MA933 - Stochastic Modelling and Random Processes

MSc in Mathematics of Systems

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These notes and other information about the course are available on

`www2.warwick.ac.uk/fac/sci/mathsys/courses/msc/ma933/`

Contents

References

- G. Grimmett, D. Stirzaker: Probability and Random Processes (3rd edition), Oxford 2001
- C.W. Gardiner: Handbook of Stochastic Methods (3rd edition), Springer 2004
- G. Grimmett: Probability on Graphs, CUP 2010
  `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`
- M.E.J. Newman: Networks: An Introduction, OUP 2010

1. Probability

- **sample space** Ω $\big($ e.g. $\{H, T\}$, $\{H, T\}^N$, $\{$paths of a stoch. process$\}\big)$
- **events** $A \subseteq \Omega$ (measurable) subsets (e.g. odd numbers on a die)
  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of all events (subset of the powerset)

Definition 1.1

A **probability distribution** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ which is

(i) normalized, i.e. $\mathbb{P}[\emptyset] = 0$ and $\mathbb{P}[\Omega] = 1$

(ii) additive, i.e. $\mathbb{P}\big[\cup_i A_i\big] = \sum_i \mathbb{P}[A_i]$,
where $A_1, A_2, \ldots$ is a collection of disjoint events, i.e. $A_i \cap A_j = \emptyset$ for all i, j.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- For **discrete** Ω: $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$
  e.g. $\mathbb{P}[\text{even number on a die}] = \mathbb{P}[2] + \mathbb{P}[4] + \mathbb{P}[6] = 1/2$
- For **continuous** Ω (e.g. $[0, 1]$): $\mathcal{F} \subsetneq \mathcal{P}(\Omega)$

1. Independence and conditional probability

- Two events $A, B \subseteq \Omega$ are called **independent** if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$.

  Example. rolling a die repeatedly
- If $\mathbb{P}[B] > 0$ then the **conditional probability** of A given B is

$$\mathbb{P}[A|B] := \mathbb{P}[A \cap B]/\mathbb{P}[B] \ .$$

  If A and B are independent, then $\mathbb{P}[A|B] = \mathbb{P}[A]$.

Lemma 1.1 (Law of total probability)

Let $B_1, \ldots, B_n$ be a **partition** of Ω such that $\mathbb{P}[B_i] > 0$ for all i. Then

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \cap B_i] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\, \mathbb{P}[B_i] \ .$$

Note that also $\mathbb{P}[A|C] = \sum_{i=1}^{n} \mathbb{P}[A|C \cap B_i]\, \mathbb{P}[B_i|C]$ provided $\mathbb{P}[C] > 0$.

1. Random variables

Definition 1.2

A **random variable** X is a (measurable) function $X : \Omega \to \mathbb{R}$.

The **distribution function** of the random variable is

$$F(x) = \mathbb{P}[X \leq x] = \mathbb{P}\big[\{\omega : X(\omega) \leq x\}\big] \, .$$

X is called **discrete**, if it only takes values in a countable subset $\Delta = \{x_1, x_2, \ldots\} \subseteq \mathbb{R}$, and its distribution is characterized by the **probability mass function**

$$\pi(x) := \mathbb{P}[X = x] \, , \quad x \in \Delta \, .$$

X is called **continuous**, if its distribution function is

$$F(x) = \int_{-\infty}^{x} f(y) \, dy \quad \text{for all } x \in \mathbb{R} \, ,$$

where $f : \mathbb{R} \to [0, \infty)$ is the **probability density function (PDF)** of X.

1. Random variables

- In general, $f = F'$ is given by the derivative (exists for cont. rv's).
  For discrete rv's, F is a step function with 'PDF'

$$f(x) = F'(x) = \sum_{y\in\Delta} \pi(y)\delta(x-y) \ .$$

- The **expected value** of X is given by $\mathbb{E}[X] = \begin{cases} \sum_{x\in\Delta} x\pi(x) \\ \int_{\mathbb{R}} x f(x)\,dx \end{cases}$
- The **variance** is given by $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$,
  the **covariance** of two r.v.s by $\mathrm{Cov}[X,Y] := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.
- Two random variables X, Y are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all $x, y \in \mathbb{R}$. This implies for **joint distributions**

$$f(x,y) = f^X(x) f^Y(y) \quad \text{or} \quad \pi(x,y) = \pi^X(x)\,\pi^Y(y)$$

  with **marginals** $f^X(x) = \int_{\mathbb{R}} f(x,y)\,dy$ and $\pi^X(x) = \sum_{y\in\Delta_y} \pi(x,y)$.
- Independence implies $\mathrm{Cov}[X,Y] = 0$, i.e. X and Y are **uncorrelated**.
  The inverse is in general false, but holds if X and Y are Gaussian.

1. Simple random walk

Definition 1.3

Let $X_1, X_2, \ldots \in \{-1, 1\}$ be a sequence of independent, identically distributed random variables **(iidrv's)** with

$$p = \mathbb{P}[X_i = 1] \quad \text{and} \quad q = \mathbb{P}[X_i = -1] = 1 - p \, .$$

The sequence $Y_0, Y_1, \ldots$ defined as $\qquad Y_0 = 0 \quad \text{and} \quad Y_n = \sum_{k=1}^n X_k$

is called the **simple random walk (SRW)** on $\mathbb{Z}$.

- for a single **increment** X_k we have

$$\mathbb{E}[X_k] = p - q = 2p - 1 \, , \quad \mathrm{var}[X_k] = p + q - (p - q)^2 = 4p(1 - p)$$

- $\mathbb{E}[Y_n] = \mathbb{E}\Big[\sum_{k=1}^n X_k\Big] = \sum_{k=1}^n \mathbb{E}[X_k] = n(2p - 1)$

  (expectation is a linear operation)

- $\mathrm{var}[Y_n] = \mathrm{var}\Big[\sum_{k=1}^n X_k\Big] = \sum_{k=1}^n \mathrm{var}[X_k] = 4np(1 - p)$

  (for a sum of **independent** rv's the variance is additive)

1. LLN and CLT

Theorem 1.2 (Weak law of large numbers (LLN))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\mathbb{E}[|X_k|] < \infty$. Then

$$\frac{1}{n} Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k \to \mu \quad \text{as } n \to \infty$$

in distribution (i.e. the distr. fct. of Y_n converges to $\mathbb{1}_{[\mu,\infty)}(x)$ for $x \neq \mu$) .

Theorem 1.3 (Central limit theorem (CLT))

Let $X_1, X_2, \ldots \in \mathbb{R}$ be a sequence of iidrv's with $\mu := \mathbb{E}[X_k] < \infty$ and $\sigma^2 := \mathrm{var}[X_k] < \infty$. Then

$$\frac{Y_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} (X_k - \mu) \to \xi \quad \text{as } n \to \infty$$

in distr., where $\xi \sim N(0,1)$ is a **standard Gaussian** with PDF $\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Expansion. as $n \to \infty$, $\quad \sum_{k=1}^{n} X_k = n\mu + \sqrt{n}\sigma\xi + o(\sqrt{n})$, $\quad \xi \sim N(0,1)$

Definition 1.4

A **discrete-time stochastic process** with **state space** S is a sequence $Y_0, Y_1, \ldots = (Y_n : n \in \mathbb{N}_0)$ of random variables taking values in S.

The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s_0, \ldots, s_n \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s_n, \ldots, Y_0 = s_0) = \mathbb{P}(Y_{n+1} \in A | Y_n = s_n) \ .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $n \in \mathbb{N}_0$ and $s \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s) = \mathbb{P}(Y_1 \in A | Y_0 = s) \ .$$

If S is discrete, the MP is called a **Markov chain (MC)**.

The generic probability space Ω is the **path space**

$$\Omega = D(\mathbb{N}_0, S) := S^{\mathbb{N}_0} = S \times S \times \ldots$$

which is uncountable even when S is finite. For a given $\omega \in \Omega$ the function $n \mapsto Y_n(\omega)$ is called a **sample path**.

Up to finite time N and with finite S, $\Omega_N = S^{N+1}$ is finite.

1. Discrete-time Markov processes

Examples.

- For the simple random walk we have state space $S = \mathbb{Z}$ and $Y_0 = 0$. Up to time N, $\mathbb{P}$ is a distribution on the finite path space Ω_N with

$$\mathbb{P}(\omega) = \begin{cases} p^{\text{\# of up-steps}} \, q^{\text{\# of down-steps}} & , \text{ path } \omega \text{ possible} \\ 0 & , \text{ path } \omega \text{ not possible} \end{cases}$$

  There are only 2^N paths in Ω_N with non-zero probability.
  For $p = q = 1/2$ they all have the same probability $(1/2)^N$.

- For the generalized random walk with $Y_0 = 0$ and increments $Y_{n+1} - Y_n \in \mathbb{R}$, we have $S = \mathbb{R}$ and $\Omega_N = \mathbb{R}^N$ with an uncountable number of possible paths.

- A sequence $Y_0, Y_1, \ldots \in S$ of iidrv's is also a Markov process with state space S.

- Let $S = \{1, \ldots, 52\}$ be a deck of cards, and $Y_1, \ldots, Y_{52}$ be the cards drawn at random without replacement. Is this a Markov process?

1. Discrete-time Markov chains

Proposition 1.4

Let $(X_n : n \in \mathbb{N}_0)$ by a homogeneous DTMC with **discrete** state space S. Then the **transition function**

$$p_n(x,y) := \mathbb{P}[X_n = y | X_0 = x] = \mathbb{P}[X_{k+n} = y | X_k = x] \quad \text{for all } k \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{k+n}(x,y) = \sum_{z \in S} p_k(x,z)\, p_n(z,y) \quad \text{for all } k,n \geq 0,\ x,y \in S\,.$$

Proof. We use the law of total probability, the Markov property and homogeneity

$$\begin{aligned}
\mathbb{P}[X_{k+n} = y | X_0 = x] &= \sum_{z \in S} \mathbb{P}[X_{k+n} = y | X_k = z,\ X_0 = x]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_{k+n} = y | X_k = z]\, \mathbb{P}[X_k = z | X_0 = x] \\
&= \sum_{z \in S} \mathbb{P}[X_n = y | X_0 = z]\, \mathbb{P}[X_k = z | X_0 = x]
\end{aligned}$$

1. Markov chains

- In matrix form with $P_n = \big(p_n(x,y) : x,y \in S\big)$ the Chapman Kolmogorov equations read

$$P_{n+k} = P_n\, P_k \quad \text{and in particular} \quad P_{n+1} = P_n\, P_1 \ .$$

With $P_0 = \mathbb{I}$, the obvious solution to this recursion is

$$P_n = P^n \quad \text{where we write} \quad P_1 = P = \big(p(x,y) : x,y \in S\big) \ .$$

- The **transition matrix** P and the initial condition $X_0 \in S$ completely determine a homogeneous DTMC, since for all $k \geq 1$ and all events $A_1, \ldots, A_k \subseteq S$

$$\mathbb{P}[X_1 \in A_1, \ldots, X_k \in A_k] = \sum_{s_1 \in A_1} \cdots \sum_{s_k \in A_k} p(X_0, s_1) p(s_1, s_2) \cdots p(s_{k-1}, s_k) \ .$$

- Fixed X_0 can be replaced by an **initial distribution** $\quad \pi_0(x) := \mathbb{P}[X_0 = x]$.

