Warwick 8 June '09 Euler Day

Relaxation Routes to Steady Euler Flows of Complex Topology

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The Euler equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p \qquad (*)$$

Restrict attention to incompressible flow

$$\nabla\cdot\mathbf{u} = 0$$

and take $\rho = \text{cst.} = 1$.

$$\boldsymbol{\omega} = \nabla\wedge\mathbf{u} \qquad \text{Vorticity field}$$

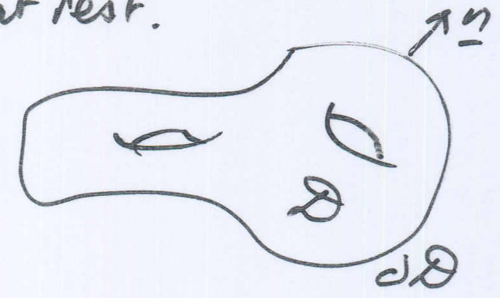
Alternative form of (*)

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\wedge\boldsymbol{\omega} - \nabla h$$

where $h = p + \frac{1}{2}u^2$

Consider flows in a finite domain $\mathcal{D}$, smooth bdy $\partial\mathcal{D}$, fixed & at rest.

but $\mathcal{D}$ may have non-trivial topology.



$\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$ on $\partial\mathcal{D}$

Under Euler evolution

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} = \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{\omega})$$

Vortex lines are frozen in the fluid.

$\underset{\sim}{\omega}$-lines: ~~are~~ Helmholtz got this wrong (1858)

i) closed curves in $\mathcal{D}$

or ii) lie on surfaces

or iii) are chaotic in subdomains of $\mathcal{D}$ generic behaviour

or iv) may terminate on $\partial\mathcal{D}$

If $\underset{\sim}{u} = 0$ on $\partial\mathcal{D}$ (i.e. no-slip)

then $\underset{\sim}{\omega} \cdot \underset{\sim}{n} = 0$ on $\partial\mathcal{D}$ also

(and there is no vortex sheet on $\partial\mathcal{D}$)

If $\underset{\sim}{\omega} \cdot \underset{\sim}{n} = 0$ on $\partial\mathcal{D}$ at $t = 0$, then this holds for $t > 0$ also ($\partial\mathcal{D}$ is a 'vorticity surface')

Invariants

Suppose $\underset{\sim}{u}(\underset{\sim}{x}, t)$ is smooth (regular, C^n, $n \geq 1$)

$$E = \frac{1}{2}\int_{\mathcal{D}} \underset{\sim}{u}^2 \, dV \quad = \text{cst.} \qquad \underline{\text{Energy}}$$

$$\mathcal{H} = \int_{\mathcal{D}} \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV \qquad \underline{\text{Helicity}}.$$

$\underline{\mathcal{H} = \text{cst. only if } \underset{\sim}{\omega} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\mathcal{D}}$ *

– a measure of the 'net' linkage of $\underset{\sim}{\omega}$-lines in $\mathcal{D}$.

There may be Lagrangian subdomains $\mathcal{D}_L(t)$ s.t.

$$\underset{\sim}{\omega} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\mathcal{D}_L$$

If so then $\mathcal{H}_L = \int_{\mathcal{D}_L} \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV = \text{cst.}$

for each such subdomain.

* If $\underset{\sim}{\omega} \cdot \underset{\sim}{n} \neq 0$ on $\partial\mathcal{D}$, then any braid of $\underset{\sim}{\omega}$-lines can be untangled by tangential (slip) motion on $\partial\mathcal{D}$.

Momentum

$$\underset{\sim}{P} = \int \underset{\sim}{u} \, dV = 0$$

$$\left(\int_{\mathcal{D}} u_i \, dV = \int_{\mathcal{D}} \frac{\partial}{\partial x_j}(x_i u_j) \, dV = \int_{\partial\mathcal{D}} (\underline{n} \cdot \underline{u}) \, x_i \, dS = 0 \right)$$

Angular momentum ?.

$$\underset{\sim}{M} = \int \underline{x} \wedge \underline{u} \, dV \neq 0 \text{ in general}$$

but $\dfrac{d\underset{\sim}{M}}{dt} \neq 0$ also in general

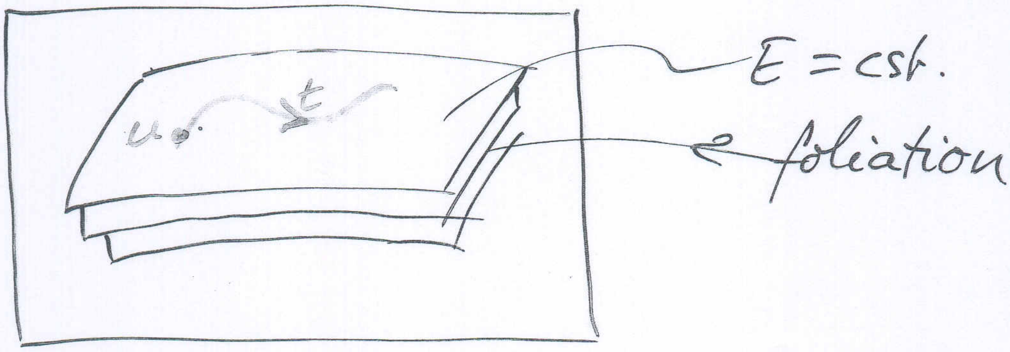
(the flow induces a pressure field which in general exerts a torque on the boundary; $\partial\mathcal{D}$ exerts an equal and opposite torque $\underline{G}$ on the fluid

and $\dfrac{d\underset{\sim}{M}}{dt} = \underset{\sim}{G}$)

allow vortex sheets, but not concentrated line vortices!

