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Stochastic Modelling and Random Processes

Assignment 3

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates the weight of each question. You need to **justify all your answers** unless it is clear you do not need to. All plots must contain axis labels and a legend (use your own judgement to find reasonable and relevant plot ranges), and corresponding comments on the text of your answers.

The written part of your assignment should be submitted as a pdf file (either latex, scan of handwritten answers, or extract a pdf version of a Jupyter notebook or similar) and you should also include your code (Jupyter notebook, .m file, or any other format you choose to use). Your files should be named MA933_assignment3_1234567 (where 1234567 is replaced by your university ID number).

You should submit your solutions via this link by **Friday, 15.12.2023, 12 noon UK time**.

1 Voter model [40]

Consider the voter model $(\eta_t : t \geq 0)$ on the state space $\{0,1\}^\Lambda$ with $\Lambda = \{1, \dots, L\}$ and transition rates

$$c(\eta, \eta^i) = \sum_{j \neq i} q(j,i)\Big(\eta(i)\big(1-\eta(j)\big) + \eta(j)\big(1-\eta(i)\big)\Big) \quad \text{for all } i \in \Lambda\ ,$$

where individual j influences the opinion of individual i with rate $q(j,i) \geq 0$. We use the standard notation

$$\eta^i(k) = \begin{cases} \eta(k) & , k \neq i \\ 1-\eta(k)\ , & k = i \end{cases} \quad \text{for configurations where the opinion of individual } i \text{ is flipped.}$$

(a) Is this process ergodic? Give a formula for all stationary distributions of the process, assuming that $q(j,i)$ is irreducible. Explain how this formula has to be adapted if $q(j,i)$ is not irreducible. Justify your answers.

(b) Consider the process on the complete graph with L individuals, i.e. $q(j,i) = 1$ for all $i \neq j$ and let

$$N_t := \sum_{i=1}^{L} \eta_t(i)$$

be the number of individuals of opinion 1 at time t. Derive the transition rates $g(n,m)$ for $n, m \in \{0, \dots, L\}$ for the process $(N_t : t \geq 0)$
Hint: a computation from $c(\eta, \eta^i)$ or intuitive explanation is okay.

(c) Give the state space S and the absorbing states of the process $(N_t : t \geq 0)$ and write down the master equation for $p_t(i) := \mathbb{P}(N_t = i)$ for all $i \in S$. Give a formula for all stationary distributions.

(d) Use the symmetry of the rates $g(n,m)$ to argue that $\mathbb{E}(N_t)$ does not change in time. Starting with the initial condition $N_0 = L/2$, how can this be interpreted in the context of absorption and the stationary distributions?

2 Erdős Rényi random graphs [30]

Consider the Erdős-Rényi random graph model and simulate at least 20 realisations of $\mathcal{G}_{N,p}$ graphs with $p = p_N = z/N$, for $z = 0.1, 0.2, \ldots, 3.0$ for $N = 100$ and $N = 1000$.

Throughout this question, comment on your results, comparing with theoretical results which you would expect from lectures.

(a) Plot the average size of the two largest components in each realisation divided by N, against z for both values of N in a single plot (4 data series in total, use different colours). Use all 20 (or more) realisations and include error bars indicating the standard deviation.

(b) For $N = 1000$, plot the average local clustering coefficient $\langle C_i \rangle$ against z using all 20 realisations and $i = 1, \ldots, N$ for averaging, and including error bars indicating the standard deviation for all $20N$ data points.

(c) Use results in lectures to state what is the expected number of edges, as well as the expected average degree as a function of z and N and plot these as a function of z for the two values of N, comparing your results with the expected ones. Include error bars to indicate the standard deviation for your data points.

(d) For $N = 1000$ and your favourite value of $z \in [0.5, 2]$, plot the degree distribution $p(k)$ against $k = 0, 1, \ldots$ using all 20 realisations, and compare it to the mass function of the Poi(z) Poisson distribution in a single plot.

(e) Consider $z = 0.5, 1.5, 5$ and 10. Plot the spectrum of the adjacency matrix A using all 20 realisations with a kernel density estimate, and compare it to the Wigner semi-circle law. Comment on your results based on what you expect from the lectures.

3 Barabási-Albert model [30]

Consider the Barabási-Albert model starting with $m_0 = 5$ connected nodes, adding in each timestep a node linked to $m = 5$ existing distinct nodes according to the preferential attachment rule. Simulate the model for $N = |V| = 1000$, with at least 20 independent realisations.

Throughout this question, comment on your results, comparing with theoretical results which you would expect from lectures.

(a) Plot the tail of the degree distribution in a double logarithmic plot for a single realisation and for the average of all 20 realisations, and compare to the power law with exponent -2 (all in a single plot).

(b) Compute the nearest neighbour

$$k_{nn}(k) = \mathbb{E}\Big[\sum_{i\in V} k_{nn,i}\delta_{k_i,k} \Big/ \sum_{i\in V} \delta_{k_i,k}\Big], \qquad \text{where} \qquad k_{nn,i} = \frac{1}{k_i}\sum_{j\in V} a_{ij}k_j,$$

and decide whether the graphs are typically uncorrelated or (dis-)assortative.

(c) Plot the spectrum of the adjacency matrix $A = (a_{ij})$ using all realisations with a kernel density estimate, and compare it to the Wigner semi-circle law with $\sigma^2 = \mathrm{var}[a_{ij}]$. Comment on your results.