  The distribution at time n is then

$$\pi_n(x) = \sum_{y \in S} \sum_{s_1 \in S} \cdots \sum_{s_{n-1} \in S} \pi_0(y) p(y, s_1) \cdots p(s_{n-1}, x) \quad \text{or} \quad \langle \boldsymbol{\pi}_n | = \langle \boldsymbol{\pi}_0 | P^n \ .$$

1. Transition matrices

The transition matrix P is **stochastic**, i.e.

$$p(x,y) \in [0,1] \quad \text{and} \quad \sum_y p(x,y) = 1 \ ,$$

or equivalently, the column vector $|1\rangle = (1,\ldots,1)^T$

is **eigenvector** with **eigenvalue** 1: $\quad P|1\rangle = |1\rangle$

Example 1 (Random walk with boundaries)

Let $(X_n : n \in \mathbb{N}_0)$ be a SRW on $S = \{1,\ldots,L\}$ with $p(x,y) = p\delta_{y,x+1} + q\delta_{y,x-1}$.

The boundary conditions are

- **periodic** if $\quad p(L,1) = p\ , \quad p(1,L) = q\ ,$
- **absorbing** if $\quad p(L,L) = 1\ , \quad p(1,1) = 1\ ,$
- **closed** if $\quad p(1,1) = q\ , \quad p(L,L) = p\ ,$
- **reflecting** if $\quad p(1,2) = 1\ , \quad p(L,L-1) = 1\ .$

1. Stationary distributions

Definition 1.5

Let $(X_n : n \in \mathbb{N}_0)$ be a homogeneous DTMC with state space S. The distribution $\pi(x),\ x \in S$ is called **stationary** if for all $y \in S$

$$\sum_{x\in S} \pi(x)p(x,y) = \pi(y) \quad \text{or} \quad \langle \boldsymbol{\pi}|P = \langle \boldsymbol{\pi}| \ .$$

π is called **reversible** if it fulfills the **detailed balance** conditions

$$\pi(x)p(x,y) = \pi(y)p(y,x) \quad \text{for all } x, y \in S \ .$$

- reversibility implies stationarity, since

$$\sum_{x\in S} \pi(x)p(x,y) = \sum_{x\in S} \pi(y)p(y,x) = \pi(y) \ .$$

- Stationary distributions as row vectors $\langle\pi| = (\pi(x) : x \in S)$ are **left eigenvectors** with **eigenvalue** 1: $\quad \langle\pi| = \langle\pi|P$.

1. Absorbing states

Definition 1.6

A state $s \in S$ is called **absorbing** for a DTMC with transition matrix $p(x,y)$, if

$$p(s,y) = \delta_{s,y} \quad \text{for all } y \in S\ .$$

RW with absorbing BC.

Let h_k be the **absorption probability** for $X_0 = k \in S = \{1, \ldots, L\}$,

$$h_k = \mathbb{P}[\text{absorption}|X_0 = k] = \mathbb{P}[X_n \in \{1, L\} \text{ for some } n \geq 0 | X_0 = k]\ .$$

Conditioning on the first jump and using Markov, we have the recursion

$$h_k = p h_{k+1} + q h_{k-1} \quad \text{for } k = 2, \ldots, L-1\ ; \quad h_1 = h_L = 1\ .$$

Ansatz for solution $\quad h_k = \lambda^k\ , \quad \lambda \in \mathbb{C}$:

$$\lambda = p\lambda^2 + q \quad \Rightarrow \quad \lambda_1 = 1\ , \quad \lambda_2 = q/p$$

General solution of 2nd order linear recursion

$$h_k = a\lambda_1^k + b\lambda_2^k = a + b(q/p)^k\ , \quad a, b \in \mathbb{R}\ .$$

Determine coefficients from boundary condition $\quad \Rightarrow \quad h_k \equiv 1$

1. Distribution at time n

Consider a DTMC on a finite state space with $|S| = L$, and let $\lambda_1, \ldots, \lambda_L \in \mathbb{C}$ be the **eigenvalues** of the transition matrix P with corresponding

left (row) eigenvectors $\langle u_i|$ and **right (column) eigenvectors** $|v_i\rangle$

in bra-ket notation. Assuming that **all eigenvalues are distinct** we have

$$P = \sum_{i=1}^{L} \lambda_i |v_i\rangle\langle u_i| \quad \text{and} \quad P^n = \sum_{i=1}^{L} \lambda_i{}^n |v_i\rangle\langle u_i|$$

since eigenvectors can be chosen **orthonormal** $\langle u_i|v_j\rangle = \delta_{i,j}$.

Since $\langle \boldsymbol{\pi}_n| = \langle \boldsymbol{\pi}_0|P^n$ we get

$$\langle \boldsymbol{\pi}_n| = \langle \boldsymbol{\pi}_0|v_1\rangle \lambda_1{}^n \langle u_1| + \ldots + \langle \boldsymbol{\pi}_0|v_L\rangle \lambda_L{}^n \langle u_L| \, .$$

- The **Gershgorin theorem** implies that $|\lambda_i| \leq 1$ and contributions with $|\lambda_i| < 1$ decay exponentially (see hand-out 1).
- $\lambda_1 = 1$ corresponds to the **stationary distribution** $\langle \boldsymbol{\pi}| = \langle u_1|$ and $|v_1\rangle = |1\rangle$.
- Other $\mathbb{C} \ni \lambda_i \neq 1$ with $|\lambda_i| = 1$ correspond to **persistent oscillations**.

1. Lazy Markov chains

Definition 1.7

Let $(X_n : n \in \mathbb{N}_0)$ be a DTMC with transition matrix $p(x, y)$. The DTMC with transition matrix

$$p^\epsilon(x, y) = \epsilon\delta_{x,y} + (1 - \epsilon)\, p(x, y)\ , \quad \epsilon \in (0, 1)$$

is called a **lazy version** of the original chain.

- P^ϵ has the **same eigenvectors** as P with eigenvalues $\lambda_i^\epsilon = \lambda_i(1-\epsilon) + \epsilon$ since

$$\langle u_i|P^\epsilon = \epsilon\langle u_i| + \lambda_i(1-\epsilon)\langle u_i| \qquad \big(\text{analogously for } |v_i\rangle\big)$$

- This implies $|\lambda_i^\epsilon| < |\lambda_i| \leq 1$ unless $\lambda_i = 1$.
  Such a matrix P^ϵ is called **aperiodic**, and there are no persistent oscillations.

- The stationary distribution is unique if and only if the eigenvalue $\lambda = 1$ is unique (has multiplicity 1), which is independent of lazyness (discussed later).

2. Continuous-time Markov chains

Definition 2.1

A **continuous-time stochastic process** with **state space** S is a family $(X_t : t \geq 0)$ of random variables taking values in S. The process is called **Markov**, if for all $A \subseteq S$, $n \in \mathbb{N}$, $t_1 < \ldots < t_{n+1} \in [0, \infty)$ and $s_1, \ldots, s_n \in S$

$$\mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n, \ldots, X_{t_1} = s_1) = \mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n) \ .$$

A Markov process (MP) is called **homogeneous** if for all $A \subseteq S$, $t, u > 0$ and $s \in S$

$$\mathbb{P}(X_{t+u} \in A | X_u = s) = \mathbb{P}(X_t \in A | X_0 = s) \ .$$

If S is discrete, the MP is called a continuous-time **Markov chain (CTMC)**.

The generic probability space Ω of a CTMC is the space of **right-continuous paths**

$$\Omega = D([0, \infty), S) := \left\{ X : [0, \infty) \to S \,\middle|\, X_t = \lim_{u \searrow t} X_u \right\}$$

$\mathbb{P}$ is a probability distribution on Ω, which by **Kolmogorov's extension theorem** is fully specified by its **finite dimensional distributions (FDDs)** of the form

$$\mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] \ , \quad n \in \mathbb{N},\ t_i \in [0, \infty),\ A_i \subseteq S \ .$$

2. Continuous-time Markov chains

Proposition 2.1

Let $(X_t : t \geq 0)$ by a homogeneous CTMC with state space S. Then for all $t \geq 0$ the **transition function**

$$p_t(x,y) := \mathbb{P}[X_t = y | X_0 = x] = \mathbb{P}[X_{t+u} = y | X_u = x] \quad \text{for all } u \geq 0$$

is well defined and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x,y) = \sum_{z \in S} p_t(x,z)\, p_u(z,y) \quad \text{for all } t,u \geq 0,\ x,y \in S\,.$$

In matrix notation $P_t = \big(p_t(x,y) : x,y \in S\big)$ we get

$$P_{t+u} = P_t\, P_u \quad \text{with} \quad P_0 = \mathbb{I}\,.$$

In particular $\quad \frac{P_{t+\Delta t} - P_t}{\Delta t} = P_t \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} = \frac{P_{\Delta t} - \mathbb{I}}{\Delta t} P_t\,,$

taking $\Delta t \searrow 0$ we get the so-called **forward and backward equations**

$$\frac{d}{dt} P_t = P_t G = G P_t\,, \quad \text{where} \quad G = \frac{dP_t}{dt}\Big|_{t=0}$$

is called the **generator** of the process (sometimes also Q-matrix).

2. Continuous-time Markov chains

- The solution is given by the matrix exponential

$$P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k = \mathbb{I} + tG + \frac{t^2}{2} G^2 + \ldots \tag{2.1}$$

- The **distribution $\boldsymbol{\pi}_t$ at time** $t > 0$ is then given by

$$\langle \boldsymbol{\pi}_t | = \langle \boldsymbol{\pi}_0 | \exp(tG) \quad \text{which solves} \quad \frac{d}{dt} \langle \boldsymbol{\pi}_t | = \langle \boldsymbol{\pi}_t | G \; . \tag{2.2}$$

- On a finite state space with $\lambda_1, \ldots, \lambda_L \in \mathbb{C}$ eigenvalues of G, P_t **has eigenvalues** $\exp(t\lambda_i)$ with the same eigenvectors $\langle v_i |, \; | u_i \rangle$.

  If the λ_i are distinct, we can expand the initial condition in the eigenvector basis

$$\langle \boldsymbol{\pi}_0 | = \alpha_1 \langle v_1 | + \ldots + \alpha_L \langle v_L | \; ,$$

  where $\alpha_i = \langle \boldsymbol{\pi}_0 | u_i \rangle$. This leads to

$$\langle \boldsymbol{\pi}_t | = \alpha_1 \langle v_1 | e^{\lambda_1 t} + \ldots + \alpha_L \langle v_L | e^{\lambda_L t} \; . \tag{2.3}$$

2. Continuous-time Markov chains

- using (2.1) we have for $G = \big(g(x,y) : x,y \in S\big)$

$$p_{\Delta t}(x,y) = g(x,y)\Delta t + o(\Delta t) \quad \text{for all } x \neq y \in S\ .$$

So $g(x,y) \geq 0$ can be interpreted as **transition rates**.

$$p_{\Delta t}(x,x) = 1 + g(x,x)\Delta t + o(\Delta t) \quad \text{for all } x \in S\ ,$$

and since $\sum_y p_{\Delta t}(x,y) = 1$ this implies that

$$g(x,x) = -\sum_{y \neq x} g(x,y) \leq 0 \quad \text{for all } x \in S\ .$$

- (2.2) can then be written intuitively as the **Master equation**

$$\frac{d}{dt}\pi_t(x) = \underbrace{\sum_{y \neq x} \pi_t(y) g(y,x)}_{\text{gain term}} - \underbrace{\sum_{y \neq x} \pi_t(x) g(x,y)}_{\text{loss term}} \quad \text{for all } x \in S\ .$$

- The Gershgorin theorem now implies that either $\lambda_i = 0$ or $\mathrm{Re}(\lambda_i) < 0$ for the eigenvalues of G, so there are **no persistent oscillations for CTMCs**.