Function space $\mathcal{F}$

$$\{\mathbf{u} : \nabla\cdot\mathbf{u} = 0,\ E < \infty,\ \cdots\ ?\}$$



A velocity field $\mathbf{u}$ is represented by a point in $\mathcal{F}$.

Under Euler evolution, this point follows a trajectory on a ~~the~~ 'folium' $E = $ cst.

Dynamical systems point of view.

Fixed points of the Euler system are steady flows $\mathbf{u}(\mathbf{x})$ satisfying

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h$$

$\mathbf{u}\cdot\nabla h = 0 \qquad \boldsymbol{\omega}\cdot\nabla h = 0$

so $\mathbf{u}$-lines and $\boldsymbol{\omega}$-lines lie on surfaces $h = $ cst.

(Only if $h \equiv$ cst. can $\boldsymbol{\omega}$-lines be chaotic for a steady Euler flow.)

Kelvin: $\delta' E = 0$ for isovortical perturbations

Analogy with magnetostatics :

Steady Euler : **E**: $\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h$ $\qquad \boldsymbol{\omega} = \nabla \wedge \mathbf{u}, \ \nabla . \mathbf{u} = 0$

Magnetostatics : **M**: $\mathbf{j} \wedge \mathbf{B} = \nabla p \ , \ \mathbf{j} = \nabla \wedge \mathbf{B}, \quad \nabla . \mathbf{B} = 0$

$$\mathbf{u} \longleftrightarrow \mathbf{B}$$

$$\boldsymbol{\omega} \longleftrightarrow \mathbf{j}$$

$$h \longleftrightarrow p_0 - p$$

So if, by any means, we can find a solution of the problem **M**, we have, via this analogy, found a corresponding solution of the problem **E**

Note that current sheets in **M** correspond to vortex sheets in **E**.

Magnetic Relaxation

Let $\underline{B}_0(\underline{x})$ $\left(\nabla\cdot\underline{B}_0 = 0,\ \underline{n}\cdot\underline{B}_0 = 0 \text{ on } \partial\mathcal{D}\right)$ be a field of arbitrarily complex topology in $\mathcal{D}$.

viscous perfectly cond^{ng} fluid

This relaxes to equilibrium via **MHD evolution**:

$$\left.\begin{aligned} \frac{\partial \underline{B}}{\partial t} &= \nabla\wedge(\underline{v}\wedge\underline{B}) \\ \frac{D\underline{v}}{Dt} &= -\nabla p + \underline{j}\wedge\underline{B} + \nu\nabla^2\underline{v} \end{aligned}\right\} \textbf{N.S.}$$

$$\underline{v} = 0,\ \underline{n}\cdot\underline{B} = 0 \quad \text{on } \partial\mathcal{D}$$

$$\nabla\cdot\underline{v} = 0,\quad \nabla\cdot\underline{B} = 0$$

We are free to make whatever dynamical model we like, provided it dissipates energy

e.g. ($Re \ll 1$: neglect $D\underline{v}/Dt$; 'Stokes')

Then we have a simple energy eqn:

$$\frac{dM}{dt} = \frac{d}{dt}\frac{1}{2}\int_{\mathcal{D}} \underline{B}^2\, dV = -\nu\int_{\mathcal{D}} (\nabla\wedge\underline{v})^2\, dV$$

Monotonic decrease of M

Lower Bound on E

Magnetic helicity $\mathcal{H}_M = \int \underline{A}.\underline{B}\, dV$ $\qquad \underline{B} = \nabla \wedge \underline{A}$

is conserved $\qquad \nabla . \underline{A} = 0$

$(\underline{B}.\underline{n} = 0$ on $\partial\mathcal{D})$

Schwarz $\displaystyle \int_{\mathcal{D}} B^2 dV \int_{\mathcal{D}} A^2 dV \geq \left(\int \underline{A}.\underline{B}\, dV \right)^2$

