Networks and Random Processes

Class test

The class test counts $25/100$ module marks, [x] indicates weight of each question.
Attempt all 4 questions.

1. (a) State the weak law of large numbers and the central limit theorem.

   (b) Consider the following degree sequences D,

   $$(0,1,2,3)\ ,\quad (3,3,3,3)\ ,\quad (0,1,1,2)\ ,\quad (2,3,3,2)\ ,\quad (1,1,1,1)\ .$$

   Decide whether D is graphical. If yes, draw a (simple) graph with that degree sequence.

   (c) Give the definition of a diffusion process $(X_t : t \geq 0)$ on $\mathbb{R}$ and write down its generator and the corresponding stochastic differential equation (SDE).
   State Itô's formula for $(X_t : t \geq 0)$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$.

   (d) Let $(B_t : t \geq 0)$ be standard Brownian motion with diffusion coefficient $\sigma^2 = 1$.
   Use Itô's formula to derive the SDE for $Y_t := B_t^2$.
   Write down the generator for the process $(Y_t : t \geq 0)$. [25]

2. (a) Write down the generator and the heat equation for Brownian motion with diffusion coefficient $\sigma^2 > 0$.

   Let $(X_t : t \geq 0)$ be a continuous-time random walk on $S = \mathbb{Z}$ with generator

   $$\mathcal{L}f(n) = \big(f(n+2) - f(n)\big) + 2\big(f(n-1) - f(n)\big)\ ,\quad f : \mathbb{Z} \to \mathbb{R}\ ,$$

   jumping two steps to the right with rate 1, and one step to the left with rate 2. For a small parameter $\epsilon > 0$, consider the rescaled process $X_t^\epsilon := \epsilon X_{t/\epsilon^\alpha}$ for some exponent $\alpha \in \mathbb{R}$.

   (b) Write down the generator $\mathcal{L}^\epsilon f(x)$ of the rescaled proecess $(X_t^\epsilon : t \geq 0)$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$ evaluated at $x \in \mathbb{R}$.

   (c) Show that as $\epsilon \searrow 0$, $\mathcal{L}^\epsilon$ converges to the generator of Brownian motion for an appropriately chosen α which you should identify.
   What is the diffusion coefficient of the limiting Brownian motion?

   (d) Consider the modified process $(X_t : t \geq 0)$ with generator

   $$\mathcal{L}f(n) = \big(f(n+2) - f(n)\big) + \big(f(n-1) - f(n)\big)\ ,\quad f : \mathbb{Z} \to \mathbb{R}\ ,$$

   where jump lengths are unchanged, but both jumps have rate 1.
   Show that now the rescaled process $(X_t^\epsilon : t \geq 0)$ converges to a deterministic limit process $(Y_t : t \geq 0)$, with a different choice of α which you should identify.
   Give the generator of the limit process and solve for Y_t with initial condition $Y_0 = 0$. [25]

3. Consider an undirected graph G with degree sequence $D = (2, 3, 4, 3, 2)$.

   (a) Draw a graph G with this degree sequence (the vertices should be numbered 1 to 5) and write its adjacency matrix.

   (b) Identify all cliques of vertices in the graph.

   (c) Give the matrix of vertex distances d_{ij} and compute the characteristic path length $L(G)$ and the diameter $\mathrm{diam}(G)$ of G.

   (d) Compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.

   (e) Compute the global clustering coefficient C and the average $\langle C_i \rangle$ of the local clustering coefficients C_i.

   (f) Give all non-zero entries of the joint degree distribution $q(k, k')$. (You can write this directly going through all the edges of the graph).

[25]

4. A general birth-death process $(X_t : t \geq 0)$ is a continuous-time Markov chain with state space $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and jump rates

$$x \xrightarrow{\alpha_x} x+1 \quad \text{for all } x \in S\ , \quad x \xrightarrow{\beta_x} x-1 \quad \text{for all } x \geq 1\ .$$

   (a) Write down the generator as a matrix G and as an operator $\mathcal{L}$, and write the master equation in explicit form, i.e. $\frac{d}{dt}\pi_t(x) = \ldots$ $(x = 0$ may need special consideration).
   Under which conditions on the jump rates is the process irreducible?

   (b) Using detailed balance, find a formula for the stationary probabities $\pi(x)$ in terms of the jump rates and $\pi(0)$, normalization is not required.

From now on take $\alpha_x = \alpha(x+1)$ for $x \geq 0$ and $\beta_x = \beta x$ for $x \geq 1$ with $\alpha, \beta > 0$.

   (c) Under which conditions on α and β can the stationary probabilities $\pi(x)$ you found in (b) be normalized?
   In that case compute the normalization and give a formula for $\pi(x)$.

   (d) Using the evolution equation $\frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \mathbb{E}\big[\mathcal{L}f(X_t)\big]$ derive an equation for the mean $\mu(t) := \mathbb{E}[X_t]$ and solve it with initial condition $\mu(0) = 0$. Consider the case $\alpha = \beta$ separately from $\alpha \neq \beta$.
   Sketch the solution $t \mapsto \mu(t)$ for the three cases $\alpha < \beta$, $\alpha = \beta$ and $\alpha > \beta$.

   (e) Using the same approach derive an equation for the second moment $m_2(t) := \mathbb{E}[X_t^2]$ which involves the first moment $\mu(t)$.
   For the case $\alpha = \beta = 1$ solve this equation with initial condition $m_2(0) = 0$, and compute the variance $\mathrm{Var}[X_t]$ as a function of t.

[25]