2. Stationary distributions

Definition 2.2

Let $(X_t : t \geq 0)$ be a homogeneous CTMC with state space S. The distribution $\pi(x),\ x \in S$ is called **stationary** if $\langle \boldsymbol{\pi}|G = \langle 0|$, or for all $y \in S$

$$\sum_{x\in S} \pi(x)g(x,y) = \sum_{x\neq y} \big(\pi(x)g(x,y) - \pi(y)g(y,x)\big) = 0\ . \tag{2.4}$$

π is called **reversible** if it fulfills the **detailed balance conditions**

$$\pi(x)g(x,y) = \pi(y)g(y,x) \quad \text{for all } x,y \in S\ . \tag{2.5}$$

- again, **reversibility implies stationarity**, since with (2.5) every single term in the sum (2.4) vanishes
- Stationary distributions are left **eigenvectors** of G with **eigenvalue** 0 .
- $\langle \boldsymbol{\pi}|G = \langle 0| \quad \text{implies} \quad \langle \boldsymbol{\pi}|P_t = \langle \boldsymbol{\pi}|\big(\mathbb{I} + \sum_{k\geq 1} t^k G^k/k!\big) = \langle \boldsymbol{\pi}| \quad \text{for all } t \geq 0$

2. Stationary distributions

Proposition 2.2 (Existence)

A DTMC or CTMC with **finite** state space S has **at least one** stationary distribution.

Proof. Since P and G have row sum 1 and 0 we have $\quad P|\mathbf{1}\rangle = |\mathbf{1}\rangle \quad$ and $\quad G|\mathbf{1}\rangle = |\mathbf{0}\rangle$
So 1 and 0 are eigenvalues, and left eigenvectors can be shown to have non-negative entries and thus can be normalized to be stationary distributions $\langle \boldsymbol{\pi}|$. $\square$

Remark. If S is countably infinite, stationary distributions may not exist, as for example for the SRW on $\mathbb{Z}$ or the Poisson process on $\mathbb{N}$ (see later).

Definition 2.3

A CTMC (or DTMC) is called **irreducible**, if for all $x, y \in S$

$$p_t(x,y) > 0 \text{ for some } t > 0 \qquad \big(p_n(x,y) > 0 \text{ for some } n \in \mathbb{N}\big)\ .$$

Remark. For continuous time irreducibility implies $p_t(x,y) > 0$ for **all** $t > 0$.

2. Stationary distributions

Proposition 2.3 (Uniqueness)

An **irreducible** Markov chain has **at most one** stationary distribution.

Proof. Follows from the **Perron Frobenius theorem:**
Let P be a stochastic matrix ($P = P_t$ for any $t \geq 0$ for CTMCs). Then

1. $\lambda_1 = 1$ is an eigenvalue of P, it is singular if and only if the chain is irreducible. Corresponding left and right eigenvectors have non-negative entries.
2. if the chain is continuous-time or discrete-time aperiodic, all remaining eigenvalues $\lambda_i \in \mathbb{C}$, $i \neq 1$ satisfy $\mathrm{Re}(\lambda_i) < 0$ or $|\lambda_i| < 1$, respectively

The second part of the Perron Frobenius theorem also implies convergence of the transition functions to the stationary distribution, since

$$p_t(x,y) = \sum_{i=1}^{|S|} \langle \delta_x | u_i \rangle \langle v_i | \, e^{\lambda_i t} \to \langle v_1 | = \langle \boldsymbol{\pi} | \quad \text{as } t \to \infty \,.$$

2. Sample paths

Sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous by convention.

For $X_0 = x$, define the **holding time** $\quad W_x := \inf\{t > 0 : X_t \neq x\}$.

Proposition 2.4

$W_x \sim \mathrm{Exp}(|g(x,x)|)$, i.e. it is **exponentially distributed** with mean $1/|g(x,x)|$, and if $|g(x,x)| > 0$ the chain jumps to $y \neq x$ after time W_x with probability $g(x,y)/|g(x,x)|$.

Proof. W_x has the **memoryless property**, i.e. for all $t, u > 0$

$$\mathbb{P}(W_x > t+u | W_x > t) = \mathbb{P}(W_x > t+u | X_t = x) = \mathbb{P}(W_x > u)$$

where we used the Markov property and homogeneity. Therefore

$$\mathbb{P}(W_x > t+u) = \mathbb{P}(W_x > u)\mathbb{P}(W_x > t) \quad \Rightarrow \quad \mathbb{P}(W_x > t) = e^{\gamma t}$$

where $\quad \gamma = \frac{d}{dt}\mathbb{P}(W_x > t)\Big|_{t=0} = \lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,x) + o(\Delta t) - 1}{\Delta t} = g(x,x) \leq 0$.

Conditioned on leaving the current state shortly, the probability to jump to y is

$$\lim_{\Delta t \searrow 0} \frac{p_{\Delta t}(x,y)}{1 - p_{\Delta t}(x,x)} = \lim_{\Delta t \searrow 0} \frac{\Delta t\, g(x,y)}{1 - 1 - \Delta t\, g(x,x)} = \frac{g(x,y)}{-g(x,x)} .$$

2. Sample paths

- the **jump times** $J_0, J_1, \ldots$ are defined recursively as

$$J_0 = 0 \quad \text{and} \quad J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\} \ .$$

- due to right-continuous paths, jump times are **stopping times**, i.e. for all $t \geq 0$, the event $\{J_n \leq t\}$ depends only on $(X_s : 0 \leq s \leq t)$.
- By the **strong Markov property** (allows conditioning on state at stopping time), subsequent holding times and jump probabilities are all independent.
- The **jump chain** $\quad (Y_n : n \in N_0) \quad$ with $\quad Y_n := X_{J_n}$

  is then a discrete-time Markov chain with transition matrix

$$p^Y(x,y) = \begin{cases} 0 & , \ x = y \\ g(x,y)/|g(x,x)| & , \ x \neq y \end{cases} \text{ if } g(x,x) < 0 \quad \text{and}$$
$$p^Y(x,y) = \delta_{x,y} \text{ if } g(x,x) = 0 \qquad \text{(by convention)} \ .$$

- A **sample path** is constructed by simulating the jump chain $(Y_n : n \in \mathbb{N}_0)$ together with independent **holding times** $(W_{Y_n} : n \in \mathbb{N}_0)$, so that $J_n = \sum_{k=0}^{n-1} W_{Y_k}$

2. Examples

- A **Poisson process** with **rate** λ (short PP(λ)) is a CTMC with

$$S = \mathbb{N}_0,\ X_0 = 0 \quad \text{and} \quad g(x,y) = \lambda\delta_{x+1,y} - \lambda\delta_{x,y}\ .$$

The PP(λ) has **stationary and independent increments** with

$$\mathbb{P}[X_{t+u} = n + k | X_u = n] = p_t(0,k) = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} \quad \text{for all } u,t > 0,\ k,n \in \mathbb{N}_0$$

since $\pi_t(k) = p_t(0,k)$ solves the Master equation $\frac{d}{dt}\pi_t(k) = (\boldsymbol{\pi}_t G)(k)$.

- A **birth-death chain** with **birth rates** α_x and **death rates** β_x is a CTMC with

$$S = \mathbb{N}_0 \quad \text{and} \quad g(x,y) = \alpha_x\delta_{x+1,y} + \beta_x\delta_{x-1,y} - (\alpha_x + \beta_x)\delta_{x,y}\ ,$$

where $\beta_0 = 0$.

Special cases include

  - **M/M/1 server queues**: $\alpha_x \equiv \alpha > 0$, $\beta_x \equiv \beta > 0$ for $x > 1$
  - **M/M/∞ server queues**: $\alpha_x \equiv \alpha > 0$, $\beta_x = x\beta$
  - **population growth model**: $\alpha_x = x\alpha$, $\beta_x = x\beta$

2. Ergodicity

Definition 2.4

A Markov process is called **ergodic** if it has a unique stationary distribution $\boldsymbol{\pi}$ and

$$p_t(x,y) = \mathbb{P}[X_t = y | X_0 = x] \to \pi(y) \quad \text{as } t \to \infty\,, \quad \text{for all } x,y \in S\,.$$

Theorem 2.5

An **irreducible** (aperiodic) MC with finite state space is **ergodic**.

Theorem 2.6 (Ergodic Theorem)

Consider an **ergodic Markov chain** with unique stationary distribution $\boldsymbol{\pi}$. Then for every bounded function $f : S \to \mathbb{R}$ we have with probability 1

$$\frac{1}{T}\int_0^T f(X_t)\,dt \quad \text{or} \quad \frac{1}{N}\sum_{n=1}^N f(X_n) \;\to\; \mathbb{E}_{\boldsymbol{\pi}}[f] \quad \text{as } T,N \to \infty\,.$$

- for a proof see e.g. [GS], chapter 9.5
- in practice, use relaxation/burn-in time before computing time averages

2. Markov Chain Monte Carlo (MCMC)

Typical problems related to sampling from π on a very large state space S

- Compute **expectations** $\mathbb{E}_{\boldsymbol{\pi}}[f] = \sum_{x\in S} f(x)\pi(x)$
- for **Gibbs measures** $\pi(x) = \frac{1}{Z(\beta)} e^{-\beta H(x)}$ (stach. mech. problems), compute **partition function** $Z(\beta) = \sum_{x\in S} e^{-\beta H(x)}$

Use the **ergodic theorem** to estimate expectations by time averages

- assume $\pi(x) > 0$ for all $x \in S$ (otherwise restrict S)
- invent CTMC/DTMC such that π is stationary, e.g. via **detailed balance**

$$\pi(x)g(x,y) = \pi(y)g(y,x) \quad \text{or} \quad \pi(x)p(x,y) = \pi(y)p(y,x)$$

for Gibbs measures $\quad e^{-\beta H(x)} g(x,y) = e^{-\beta H(y)} g(y,x)$

Typically $g(x,y) = q(x,y)\, a(x,y)$, i.e. **propose move** from x to y with rate $q(x,y) = q(y,x)$ (irreducible on S but 'local'), **accept** with probability $a(x,y)$

- **Heat bath algorithm:** $\quad a(x,y) = \dfrac{e^{-\beta H(y)}}{e^{-\beta H(x)} + e^{-\beta H(y)}}$
- **Metropolis-Hastings:** $\quad a(x,y) = \begin{cases} 1 & , \text{ if } H(y) \le H(x) \\ e^{\beta(H(x)-H(y))} & , \text{ if } H(y) > H(x) \end{cases}$

2. Reversibility

Proposition 2.7 (Time reversal)

Let $(X_t : t \in [0,T])$ be a finite state, irreducible CTMC with generator G^X on a compact time interval which is **stationary**, i.e. $X_t \sim \boldsymbol{\pi}$ for $t \in [0,T]$. Then the **time reversed chain**

$$(Y_t : t \in [0,T]) \quad \text{with} \quad Y_t := X_{T-t}$$

is a stationary CTMC with generator $\quad g^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, g^X(y,x)$ and stat. prob. $\boldsymbol{\pi}$.