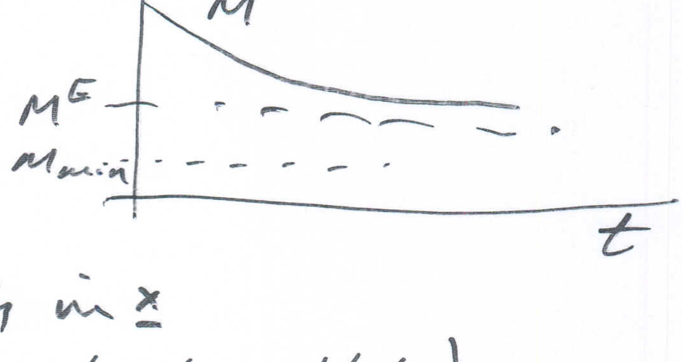
Poincaré $\displaystyle \frac{\int B^2 dV}{\int A^2 dV} \geq q_0^2 \quad \left(\sim \frac{1}{L_{\mathcal{D}}^2} \,? \right)$

so $\displaystyle \frac{1}{2}\int B^2 dV \geq \frac{1}{2} q_0 \left| \mathcal{H}_M \right| = M_{min}$

Positive lower bound on M if $\mathcal{H}_M \neq 0$

(Actually there is a lower bound even if $\mathcal{H}_M = 0$; it is sufficient that the topology of $\underset{\sim}{B}$ be non-trivial : Freedman ~1989 ?)

so $M \downarrow M^E$ as $t \to \infty$



$\int (\nabla \wedge v)^2 dV \to 0$

$\therefore \int v^2 dV \to 0$

$\therefore (?)$ $\underline{v} \to 0$ uniformly in $\underline{x}$

(unproved, but extremely plausible!)

<u>The Physical Mechanism of Relaxation</u> is just contraction of $\underline{B}$-lines in response to Maxwell tension, e.g. for two linked flux tubes

Volumes V_1, V_2 and fluxes Φ_1, Φ_2 are conserved

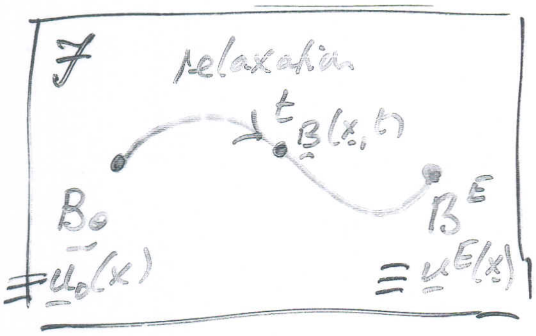
Linkage limits the relaxation process
Tangential discontinuity of $\underline{B}$ (i.e. current sheet) appears in the limit

If (as seems inevitable!) $\underline{v} \to 0$
then in the limit $\underset{\sim}{j} \wedge \underset{\sim}{B} = \nabla p$

Magnetostatic eqn.

and the equilibrium field $\underset{\sim}{B}(\underline{x})$ is <u>topologically accessible</u> from the initial arbitrarily complex field $\underset{\sim}{B_0}(\underline{x})$

Back to the function space $\mathcal{F}$:



So by this means, we have found a steady Euler flow $\underset{\sim}{u}^E(\underline{x})$ whose streamline structure is 'topologically accessible' from that of the arbitrary flow $\underline{u}_0(\underline{x})$.

(not topologically equivalent, because tangential discontinuities can, and in general, do appear.)

Question: How 'bad' can the singularities of $\underset{\sim}{u}^E$ be? What is the 'natural' function space $\mathcal{F}$ that includes all $\underset{\sim}{u}^E$ resulting from C^∞ $\underset{\sim}{u}_0$'s ?

Comments:

1. VCY relaxation (Vallis, Carnevale, Young) JFM ~1990

$$\mathbf{v} = \mathbf{u} + \alpha\, \partial \mathbf{u}/\partial t$$

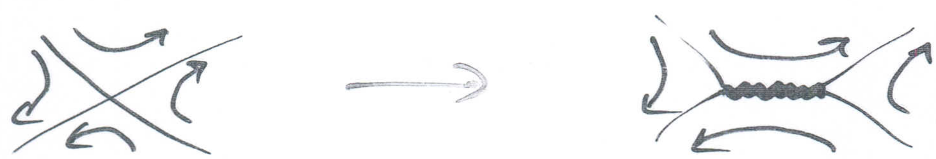
$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v}\wedge\boldsymbol{\omega} - \nabla h \qquad \boldsymbol{\omega} = \nabla\wedge\mathbf{u}$$

$$\Rightarrow \quad \frac{dE}{dt} = -\alpha \int \left(\frac{\partial \mathbf{u}}{\partial t}\right)^2 dV$$

But no bound on E for general 3D flows.

Upper bound on E for axisymm. flows; $\therefore$ useful with $\alpha < 0$.

2. In 2D or axisymm. magnetic relaxation, with no saddle points, current sheets can't form.
But saddle-point collapse can lead to current sheets:



3. Presence of vortex sheets in analogous Euler flows $\Rightarrow$ Kelvin-Helmholtz instability

3D Euler flows do not satisfy Arnold's (1966) sufficient cond^n. for stability

$$\delta^2 E > 0 \quad \text{or} \quad \delta^2 E < 0$$

(all isovortical pert^{ns}.)