

If $(x,t)\in S$ then

$$r^{-2/3}\left(\int_{Q_r}|u|^3\right)^{1/3}+r^{-4/3}\left(\int_{Q_r}|p|^{3/2}\right)^{2/3}>\alpha$$

for all $r\ge 0$. By Hölder's inequality

$$\int_{Q_r}|u|^{10/3}+\int_{Q_r}|p|^{5/3}>c_3r^{5/3}.$$

Note: $u\in L^{10/3}(\Omega\times(0,T))$ for any weak solution.

If $d_{\text{box}}(S \cap K) > 5/3$ fix $d$ with $5/3 < d < d_{\text{box}}(S)$:

- there exists a decreasing sequence $\epsilon_j \to 0$ such that $N_j = N(S \cap K, \epsilon_j) \geq \epsilon_j^{-d}$;
- there exists an $r_0 > 0$ such that $Q_r(x,t) \subset \Omega \times (0,T)$ for all $r < r_0$ and every $(x,t) \in K$.

Let $\{(x_i,t_i)\}_{i=1}^{N_j}$ be a collection of $\epsilon$-separated points in $S \cap K$.

Take $j$ large enough that $\epsilon_j < r_0$, and then

$$\underbrace{\int_{\Omega\times(0,T)} |u|^{10/3} + |p|^{5/3}}_{\text{finite}} \geq \sum_{i=1}^{N_j} \int_{Q_\epsilon(x_i,t_i)} |u|^{10/3} + |p|^{5/3} \geq \underbrace{\epsilon_j^{-d} \times c_3 \epsilon_j^{5/3}}_{\to\infty \text{ as } j\to\infty}.$$

The contradiction implies that $d_{\text{box}}(S \cap K) \leq 5/3$ as claimed.

**Theorem (JCR & WS, after Aizenman, 1978; Cipriano & Cruzeiro, 2005)**

*Let $\Omega \subset \mathbb{R}^d$, and let $\Phi : \Omega \times [0,T] \to \Omega$ be a volume-preserving solution mapping corresponding to a vector field $u$ with $u \in L^1(0,T;L^\infty(\Omega))$ for every $T > 0$. If $X$ is a compact subset of $\Omega$ with $d_{\rm box}(X) < d-1$ then for almost every initial condition $a \in \Omega$, $\Phi(t,a) \notin X$ for all $t \geq 0$.*

Choose $T > 0$ and fix $N \in \mathbb{N}$; write $t_j = jT/N$, and consider the problem of avoiding $X$ on the time interval $[t_k, t_{k+1}]$ for some $k \in \{0,\ldots,N-1\}$. Since

$$\xi_a(t) - \xi_a(s) = \int_s^t u(\xi_a(r),r)\,\mathrm{d}r,$$

it follows that for all $t \in [t_k, t_{k+1}]$,

$$|\xi_a(t) - \xi_a(t_k)| \leq \delta_k := \int_{t_k}^{t_{k+1}} \|u(t)\|_\infty\,\mathrm{d}t.$$

So if $\xi_a(t) \in X$ for some $t \in [t_k, t_{k+1}]$, we must have

$$\xi_a(t_k) \in O(X, \delta_k).$$

What is $\mu(O(X, \delta))$?

What is $\mu(O(X,\delta))$?

Recall

$$d_{\text{box}}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the minimum number of $\epsilon$-balls that cover $X$.

What is $\mu(O(X,\delta))$?

Recall

$$d_{\text{box}}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the minimum number of $\epsilon$-balls that cover $X$.

For any $\rho > d_{\text{box}}(X)$, for all $\delta$ sufficiently small

$$N(X,\delta) < \delta^{-\rho}$$

and

$$X \subset \bigcup_{j=1}^{N(X,\delta)} B(x_j,\delta) \quad \Rightarrow \quad O(X,\delta) \subset \bigcup_{j=1}^{N(X,\delta)} B(x_j,2\delta).$$

Thus

$$\mu(O(X,\delta)) \leq \delta^{-\rho}\omega_n(2\delta)^n = c_n\delta^{n-\rho}.$$

What is $\mu(O(X,\delta))$?

Recall

$$d_{\mathrm{box}}(X) = \limsup_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the minimum number of $\epsilon$-balls that cover $X$.