- An analogous statement holds for stationary, finite state, irreducible DTMCs with $\quad p^Y(x,y) = \dfrac{\pi(y)}{\pi(x)}\, p^X(y,x)$.
- Stationary chains with reversible $\boldsymbol{\pi}$ are **time-reversible**, $g^Y(x,y) = g^X(x,y)$.
- The definition of stationary chains can be extended to negative times, $(X_t : t \in \mathbb{R})$, with the time reversed chain given by $Y_t := X_{-t}$.
- The time reversal of non-stationary MCs is in general **not** a homogeneous MC, for DTMCs using Bayes' Theorem we get $\quad p^Y(x,y;n) = \frac{\pi_{N-n-1}(y)}{\pi_{N-n}(x)} p^X(y,x)$

2. Countably infinite state space

For infinite state space, Markov chains can get 'lost at infinity' and have no stationary distribution. Let $T_x := \inf\{t > J_1 : X_t = x\}$ be the first **return time** to a state x.

$\big($ For DTMCs return times are defined as $T_x := \inf\{n \geq 1 : X_n = x\}\big)$

Definition 2.5

A state $x \in S$ is called

- **transient**, if $\mathbb{P}[T_x = \infty | X_0 = x] > 0$
- **null recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] = \infty$
- **positive recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] < \infty$

and these properties partition S into **communicating classes**.

- For an irreducible MC all states are either transient, null or positive recurrent.
- A MC has a unique stationary distribution if and only if it is positive recurrent

$$\text{and in this case} \quad \pi(x) = \frac{1}{\mathbb{E}[T_x | X_0 = x]} \mathbb{E}\Big[\int_0^{T_x} \mathbb{1}_x(X_s)\, ds \Big| X_0 = x\Big] .$$

2. Countably infinite state space

A CTMC with an infinite transient component in S can exhibit **explosion**.

Definition 2.6

For a CTMC define the **explosion time**

$$J_\infty := \lim_{n\to\infty} J_n \in (0, \infty] \quad \text{where } J_n \text{ are the jump times of the chain .}$$

The chain is called **non-explosive** if $\quad \mathbb{P}[J_\infty = \infty] = 1 \,, \quad$ otherwise it is **explosive**.

- If the exit rates are uniformly bounded, i.e. $\quad \sup_{x\in S} |g(x,x)| < \infty$, then the chain is non-explosive, which is always the case if S is finite.
- As an example, consider a **pure birth chain** with $X_0 = 1$ and rates

  $$g(x,y) = \alpha_x \delta_{y,x+1} - \alpha_x \delta_{y,x} \,, \quad x, y \in S = \mathbb{N}_0 \,.$$

  If $\alpha_x \to \infty$ fast enough (e.g. $\alpha_x = x^2$) we get

  $$\mathbb{E}[J_\infty] = \sum_{x=1}^{\infty} \mathbb{E}[W_x] = \sum_{x=1}^{\infty} \frac{1}{\alpha_x} < \infty$$

  since holding times $W_x \sim \mathrm{Exp}(\alpha_x)$. This implies $\mathbb{P}[J_\infty = \infty] = 0 < 1$.

3. Markov processes with $S = \mathbb{R}$

Proposition 3.1

Let $(X_t : t \geq 0)$ by a homogeneous MP as in Definition 18 with state space $S = \mathbb{R}$. Then for all $t \geq 0$ and (measurable) $A \subseteq \mathbb{R}$ the **transition kernel** for all $x \in \mathbb{R}$

$$P_t(x, A) := \mathbb{P}[X_t \in A | X_0 = x] = \mathbb{P}[X_{t+u} \in A | X_u = x] \quad \text{for all } u \geq 0$$

is well defined. If it is absolutely continuous the **transition density** p_t with

$$P_t(x, A) = \int_A p_t(x, y)\, dy$$

exists and fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x, y) = \int_{\mathbb{R}} p_t(x, z)\, p_u(z, y)\, dz \quad \text{for all } t, u \geq 0,\ x, y \in \mathbb{R}\ .$$

As for CTMCs, the transition densities and the initial distribution $p_0(x)$ describe all **finite dimensional distributions (fdds)**

$$\mathbb{P}[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = \int_R dz_0 p_0(z_0) \int_{-\infty}^{x_1} dz_1 p_{t_1}(z_0, z_1) \cdots \int_{-\infty}^{x_n} dz_n p_{t_n - t_{n-1}}(z_{n-1}, z_n)$$

3. Jump processes

$(X_t : t \geq 0)$ is a **jump process** with state space $S = \mathbb{R}$ characterized by a
jump rate density $\quad r(x, y) \geq 0 \quad$ with a uniformly bounded
total exit rate $\quad R(x) = \int_{\mathbb{R}} r(x, y)\, dy < \bar{R} < \infty \quad$ for all $x \in \mathbb{R}$.
Ansatz for transition function as $\Delta t \to 0$:

$$p_{\Delta t}(z, y) = r(z, y)\Delta t + \big(1 - R(z)\Delta t\big)\delta(y - z)$$

Then use the Chapman Kolmogorov equations

$$p_{t+\Delta t}(x, y) - p_t(x, y) = \int_{\mathbb{R}} p_t(x, z)\, p_{\Delta t}(z, y)\, dz - p_t(x, y) =$$
$$= \int_{\mathbb{R}} p_t(x, z) r(z, y) \Delta t\, dz + \int_{\mathbb{R}} \big(1 - R(z)\Delta t - 1\big) p_t(x, z)\delta(y - z) dz$$

to get the **Kolmogorov-Feller equation** $\quad$ (x is a fixed initial condition)

$$\frac{\partial}{\partial t} p_t(x, y) = \int_{\mathbb{R}} \Big(p_t(x, z) r(z, y) - p_t(x, y) r(y, z)\Big)\, dz\, .$$

As for CTMC sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous.

3. Gaussian processes

$\mathbf{X} = (X_1, \ldots, X_n) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ is a **multivariate Gaussian** in $\mathbb{R}^n$ if it has PDF

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\Big(-\frac{1}{2} \langle \mathbf{x} - \boldsymbol{\mu} | \, \Sigma^{-1} \, | \mathbf{x} - \boldsymbol{\mu} \rangle \Big) ,$$

with **mean** $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and **covariance matrix**

$$\Sigma = (\sigma_{ij} : i, j = 1, \ldots, n) , \quad \sigma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big] .$$

Definition 3.1

A stochastic process $(X_t : t \geq 0)$ with state space $S = \mathbb{R}$ is a **Gaussian process** if for all $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n$ the vector $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian.

Proposition 3.2

All fdds of a Gaussian process $(X_t : t \geq 0)$ are fully characterized by the **mean** and the **covariance function**

$$m(t) := \mathbb{E}[X_t] \quad \text{and} \quad \sigma(s, t) := \mathrm{Cov}[X_s, X_t] .$$

3. Stationary independent increments

Definition 3.2

A stochastic process $(X_t : t \geq 0)$ has **stationary increments** if

$$X_t - X_s \sim X_{t-s} - X_0 \quad \text{for all } 0 \leq s \leq t\,.$$

It has **independent increments** if for all $n \geq 1$ and $0 \leq t_1 < \cdots < t_n$

$$\big\{X_{t_{k+1}} - X_{t_k} : 1 \leq k < n\big\} \quad \text{are independent}\,.$$

Example. The Poisson process $(N_t : t \geq 0) \sim PP(\lambda)$ has stationary independent increments with $N_t - N_s \sim \mathrm{Poi}\big(\lambda(t-s)\big)$.

Proposition 3.3

The following two statements are equivalent for a stochastic process $(X_t : t \geq 0)$:

- X_t has stationary independent increments and $X_t \sim \mathcal{N}(0, t)$ for all $t \geq 0$.
- X_t is a Gaussian process with $m(t) = 0$ and $\sigma(s,t) = \min\{s, t\}$.

Stationary independent incr. have **stable distributions** such as Gaussian or Poisson.

3. Brownian motion

Definition 3.3

Standard Brownian motion $(B_t : t \geq 0)$ is a stochastic process that satisfies either of the two equivalent properties in Proposition 3.3 and has **continuous paths**, i.e.

$$\mathbb{P}\big[\{\omega : t \mapsto B_t(\omega) \text{ is continuous in } t \geq 0\}\big] = 1 \ .$$

Theorem 3.4 (Wiener 1923)

There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which standard Brownian motion exists.

Proof idea.* Construction on $\Omega = \mathbb{R}^{[0,\infty)}$, using **Kolmogorov's extension theorem**:
For every 'consistent' description of finite dimensional distributions (fdds) there exists a 'canonical' process $X_t[\omega] = \omega(t)$ characterized by a law $\mathbb{P}$ on Ω.
The main problem is to show that there exists a 'version' of the process that has continuous paths, i.e. $\mathbb{P}$ can be chosen to concentrate on continuous paths ω.

Remark.* Construction of $(N_t : t \geq 0) \sim PP(\lambda)$ is

$$N_t := \max\big\{k \geq 1 : \tau_1 + \cdots + \tau_k \leq t\big\} \ , \quad \tau_1, \tau_2, \cdots \sim \text{Exp}(\lambda) \text{ iidrvs}$$

3. Properties of Brownian motion

- SBM is a time-homogeneous MP with $B_0 = 0$.
- $\sigma B_t + x$ with $\sigma > 0$ is a (general) BM with $B_t \sim \mathcal{N}(x, \sigma^2 t)$.
  The transition density is given by a Gaussion PDF

$$p_t(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\Big(-\frac{(y-x)^2}{2\sigma^2 t}\Big)$$

  This is also called the **heat kernel**, since it solves the **heat/diffusion equation**

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} p_t(x,y) \quad \text{with} \quad p_0(x,y) = \delta(y-x)\ .$$

- SBM is **self-similar** with **Hurst exponent** $H = 1/2$, i.e.

$$(B_{\lambda t} : t \geq 0) \sim \lambda^H (B_t : t \geq 0) \quad \text{for all } \lambda > 0\ .$$

- $t \mapsto B_t$ is $\mathbb{P} - a.s.$ **not differentiable** at t for all $t \geq 0$.
  For fixed $h > 0$ define $\xi_t^h := (B_{t+h} - B_t)/h \sim \mathcal{N}(0, 1/h)$, which is a mean-0 Gaussian process with covariance $\sigma(s,t) = \begin{cases} 0 & , \ |t-s| > h \\ (h - |t-s|)/h^2 & , \ |t-s| < h \end{cases}$.
  The (non-existent) derivative $\xi_t := \lim_{h \to 0} \xi_t^h$ is called **white noise** and is formally a mean-0 Gaussian process with covariance $\sigma(s,t) = \delta(t-s)$.

