Networks and Random Processes

Class test

The class test counts $25/100$ module marks, [x] indicates weight of each question.

1. (a) State the weak law of large numbers and the central limit theorem.
   (b) Define the Erdős-Rényi random graph models $\mathcal{G}_{N,K}$ and $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.
   For both models, give the distribution of the total number of edges. [12]

2. Consider the undirected graph G with adjacency matrix $A = \begin{pmatrix} 0&1&1&0 \\ 1&0&1&0 \\ 1&1&0&1 \\ 0&0&1&0 \end{pmatrix}$.

   (a) Draw the graph G. Identify a clique of vertices and draw a spanning tree of G.
   (b) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.
   (c) Give the degree sequence $(k_1, \ldots, k_4)$ and compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.
   (d) Compute the global clustering coefficient C, and the average $\langle C_i \rangle$ of the local clustering coefficients C_i.

   [12]

3. (a) Define the configuration model with N vertices and degree sequence D.
   (b) Consider the following degree sequences

$$(1,2,3,1)\ , \quad (1,2,3,4)\ , \quad (4,4,4,4)\ , \quad (1,2,2,1)\ , \quad (1,1,1,1)\ .$$

   Decide whether the sequence is graphical. If yes, draw a planar graph with that sequence. If the graph is connected, draw the dual (multi-)graph.
   (c) Which of the graphs you drew in (b) is a triangulation?
   Give an example of a non-planar graph by drawing it and giving its adjacency matrix.

   [12]

4. Consider the simple symmetric ($p = 1/2$) random walk $(X_n : n \in \mathbb{N}_0)$ in discrete time, on the state space $S = \{-N, \ldots, N\}$ with absorbing boundary conditions.

   (a) Sketch the one-step transition matrix P. Is the process ergodic (justify your answer)? Give a formula for all stationary distributions π. Are they reversible?
   (b) For $A = \{-N, N\}$ we know that $h_k^A := \mathbb{P}[X_n \in A \text{ for some } n \geq 0 | X_0 = k] = 1$, i.e. the process gets absorbed with probability 1 in a point of A for all initial conditions k. Let $T^A = \min\{n \geq 0 : X_n \in A\}$ be the corresponding absorption time, and $\tau_k^A = \mathbb{E}[T^A | X_0 = k]$ its expected value starting in k. Show that

$$\tau_k^A = \tfrac{1}{2}\,\tau_{k-1}^A + \tfrac{1}{2}\,\tau_{k+1}^A + 1\ , \quad k = -N+1, \ldots, N-1\ .$$

   What are the boundary conditions of this recursion?
   (c) The solution of the above recursion is of the form $\tau_k^A = ak^2 + bk + c$.
   Determine $a, b, c \in \mathbb{R}$ and compute τ_0^A. (Hint: use the symmetry of the problem.)

   [12]

5. Consider a birth-death process $(X_t : t \geq 0)$ with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and transition rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\,, \qquad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1\,,$$

where we take $\alpha_x = \alpha$ and $\beta_x = x\beta$ for all $x \geq 0$.
(This is called an $M/M/\infty$ queue, where customers arrive at rate α and each of them is served **independently** by one of the infinitely many servers with rate β.)

(a) Under which conditions on $\alpha, \beta \geq 0$ is the process irreducible?
Write down the master equation for $p_t(x) := \mathbb{P}[X_t = x]$ for all $x \in S$.

(b) Use detailed balance to find an explicit formula for the stationary distribution π, which does not contain any summations.
Under which conditions on $\alpha, \beta \geq 0$ can it be normalized?

(c) Use the master equation to show that $\mu_t := \mathbb{E}[X_t] = \sum_{x\in S} x\, p_t(x)$ fulfills

$$\frac{d}{dt}\mu_t = \alpha - \beta\mu_t\,,$$

and solve this equation for general initial condition $\mu_0 \geq 0$.

[12]

6. Consider the voter model $(\eta_t : t \geq 0)$ on the state space $\{0,1\}^\Lambda$ with $\Lambda = \{1, \ldots, L\}$ and transition rates

$$c(\eta, \eta^i) = \sum_{j\neq i} q(j,i)\Big(\eta(i)\big(1-\eta(j)\big) + \eta(j)\big(1-\eta(i)\big)\Big) \quad \text{for all } i \in \Lambda\,,$$

where individual j influences the opinion of individual i with rate $q(j,i) \geq 0$. We use the standard notation

$$\eta^i(k) = \begin{cases} \eta(k) & , k \neq i \\ 1-\eta(k)\,, & k = i \end{cases} \quad \text{for configurations where the opinion of individual } i \text{ is flipped.}$$

(a) Is the process ergodic (justify your answer)?
Give a formula for all stationary distributions of the process, assuming that $q(j,i)$ is irreducible. Explain how this formula has to be adapted if $q(j,i)$ is not irreducible.

(b) Consider the process on the complete graph with L individuals, i.e. $q(j,i) = 1$ for all $i \neq j$ and let

$$N_t := \sum_{i=1}^{L} \eta_t(i) \quad \text{be the number of individuals of opinion 1 at time } t.$$

Derive the transition rates $g(n,m)$ for $n, m \in \{0, \ldots, L\}$ for the process $(N_t : t \geq 0)$ (computation from $c(\eta, \eta^i)$ or intuitive explanation is fine).

(c) Give the state space S and the absorbing states of the process $(N_t : t \geq 0)$ and write down the master equation for $p_t(i) := \mathbb{P}[N_t = i]$ for all $i \in S$.
Give a formula for all stationary distributions.

(d) Use the symmetry of the rates $g(n,m)$ to argue that $\mathbb{E}[N_t]$ does not change in time.
Starting with the initial condition $N_0 = L/2$, how can this be interpreted in the context of absorption and the stationary distributions?

[15]