For any $\rho > d_{\mathrm{box}}(X)$, for all $\delta$ sufficiently small

$$N(X,\delta) < \delta^{-\rho}$$

and

$$X \subset \bigcup_{j=1}^{N(X,\delta)} B(x_j,\delta) \quad \Rightarrow \quad O(X,\delta) \subset \bigcup_{j=1}^{N(X,\delta)} B(x_j,2\delta).$$

Thus

$$\mu(O(X,\delta)) \le \delta^{-\rho}\omega_n(2\delta)^n = c_n\delta^{n-\rho}.$$

$$d_{\mathrm{box}}(X) < n-1 \quad \Rightarrow \quad \mu(O(X,\delta)) \le c_n\delta^r \quad \text{for some } r > 1.$$

$\Phi$ is measure-preserving, so

$$\mu\{a:\ \xi_a(t_k) \in O(X, \delta_k)\} \leq c_n \delta_k^r.$$

Thus for any choice of $N$, the measure of initial conditions $\Omega_X$ for which $\xi_a(t) \in X$ for some $t \in [0, T]$ is bounded by

$$c_n \sum_{k=0}^{N-1} \delta_k^r = c_n \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+1}} \|u(s)\|_\infty \, \mathrm{d}s \right)^r. \qquad (\dagger)$$

Since $u \in L^1(0, T; L^\infty)$ its integral is absolutely continuous. In particular, given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for each $k \in \{0, \ldots, N-1\}$,

$$\int_{t_k}^{t_{k+1}} \|u(s)\|_\infty \, \mathrm{d}s < \epsilon.$$

Using this in ($\dagger$) it follows that

$$\mu(\Omega_X) \leq c_n \epsilon^{r-1} \int_0^T \|u(s)\|_\infty \, \mathrm{d}s.$$

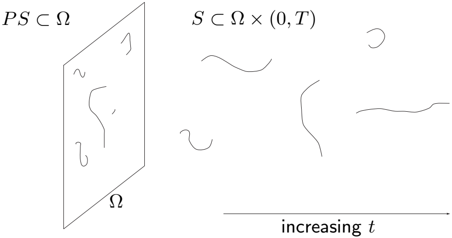
Since $r > 1$ and both $\epsilon$ and $T$ are arbitrary this completes the proof.

**Theorem**

*If $u$ is a suitable weak solution and $\Phi$ is a corresponding solution mapping then for almost every $a \in \Omega$, $\Phi(a,t) \notin S$ for all $t \geq 0$ .*

Fix $T > 0$ and let $\Omega \times (0,T) = \bigcup_{n=1}^{\infty} K_n$ with $K_n$ compact.

$d_{\mathrm{box}}(P[S \cap K_n]) \leq d_{\mathrm{box}}(S \cap K_n) \leq 5/3.$



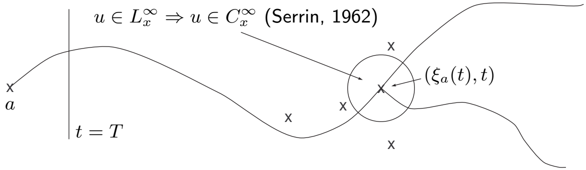

Thus almost every trajectory avoids $P(S \cap K_n)$ for all $t \geq 0$, and so in particular avoids $S \cap K_n \Rightarrow$ almost every trajectory avoids $S$.

## Corollary

*If $u$ is a suitable weak solution corresponding to $u_0 \in H \cap H^{1/2}(\Omega)$ then almost every initial condition $a \in \Omega$ gives rise to a unique particle trajectory, which is a $C^1$ function of time.*

$u_0 \in H^{1/2}(\Omega)$ implies that trajectories are unique on $[0, T)$ (for some $T > 0$).

Let $\xi_a(t)$ be a trajectory that avoids the singular set for all $t \geq 0$, and suppose that there are two trajectories that pass through the space-time point $(\xi_a(t), t)$.




The solution of $\dot{\xi} = u(\xi, t)$ is unique at $(\xi_a(t), t)$, a contradiction.

Since $u$ is also Hölder continuous in $(x, t)$ on the complement of $S$, it follows that $\xi_a(\cdot)$ is a $C^1$ function of time.

- Can one weaken the assumption on the initial condition to $u_0 \in L^2$ (problem is just 'at $t = 0$')?
- Can one improve the bound on the box-counting dimension of the singular set? (Yes, Kukavica has a finer result, $135/82 \simeq 1.646\ldots$; cf. $5/3 = 1.666\ldots$)
- What are the minimal conditions for almost everywhere uniqueness of ODEs?