3. Generators as operators

For a **CTMC** $(X_t : t \geq 0)$ with discrete state space S we have for $f : S \to \mathbb{R}$

$$\mathbb{E}\big[f(X_t)\big] = \sum_{x \in S} \pi_t(x) f(x) = \langle \pi_t | f \rangle \quad \text{and} \quad \underbrace{\frac{d}{dt}\langle \pi_t | = \langle \pi_t | G}_{\textbf{master equation}}$$

Therefore $\quad \frac{d}{dt}\mathbb{E}\big[f(X_t)\big] = \frac{d}{dt}\langle \pi_t | f \rangle = \langle \pi_t | G | f \rangle = \mathbb{E}\big[(Gf)(X_t)\big]$.

The **generator** G can be defined as an operator G acting on functions $f : S \to \mathbb{R}$

$$G|f\rangle(x) = (Gf)(x) = \sum_{y \neq x} g(x,y)\big[f(y) - f(x)\big] .$$

For **Brownian motion** use the **heat eq.** and integration by parts for $f \in C^2(\mathbb{R})$

$$\frac{d}{dt}\mathbb{E}_x\big[f(X_t)\big] = \int_{\mathbb{R}} \partial_t p_t(x,y) f(y)dy = \frac{\sigma^2}{2}\int_{\mathbb{R}} \partial_y^2 p_t(x,y) f(y)dy = \mathbb{E}_x\big[(\mathcal{L}f)(X_t)\big]$$

where the **generator of BM** is $\quad (\mathcal{L}f)(x) = \frac{\sigma^2}{2}\Delta f(x) \qquad \left(\text{or } \frac{\sigma^2}{2} f''(x)\right)$.

For **jump processes** with $S = \mathbb{R}$ and rate density $r(x,y)$ the generator is

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}} r(x,y)\big[f(y) - f(x)\big]\, dy .$$

3. Brownian motion as scaling limit

Proposition 3.5

Let $(X_t : t \geq 0)$ be a jump process on $\mathbb{R}$ with **translation invariant rates** $r(x,y) = q(y-x)$ which have

$$\textbf{mean zero} \qquad \int_{\mathbb{R}} q(z)\, z\, dz = 0 \quad \text{and}$$

$$\textbf{finite second moment} \qquad \sigma^2 := \int_{\mathbb{R}} q(z)\, z^2\, dz < \infty\,.$$

Then for all $T > 0$ the rescaled process

$$\big(\epsilon \mathbf{X}_{\mathbf{t}/\epsilon^2} : \mathbf{t} \in [\mathbf{0}, \mathbf{T}]\big) \quad \Rightarrow \quad \big(\mathbf{B}_{\mathbf{t}} : \mathbf{t} \in [\mathbf{0}, \mathbf{T}]\big) \quad \text{as } \epsilon \to \mathbf{0}$$

converges in distribution to a BM with generator $\mathcal{L} = \frac{1}{2}\sigma^2 \Delta$ for all $T > 0$.

Proof. Taylor expansion of the generator for test functions $f \in C^3(\mathbb{R})$, and tightness argument for continuity of paths (requires fixed interval $[0, T]$).

3. Diffusion processes

Definition 3.4

A **diffusion process** with **drift** $a(x,t) \in \mathbb{R}$ and **diffusion** $\sigma(x,t) > 0$ is a real-valued process with continuous paths and generator

$$(\mathcal{L}f)(x) = a(x,t) f'(x) + \frac{1}{2}\sigma^2(x,t) f''(x) .$$

Examples.

- The **Ornstein-Uhlenbeck process** is a diffusion process with generator

$$(\mathcal{L}f)(x) = -\alpha\, x f'(x) + \frac{1}{2}\sigma^2 f''(x) , \quad \alpha, \sigma^2 > 0 .$$

  It has a Gaussian stationary distribution $\mathcal{N}(0, \sigma^2/(2\alpha))$.
  If the initial distribution π_0 is Gaussian, this is a **Gaussian process**.

- The **Brownian bridge** is a Gaussian diffusion with $X_0 = 0$ and generator

$$(\mathcal{L}f)(x) = -\frac{x}{1-t} f'(x) + \frac{1}{2} f''(x) .$$

  Equivalently, it can be characterized as a SBM conditioned on $B_1 = 0$.

3. Diffusion processes

Time evolution of the mean. Use $\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[(\mathcal{L}f)(x_t)]$ with $f(x) = x$

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[a(X_t, t)]$$

Time evolution of the transition density. With $X_0 = x$ we have for $p_t(x, y)$

$$\int_{\mathbb{R}} \frac{\partial}{\partial t} p_t(x, y) f(y) dy = \frac{d}{dt}\mathbb{E}[f(X_t)] = \int_{\mathbb{R}} p_t(x, y)\mathcal{L}f(y) dy \quad \text{for any } f\ .$$

Use integration by parts to get the **Fokker-Planck equation**

$$\frac{\partial}{\partial t} p_t(x, y) = -\frac{\partial}{\partial y}\big(a(y, t) p_t(x, y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(y, t) p_t(x, y)\big)\ .$$

Stationary distributions for time-independent $a(y) \in \mathbb{R}$ and $\sigma^2(y) > 0$

$$\frac{d}{dy}\big(a(y) p^*(y)\big) = \frac{1}{2}\frac{d^2}{dy^2}\big(\sigma^2(y) p^*(y)\big)\ ,$$

leads to a stationary density (modulo normalization fixing $p^*(0)$)

$$p^*(x) = p^*(0) \exp\Big(\int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)} dy\Big)\ .$$

3. Beyond diffusion

Definition 3.5

A **Lévy process** $(X_t : t \geq 0)$ is a real-valued process with right-continuous paths and stationary, independent increments.

The generator has a part with **constant drift** $a \in \mathbb{R}$ and **diffusion** $\sigma^2 \geq 0$

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2}f''(x) + \int_{\mathbb{R}} \Big(f(x+z) - f(x) - zf'(x)\mathbb{1}_{(0,1)}(|z|)\Big)q(z)dz\,,$$

and a translation invariant **jump part** with density $q(z)$ (or measure $\nu(dz)$)

and fulfills $\displaystyle\int_{|z|>1} q(z)dz < \infty$ and $\displaystyle\int_{0<|z|<1} z^2 q(z)dz < \infty$.

- Diffusion processes, in particular **BM** with $a = 0$, $\sigma^2 > 0$ and $q(z) \equiv 0$, or jump processes, in particular **Poisson** with $a = \sigma = 0$ and $q(z) = \lambda\delta(z-1)$.
- For $a = \sigma = 0$ and heavy-tailed jump distribution

  $$q(z) = \tfrac{C}{|z|^{1+\alpha}} \quad \text{with} \quad C > 0 \text{ and } \alpha \in (0,2]$$

  the process is called **α-stable symmetric Lévy process** or **Lévy flight**.

  self-similar $(X_{\lambda t} : t \geq 0) \sim \lambda^H (X_t : t \geq 0)\,,\quad \lambda > 0$ with $H = 1/\alpha$

  $\Rightarrow$ **super-diffusive behaviour** with $\mathbb{E}[X_t^2] \propto t^{2/\alpha}$

3. Beyond diffusion

In general, a process $(X_t : t \geq 0)$ is said to exhibit **anomalous diffusion** if

$$\mathrm{Var}[X_t]/t \to \begin{cases} 0 & , \text{ sub-diffusive} \\ \infty & , \text{ super-diffusive} \end{cases} \quad \text{as } t \to \infty \,.$$

Definition 3.6

A **fractional Brownian motion (fBM)** $(B_t^H : t \geq 0)$ with **Hurst index** $H \in (0,1)$ is a **mean-zero Gaussian process** with continuous paths, $B_0^H = 0$ and **covariances**

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big) \quad \text{for all } s,t \geq 0 \,.$$

- For $H = 1/2$, fBM is standard Brownian motion.
- fBM has stationary Gaussian increments where for all $t > s \geq 0$

  $$B_t^H - B_s^H \sim B_{t-s}^H \sim \mathcal{N}\big(0, (t-s)^{2H}\big) \,,$$

  which for $H \neq 1/2$ are **not** independent and the process is **non-Markov**.
- fBM is **self-similar**, i.e. $(B_{\lambda t}^H : t \geq 0) \sim \lambda^H (B_t^H : t \geq 0)$ for all $\lambda > 0$.

3. Fractional BM and noise

- fBM exhibits **anomalous diffusion** with $\mathrm{Var}[B_t^H] = t^{2H}$
- $\mathbf{H > 1/2}$: super-diffusive with positively correlated increments
  $\mathbf{H < 1/2}$: sub-diffusive with negatively correlated increments

$$\mathbb{E}\big[B_1^H(B_{t+1}^H - B_t^H)\big] = \frac{(t+1)^{2H} - 2t^{2H} + (t-1)^{2H}}{2} \underset{t\to\infty}{\simeq} H(2H-1)t^{2(H-1)}$$

For a **stationary process** $\big(X_t : t \geq 0\big)$ on $\mathbb{R}$ define the **autocorrelation/covariance fct**

$$c(t) := \mathrm{Cov}[X_s, X_{s+t}] \quad \text{for all } s, t \in \mathbb{R} \,.$$

Its Fourier transform is the **spectral density** $S(\omega) := \int_{\mathbb{R}} c(t)e^{-i\omega t}dt$

- **white noise** $(\xi_t : t \geq 0)$, stationary GP with mean zero and

$$c(t) = \delta(t) \quad \Rightarrow \quad S(\omega) \equiv 1 \,.$$

- **fractional** or **1/f noise** $(\xi_t^H : t \geq 0)$, stationary GP with mean zero and

$$c(t) = \frac{2H(2H-1)}{|t|^{2(1-H)}} \quad \Rightarrow \quad S(\omega) \propto |\omega|^{2(1-H)-1} = \frac{1}{\omega^{2H-1}}$$

3. SDEs and Itô's formula

Let $(B_t : t \geq 0)$ be a standard BM. Then a diffusion process with drift $a(x,t)$ and diffusion $\sigma(x,t)$ solves the **Stochastic differential equation (SDE)**

$$dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t \ .$$

Here dB_t is white noise, interpreted in integrated form as

$$X_t - X_0 = \int_0^t a(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s \ .$$

Theorem 3.6 (Itô's formula for diffusions)

Let $(X_t : t \geq 0)$ be a diffusion with generator $\mathcal{L}$ and $f : \mathbb{R} \to \mathbb{R}$ a smooth. Then

$$f(X_t) - f(X_0) = \int_0^t (\mathcal{L}f)(X_s)ds + \int_0^t \sigma(X_s, s)f'(X_s)dB_s \ .$$

or, equivalently in terms of SDEs

$$df(X_t) = a(X_t, t)f'(X_t)dt + \frac{1}{2}\sigma^2(X_t, t)f''(X_t)dt + \sigma(X_t, t)f'(X_t)dB_t \ .$$

3. SDEs and Itô's formula

Itô's formula for diffusions implies the following.

Proposition 3.7

Let $(X_t : t \geq 0)$ be a diffusion process with drift $a(x,t)$ and diffusion $\sigma(x,t)$, and $f : \mathbb{R} \to \mathbb{R}$ a smooth invertible function. Then $(Y_t : t \geq 0)$ with $Y_t = f(X_t)$ is a diffusion process with $(x = f^{-1}(y))$

$$\textbf{drift} \quad a(x,t)f'(x) + \frac{1}{2}\sigma^2(x,t)f''(x) \quad \text{and } \textbf{diffusion} \quad \sigma(x,t)f'(x)\ .$$

Geometric BM. $Y_t := e^{\theta B_t}$, so $f(x) = e^{\theta x}$ with $f'(x) = \theta f(x)$ and $f''(x) = \theta^2 f(x)$, where $(B_t : t \geq 0)$ is standard BM with $a \equiv 0$, $\sigma^2 \equiv 1$ and $\theta \in \mathbb{R}$.
Then $(Y_t : t \geq 0)$ is a diffusion process with SDE $\quad dY_t = \frac{\theta}{2} Y_t dt + \theta Y_t dB_t\ .$

Exponential martingale.
$Z_t := e^{\theta B_t - \theta^2 t/2}$, so $f(x,t) = e^{\theta x - \theta^2 t/2}$ and $\partial_t f(x,t) = -\frac{\theta^2}{2} f(x,t)$
Then $\quad dZ_t = \frac{\theta}{2} Z_t dt - \frac{\theta}{2} Z_t dt - +\theta Z_t dB_t = \theta Z_t dB_t$
and $(Z_t : t \geq 0)$ is a **martingale** (see next slide) with $\mathbb{E}[Z_t] \equiv Z_0 = 1$.

3. Fluctuations and martingales

Definition 3.7

A real-valued stochastic process $(M_t : t \geq 0)$ is a **martingale** w.r.t. the process $(X_t : t \geq 0)$ if for all $t \geq 0$ we have $\mathbb{E}[|M_t|] < \infty$ and

$$\mathbb{E}\big[M_t \big| \{X_u : 0 \leq u \leq s\}\big] = M_s \quad \text{a.s. for all } s \leq t\ .$$

If in addition $\mathbb{E}[M_t^2] < \infty$, there exists a unique increasing process $([M]_t : t \geq 0)$ called the **quadratic variation**, with $[M]_0 = 0$ and such that $M_t^2 - [M]_t$ is martingale.

Theorem 3.8 (Itô's formula)

Let $(X_t : t \geq 0)$ be a Markov process on state space S with generator $\mathcal{L}$. Then for any smooth enough $f : S \times [0, \infty) \to \mathbb{R}$

$$f(X_t, t) - f(X_0, 0) = \int_0^t (\mathcal{L}f)(X_s, s)ds + \int_0^t \partial_s f(X_s, s)ds + M_t^f\ ,$$

where $(M_t^f : t \geq 0)$ is a martingale w.r.t. $(X_t : t \geq 0)$ with $M_0^f = 0$ and

$$\text{quadratic variation} \quad [M^f]_t = \int_0^t \Big((\mathcal{L}f^2)(X_s, s) - 2(f\mathcal{L}f)(X_s, s)\Big)ds\ .$$

3. Fluctuations and martingales

- For a **Poisson process** $(N_t : t \geq 0)$ with rate $\lambda > 0$ Itô's formula implies that

$$M_t := N_t - \lambda t \quad \text{is a martingale with quadr. variation} \quad [M]_t = \lambda t \ .$$

- **Watanabe's charactericatio of PP**: Let $(N_t : t \geq 0)$ be a **counting process**, i.e. a jump process on $S = \mathbb{N}$ with jump size $+1$ only. If $M_t = N_t - \lambda t$ is a martingale, then $(N_t : t \geq 0) \sim PP(\lambda)$.
- For a **diffusion process**, choosing $f(X_t, t) = X_t$ in Itô's formula leads to

$$X_t - X_0 = \int_0^t a(X_s, s)ds + M_t \quad \text{with} \quad [M]_t = \int_0^t \sigma^2(X_s, s)ds \ .$$

  In particular for BM with $a(x, t) \equiv 0$ and $\sigma^2(x, t) \equiv \sigma^2$ we have

$$(B_t : t \geq 0) \text{ is a martingale with quadratic variation} \quad [B]_t = t \ .$$

- **Lévy's charactericatio of BM**: Any continuous martingale $(M_t : t \geq 0)$ on $\mathbb{R}$ with $M_0 = 0$ and quadratic variation $[M]_t = t$ is standard Brownian motion.

  Furthermore, any **continuous martingale $(M_t : t \geq 0)$ on $\mathbb{R}$ with $M_0 = 0$ is a continuous (random) time-change of a standard BM**, i.e.

$$(M_t : t \geq 0) \sim \big(B_{[M]_t} : t \geq 0\big) \quad \text{for SBM} \quad (B_t : t \geq 0) \ .$$

3. Fluctuations and martingales

- For a **diffusion process** $(X_t : t \geq 0)$ we have

$$X_t - X_0 = \int_0^t a(X_s, s)ds + M_t \quad \text{with} \quad [M]_t = \int_0^t \sigma^2(X_s, s)ds \; .$$

with M_t a continuous martingale $\quad \Rightarrow \quad (M_t : t \geq 0) \sim (B_{[M]_t} : t \geq 0)$

- Related time-changed BMs can be written as **stochastic Itô integrals**

$$M_t = \int_0^t \sigma(X_s, s)dB_s := B_{[M]_t} \; .$$

Therefore $\sigma \equiv 0$ implies **deterministic dynamics** with $M_t \equiv 0$,
(also because $\mathcal{L}f^2 = 2f f' a = 2f\mathcal{L}f$ for all f, so $[M^f]_t \equiv 0$ in Itô's formula)
and the corresponding SDE is an ODE $\quad dX_t/dt = a(X_t, t)$.

Vanishing drift $a \equiv 0$ implies $\quad X_t - X_0 = M_t \quad$ or $\quad dX_t = \sigma(X_t, t)dB_t$
and the process $(X_t : t \geq 0)$ is a **martingale**.

- Recall the **exponential martingale** $e^{\theta B_t - \theta^2 t/2}$ as a non-trivial example.

3. Martingales and conservation laws

Consider a **CTMP** $(X_t : t \geq 0)$ on state space S with **generator** $\mathcal{L}$, and an **observable** $f : S \to \mathbb{R}$ such that $\mathcal{L}f : S \to \mathbb{R}$ is well defined (e.g. $f \in C^2(S, \mathbb{R})$ for diffusions).

Proposition 3.9

If $\mathcal{L}f(x) = 0$ for all $x \in S$, then $f(X_t)$ **is a martingale**, and is **conserved in expectation**, i.e. (for any initial condition X_0)

$$E\big[f(X_t)\big] = \mathbb{E}\big[f(X_0)\big] \quad \text{for all } t \geq 0 \,.$$

If in addition $\mathcal{L}f^2(x) = 0$ for all $x \in S$, then $f(X_t)$ is **conserved** (or **a conserved quantity**), i.e. (for any initial condition X_0)

$$f(X_t) = f(X_0) \quad \text{almost surely for all } t \geq 0 \,.$$

Proof. The first claim follows directly from Itô's formula (Theorem 3.8).
For the second claim, we have $f(X_t) = f(X_0) + M_t^f$ and M_t^f has quadratic variation

$$[M^f]_t = \int_0^t \Big((\mathcal{L}f^2)(X_s, s) - 2(f\mathcal{L}f)(X_s, s)\Big) ds = 0 \,,$$

for all $t \geq 0$, which implies $M_t^f \equiv 0$ almost surely. □

4. Stochastic particle systems

- **lattice/population**: $\Lambda = \{1, \ldots, L\}$, finite set of points
- **state space** S is given by the set of all **configurations**

$$\eta = \big(\eta(i) : i \in \Lambda\big) \in S = \{0,1\}^L \quad \text{(often also written } \{0,1\}^\Lambda) \, .$$

  $\eta(i) \in \{0,1\}$ signifies the presence of a particle/infection at site/individual i.
- Only local transitions are allowed with rates

$$\begin{array}{llll} \eta \to \eta^i & \text{with rate} & c(\eta, \eta^i) & \textbf{(reaction)} \\ \eta \to \eta^{ij} & \text{with rate} & c(\eta, \eta^{ij}) & \textbf{(transport)} \end{array}$$

where $\eta^i(k) = \begin{cases} \eta(k) & , \ k \neq i \\ 1 - \eta(k) \, , & k = i \end{cases}$ and $\eta^{ij}(k) = \begin{cases} \eta(k) \, , & k \neq i,j \\ \eta(j) \, , & k = i \\ \eta(i) \, , & k = j \end{cases}$

Definition 4.1

A **stochastic particle system** is a CTMC with state space $S = \{0,1\}^\Lambda$ and generator

$$\mathcal{L}f(\eta) = \sum_{i \in \Lambda} c(\eta, \eta^i) \big[f(\eta^i) - f(\eta)\big] \quad \text{or} \quad \mathcal{L}f(\eta) = \sum_{i,j \in \Lambda} c(\eta, \eta^{ij}) \big[f(\eta^{ij}) - f(\eta)\big] \, .$$

4. Contact process

The contact process is a simple stochastic model for the **SI epidemic** with **infection rates** $q(i,j) \geq 0$ and uniform **recovery rate** 1.

Definition 4.2

The **contact process (CP)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^i) = \underbrace{1 \cdot \delta_{\eta(i),1}}_{\text{recovery}} + \underbrace{\delta_{\eta(i),0} \sum_{j \neq i} q(j,i) \delta_{\eta(j),1}}_{\text{infection}} \quad \text{for all } i \in \Lambda .$$

Usually, $q(i,j) = q(j,i) \in \{0, \lambda\}$, i.e. connected individuals infect each other with fixed rate $\lambda > 0$.

- The CP has one absorbing state $\eta(i) = 0$ for all $i \in \Lambda$, which can be reached from every initial configuration. Therefore the process is ergodic and the infection eventually gets **extinct** with probability 1.
- Let $T := \inf\{t > 0 : \eta_t \equiv 0\}$ be the **extinction time**. Then there exists a **critical value (epidemic threshold)** $\lambda_c > 0$ such that (for irreducible $q(i,j)$)

$$\mathbb{E}[T|\eta_0 \equiv 1] \propto \log L \quad \text{for } \lambda < \lambda_c \quad \text{and} \quad \mathbb{E}[T|\eta_0 \equiv 1] \propto e^{CL} \quad \text{for } \lambda > \lambda_c .$$

4. Voter model

The voter model describes opinion dynamics with **influence rates** $q(i,j) \geq 0$ at which individual i persuades j to switch to her/his opinion.

Definition 4.3

The **linear voter model (VM)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^i) = \sum_{j \neq i} \underbrace{q(j,i)\big(\delta_{\eta(i),1}\delta_{\eta(j),0} + \delta_{\eta(i),0}\delta_{\eta(j),1}\big)}_{j \text{ influences } i \text{ if opinions differ}} \quad \text{for all } i \in \Lambda\ .$$

In non-linear versions the rates can be replaced by general (symmetric) functions.

- The VM is **symmetric** under relabelling opinions $0 \leftrightarrow 1$.
- If $q(i,j)$ is irreducible there are two absorbing states, $\eta \equiv 0, 1$, both of which can be reached from every initial condition. Therefore the VM is not ergodic, and **stationary measures** are

$$\alpha\delta_0 + (1-\alpha)\delta_1 \quad \text{with } \alpha \in [0,1] \text{ depending on the initial condition}\ .$$

- **Coexistence** of both opinions can occur on infinite lattices (e.g. $\mathbb{Z}^d$ for $d \geq 3$).

4. Exclusion process

The exclusion process describes transport of a conserved quantity (e.g. mass or energy) with **transport rates** $q(i,j) \geq 0$ site i to j.

Definition 4.4

The **exclusion process (EP)** $(\eta_t : t \geq 0)$ is an IPS with rates

$$c(\eta, \eta^{ij}) = q(i,j)\delta_{\eta(i),1}\delta_{\eta(j),0} \quad \text{for all } i,j \in \Lambda \; .$$

The EP is called **simple (SEP)** if jumps occur only between nearest neighbours on Λ. The SEP is **symmetric (SSEP)** if $q(i,j) = q(j,i)$, otherwise **asymmetric (ASEP)**.

- The SEP is mostly studied in a 1D geometry with periodic or open boundaries.
- For periodic boundary conditions the total number of particles $N = \sum_i \eta(i)$ is **conserved**. The process is ergodic on the sub-state space

  $$S_N = \Big\{ \eta \in \{0,1\}^L : \sum_i \eta(i) = N \Big\}$$

  for each value $N = 0, \ldots L$, and has a unique stationary distribution.
- For open boundaries particles can be created and destroyed at the boundary, the system is ergodic on S and has a unique stationary distribution.

4. Mean-field scaling limits

Consider the contact process $(\eta_t : t \geq 0)$ on a **complete graph**, using $\eta(i) \in \{0,1\}$ we can write the generator as

$$\mathcal{L}f(\eta) = \sum_{i\in\Lambda} \Big(\eta(i) + \lambda\big(1-\eta(i)\big)\sum_{j\in\Lambda}\eta(j)\Big)\big[f(\eta^i) - f(\eta)\big] .$$

For **mean-field observables** such as $N(\eta) := \sum_{i\in\Lambda}\eta(i)$ one can compute for $f : \mathbb{N}_0 \to \mathbb{R}$ (see problem sheet 3)

$$\mathcal{L}(f\circ N)(\eta) = \lambda\big(L-N\big)N\big[f(N+1) - f(N)\big] + N\big[f(N-1) - f(N)\big] ,$$

which shows that $t \mapsto N_t := N(\eta_t)$ is a Markov process with above generator for all L.

Mean-field scaling limit. $L \to \infty$ with $\lambda L \to \hat{\lambda}$

then $N_t/L \to X_t$, which is a **diffusion process** on $[0,1]$ with generator

$$\mathcal{L}f(x) = \big(\hat{\lambda}x(1-x) - x\big)f'(x) + \frac{1}{2L}\big(\hat{\lambda}x(1-x) + x\big)f''(x)$$

In the limit the diffusion coefficient vanishes and the process is **deterministic** (blue) with leading order diffusive correction (red) and corresponding SDE

$$dX_t = \big(\hat{\lambda}X_t(1-X_t) - X_t\big)dt + \sqrt{\tfrac{1}{L}\big(\hat{\lambda}X_t(1-X_t) + X_t\big)}dB_t$$

5. Graphs - definition

Definition 5.1

A **graph** (or **network**) $G = (V, E)$ consists of a finite set $V = \{1, \ldots, N\}$ of **vertices** (or **nodes**, **points**), and a set $E \subseteq V \times V$ of **edges** (or **links**, **lines**).
The graph is called **undirected** if $(i, j) \in E$ implies $(j, i) \in E$, otherwise **directed**.
The structure of the graph is encoded in the **adjacency** (or **connectivity**) **matrix**

$$A = (a_{ij} : i, j \in V) \quad \text{where} \quad a_{ij} = \begin{cases} 1\,, & (i,j) \in E \\ 0\,, & (i,j) \notin E \end{cases} .$$

We denote the number of edges by $K = |E|$ for directed, or $K = |E|/2$ for undirected graphs.

- Graphs we consider do not have self edges, i.e. $(i, i) \notin E$ for all $i \in V$, or multiple edges, since edges (i, j) are unique elements of E.
- **Weighted graphs** with edge weights $w_{ij} \in \mathbb{R}$ can be used to represent continuous- or discrete-time Markov chains.
- In general graphs can also be infinite, but we will focus on finite graphs. Many of the following graph characteristics only make sense in the finite case.

5. Graphs - paths and connectivity

Definition 5.2

A **path** γ_{ij} of length $l = |\gamma_{ij}|$ from vertex i to j is sequence of vertices

$$\gamma_{ij} = (v_1 = i, v_2, \ldots, v_{l+1} = j) \quad \text{with} \quad (v_k, v_{k+1}) \in E \text{ for all } k = 1, \ldots, l\,,$$

and $v_k \neq v_{k'}$ for all $k \neq k' \in \{1, \ldots, l\}$ (i.e. each vertex is visited only once).
If such a path exists, we say that vertex i is **connected** to j (write $i \to j$).
Shortest paths between vertices i, j are called **geodesics** (not necessarily unique) and their length d_{ij} is called the **distance** from i to j. If $i \not\to j$ we set $d_{ij} = \infty$.
A graph is **connected** if $d_{ij} < \infty$ for all $i, j \in V$.
The **diameter** and the **characteristic path length** of the graph G are given by

$$\mathrm{diam}(G) := \max\{d_{ij} : i, j \in V\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$
$$L = L(G) := \frac{1}{N(N-1)} \sum_{i,j \in V} d_{ij} \in [0, \infty]\,.$$

For undirected graphs we have $d_{ij} = d_{ji}$ which is finite if $i \leftrightarrow j$, and they can be decomposed into **connected components**, where we write

$$C_i = \{j \in V : j \leftrightarrow i\} \quad \text{for the component containing vertex } i\,.$$

5. Graphs - degrees

Definition 5.3

The **in-** and **out-degree** of a node $i \in V$ is defined as

$$k_i^{\text{in}} = \sum_{j \in V} a_{ji} \quad \text{and} \quad k_i^{\text{out}} = \sum_{j \in V} a_{ij} \ .$$

$k_1^{\text{in}}, \ldots k_N^{\text{in}}$ is called the **in-degree sequence** and the **in-degree distribution** is

$$\big(p^{\text{in}}(k) : k \in \{0, \ldots, K\}\big) \quad \text{with} \quad p^{\text{in}}(k) = \tfrac{1}{N} \sum_{i \in V} \delta_{k, k_i^{\text{in}}}$$

giving the fraction of vertices with in-degree k. The same holds for out-degrees, and in undirected networks we simply write $k_i = k_i^{\text{in}} = k_i^{\text{out}}$ and $p(k)$.

- Note that $\sum\limits_{i \in V} k_i = \sum\limits_{i,j \in V} a_{ij} = |E|$ (also for directed), average and variance are

$$\langle k \rangle = \tfrac{1}{N} \sum_{i \in V} k_i = |E|/N = \sum_k k p(k) \ , \quad \sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 \ .$$

- In a **regular graph** (usually undirected) all vertices have equal degree $k_i \equiv k$.
- Graphs where the degree distribution shows a power law decay, i.e. $p(k) \propto k^{-\alpha}$ for large k, are often called **scale-free**.
  Real-world networks are often scale-free with exponent around $\alpha \approx 3$.

5. Graphs - first examples

Example 2 (Some graphs)

The **complete graph** K_N with N vertices is an undirected graph where all $N(N-1)/2$ vertices $E = \big((i,j) : i \neq j \in V\big)$ are present.
Regular lattices $\mathbb{Z}^d$ with edges between nearest neighbours are examples of regular graphs with degree $k = 2d$.

Definition 5.4

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Vertices with degree 1 are called **leaves**.
In a **rooted tree** one vertex $i \in V$ is the designated **root**, and the graph can be directed, where all vertices point towards or away from the root.

A **cycle** is a closed path γ_{ii} of length $|\gamma_{ii}| > 2$. G is a tree if and only if

- it is connected and has no cycles;
- it is connected but is not connected if a single edge is removed;
- it has no cycles but a cycle is formed if any edge is added.

5. Graphs - degree correlations

Definition 5.5

For undirected graphs, the **joint degree distribution** of nodes linked by an edge is

$$q(k,k') = \frac{1}{|E|}\sum_{(i,j)\in E}\delta_{k_i,k}\delta_{k_j,k'} = \frac{\sum_{i,j\in V} a_{ij}\delta_{k_i,k}\delta_{k_j,k'}}{\sum_{i,j\in V} a_{ij}} = q(k',k)\ .$$

With the marginal $q(k') = \sum_k q(k,k')$ we have the **conditional degree distribution**

$$q(k|k') = q(k,k')/q(k') \quad \text{with average} \quad k_{nn}(k') := \sum_k kq(k|k')\ .$$

The network is called **uncorrelated** if $k_{nn}(k')$ is independent of k', **assortative** if $k_{nn}(k') \nearrow$ in k' and **disassortative** if $k_{nn}(k') \searrow$ in k'.

- The marginal $q(k)$ corresponds to **edge biased degree sampling**, i.e.

$$q(k) = \sum_{k'} q(k,k') = \frac{1}{|E|}\sum_{i,j\in V} a_{ij}\delta_{k_i,k} = \frac{N}{|E|}\frac{1}{N}\sum_{i\in V} k_i\delta_{k_i,k} = \frac{kp(k)}{\langle k\rangle}\ .$$

For uncorrelated networks $q(k|k') = q(k)$ and thus $\quad k_{nn}(k') = \langle k^2\rangle/\langle k\rangle\ .$

5. Subgraphs

- The degree of correlation can be quantified by the **correlation coefficient**

$$\chi := \frac{\langle kk' \rangle_q - \langle k \rangle_q^2}{\langle k^2 \rangle_q - \langle k \rangle_q^2} = \frac{\sum_{k,k'} kk' \big(q(k,k') - q(k)q(k') \big)}{\sum_k k^2 q(k) - (\sum_k kq(k))^2} \in [-1, 1] \,.$$

Definition 5.6

A **subgraph** $G' = (V', E')$ of $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

- Small connected subgraphs are also called **motifs**, the simplest non-trivial examples in undirected graphs are connected triples and triangles.
- Fully connected (complete) subgraphs which are maximal with respect to connectedness are called **cliques**.
- A **spanning tree** is a tree subgraph that contains all vertices of the graph.
- A subgraph G' is called a **community**, if (for example)

$$\sum_{i,j \in V'} a_{ij} > \sum_{i \in V', j \notin V'} a_{ij} \quad \text{(there are also other definitions)} \,.$$

5. Clustering

Clustering aims to quantify the probability that two neighbours of a given vertex are themselves neighbours. Two different definitions are used in the literature.

Definition 5.7

The **global clustering coefficient** for an undirected graph is defined as

$$C = \frac{3 \times \text{\# of (connected) triangles}}{\text{\# of (connected) triples}} = \frac{3\sum_{i<j<l} a_{ij}a_{jl}a_{li}}{\sum_{i<j<l}(a_{ij}a_{il} + a_{ji}a_{jl} + a_{li}a_{lj})} \in [0,1] \,.$$

Alternatively, one can define a **local clustering coefficient**

$$C_i = \frac{\text{\# of triangles containing vertex } i}{\text{\# of triples centered on vertex } i} = \frac{\sum_{j<l} a_{ij}a_{jl}a_{li}}{\sum_{j<l} a_{ij}a_{il}} \in [0,1] \,,$$

and use the average $\langle C_i \rangle = \frac{1}{N}\sum_i C_i$ to quantify clustering.

- For a tree we have $C = \langle C_i \rangle = 0$ and for the complete graph $C = \langle C_i \rangle = 1$.
- Higher-order clustering coefficients can be defined similarly, using different subgraphs as basis.

5. Graph spectra

Definition 5.8

The **spectral density** of a graph $G = (V, E)$ is

$$\rho(\lambda) := \frac{1}{N} \sum_{i \in V} \delta(\lambda - \lambda_i) \quad \text{where} \quad \lambda_1, \ldots, \lambda_N \in \mathbb{C}$$

are the eigenvalues of the adjacency matrix A.

- **Perron-Frobenius:** A has a real eigenvalue $\lambda_1 > 0$ with maximal modulus and real, non-negative eigenvector(s). If the graph is connected, it has multiplicity 1 and $|\lambda_j| < \lambda_1$ for all other eigenvalues with $j \neq 1$.
- For undirected graphs, $(A^n)_{ij}$ is equal to the **number of walks** (paths which allow repeated vertices) from i to j of length n. We also have

  $$\mathrm{Tr}(A^n) = \sum_{i=1}^N \lambda_i^n \quad \text{and} \quad \big(\mathrm{Tr}(A)\big)^n = 0 \, ,$$

  which can be used to derive statements like:

  $$\sum_{i<j} \lambda_i \lambda_j = -|E| \, , \quad \sum_{i<j<l} \lambda_i \lambda_j \lambda_l = 2 \cdot \# \text{ of triangles in } G \, .$$

5. Graph Laplacian

Definition 5.9

The **Graph Laplacian** for a graph (V, E) with adjacency matrix A is defined as

$$Q := A - D \quad \text{where} \quad D = \left(\delta_{ij} \sum_{l \neq i} a_{il} : i, j \in V\right).$$

- Q has eigenvalues in $\mathbb{C}$ with real part $\mathrm{Re}(\lambda) < 0$ except for $\lambda_1 = 0$, which follows directly from the Gershgorin theorem and vanishing row sums.
  The **multiplicity of λ_1 equals the number of connected components** in undirected graphs. Properly chosen orthogonal eigenvectors to λ_1 have non-zero entries on the individual connected components.
  The smaller the second largest real part of an eigenvalue, the harder it is to cut G into separated components by removing edges.
- Q defines a generator matrix of a continuous-time random walk on V with transition rates a_{ij}. Using weighted graphs, any finite state CTMC can be represented in this way.
- The Laplacian determines the first order linearized dynamics of many complex processes on graphs and is therefore of particular importance.

5. The Wigner semi-circle law

Theorem 5.1 (Wigner semi-circle law)

Let $A = (a_{ij} : i,j = 1,\dots,N)$ be a real, symmetric matrix with iid entries a_{ij} for $i \le j$ with finite moments, and $\mathbb{E}[a_{ij}] = 0$, $\mathrm{var}[a_{ij}] = \sigma^2$ (called a **Wigner matrix**). Then the spectral density ρ_N of the matrix $A/\sqrt{N}$ converges in distribution to

$$\rho_N(\lambda) \to \rho_{sc}(\lambda) := \begin{cases} (2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - \lambda^2} & , \text{ if } |\lambda| < 2\sigma \\ 0 & , \text{ otherwise} \end{cases} .$$

- The bulk of eigenvalues of unscaled Wigner matrices typically lies in the interval $[-2\sqrt{N}\sigma, 2\sqrt{N}\sigma]$.
- Adjacency matrices A of $\mathcal{G}_{N,p}$ **random graphs** are symmetric with iid $\mathrm{Be}(p)$ entries with $\mathbb{E}[a_{ij}] = p$ and $\mathrm{var}[a_{ij}] = p(1-p)$, so are not Wigner matrices.
  A has a maximal **Perron-Frobenius eigenvalue** of order pN, but all other eigenvalues have modulus of order $\sqrt{N}$.
  For fixed $p > 0$ or scaled $p = p_N \gg p_c = 1/N$ the Wigner semi-cirlce law holds for $N \to \infty$ with support width $4\sqrt{N}\sigma_N$.
  For $p = p_N \ll p_c = 1/N$ the asymptotic spectral density deviates from ρ_{sc}.
- There is a related **circular law** for **non-symmetric Wigner matrices**.

5. More general graphs and networks

- For **multigraphs**, multiple edges between nodes and loops ($a_{ii} > 0$) are allowed.
- **Hypergraphs** (V, E) are generalizations in which an edge can connect any number of vertices. Formally, the set of **hyperedges** $E \subseteq \mathcal{P}(V)$ is a set of non-empty subsets of V.
- In **bipartite graphs** the edge set can be partitioned into two sets $V_1, V_2 \subseteq V$ each non-empty, with no connections within themselves, i.e. $a_{ij} = a_{ji} = 0$ for all $i, j \in V_1$ and also for all $i, j \in V_2$.
  Simple undirected examples include regular lattices $\mathbb{Z}^d$ for $d \geq 1$ which are partitioned into sites with even and odd parity. Feed-forward neural networks are examples of directed graphs with bipartite or multi-partite structure.
- **Multilayer networks** $M = (\mathbf{G}, \mathbf{C})$ consist of a family of m (weighted or unweighted) graphs $G_\alpha = (V_\alpha, E_\alpha)$ (called **layers** of M), and the set of interconnections between nodes of different layers

$$\mathbf{C} = \left\{ c_{\alpha,\beta} \subseteq V_\alpha \times V_\beta : \alpha, \beta \in \{1, \ldots, m\},\ \alpha \neq \beta \right\} .$$

  Real examples include transportation networks or social networks with different types of connections.

6. E-R Random graphs

Definition 6.1

An **(Erdős-Rényi, short E-R) random graph** $G \sim \mathcal{G}_{N,K}$ has uniform distribution on the set of all undirected graphs with N vertices and $K = |E|/2$ edges, i.e.

$$\mathbb{P}_{N,k}[G = (V,E)] = 1 \Big/ \binom{N(N-1)/2}{K} .$$

An **(E-R) random graph** $G \sim \mathcal{G}_{N,p}$ has N vertices and each (undirected) edge is present independently with probability $p \in [0,1]$, i.e.

$$\mathbb{P}_{N,p}[G = (V,E)] = p^{|E|/2}(1-p)^{N(N-1)/2-|E|/2} .$$

- The ensemble $\mathcal{G}_{N,p}$ is easier to work with and is mostly used in practice, and for N, K large, $\mathcal{G}_{N,K}$ is largely equivalent to $\mathcal{G}_{N,p}$ with $p = 2K/(N(N-1))$.
- Since edges are present independently, graphs $G \in \mathcal{G}_{N,p}$ should typically be uncorrelated. Indeed, one can show that $\chi(G)$, $\mathbb{E}[\chi] \to 0$ as $N \to \infty$.

6. E-R Random graphs - properties

- The number of undirected edges for $G \sim \mathcal{G}_{N,p}$ is **random**, $K \sim \mathrm{Bi}\big(\frac{N(N-1)}{2}, p\big)$.
  For all i by homogeneity, $k_i \sim \mathrm{Bi}(N-1, p)$ and $\mathbb{E}[\langle k\rangle] = \mathbb{E}[k_i] = (N-1)p$.
- The expected number of triangles in a $\mathcal{G}_{N,p}$ graph is $\binom{N}{3}p^3$,
  and the number of triples is $\binom{N}{3}3p^2$.
  Since fluctuations are of lower order, this implies for all $G_N \sim \mathcal{G}_{N,p}$

$$C(G_N) = \frac{3\binom{N}{3}p^3(1+o(1))}{\binom{N}{3}3p^2(1+o(1))} \to p \quad \text{as } N \to \infty \ .$$

- The **expected** degree distribution for $G_N \sim \mathcal{G}_{N,p}$ is $\mathrm{Bi}(N-1, p)$. In the limit $N \to \infty$ with $p = p_N = z/(N-1)$ keeping $z = \mathbb{E}[\langle k\rangle]$ fixed we have

$$\mathbb{E}[p(k)] = \mathbb{P}[k_i = k] = \binom{N-1}{k} p_N^k (1-p_N)^{N-1-k} \to \frac{z^k}{k!} e^{-z} \ .$$

  Therefore, E-R $\mathcal{G}_{N,p}$ graphs are sometimes called **Poisson random graphs**.
- In this scaling limit E-R graphs are **locally tree-like**, i.e. connected components

$$C_i^n := \{j \in V : j \leftrightarrow i, d_{ij} \leq n\} \ , \quad n \text{ fixed}$$

  are tree subgraphs as $N \to \infty$ with probability 1.
  Vertex degrees are $k_i \sim \mathrm{Poi}(z)$ and iid $k_j \sim \mathrm{Poi}(z) + 1$.

6. Percolation

Percolation studies robustness of connectivity properties of graphs under deletion of edges or vertices (e.g. random attacks or immunization).

Definition 6.2

Consider a connected, undirected graph $G = (V, E)$. **Bond percolation** is a static probabilistic model with state space

$$S = \Omega = \{0,1\}^E = \{e_{ij} \in \{0,1\} : (i,j) \in E\} ,$$

and distribution $p = \mathbb{P}[e_{ij} = 1] = 1 - \mathbb{P}[e_{ij} = 0]$, i.e. $e_{ij} \sim \mathrm{Be}(p)$ iid with $p \in [0,1]$. Edges $(i,j) \in E$ are called **open** if $e_{ij} = 1$ and **closed** if $e_{ij} = 0$, and we denote by

$$G^o = (V, E^o) \quad \text{with} \quad E^o = \{(i,j) \in E : e_{ij} = 1\} \subseteq E$$

the (random) subgraph containing only open edges. A sequence of connected graphs G_N of increasing size $|V_N| = N$ exhibits **percolation with parameter** p if

$$|\bar{C}_N^o|/N \geq c > 0 \quad \text{as } N \to \infty \quad \text{with probability } 1 ,$$

where $|\bar{C}_N^o| = \max_{i \in V_N} |C_i^o|$ is the size of the largest connected component $\bar{C}_N^o$ of G_N^o.

6. Percolation and E-R graphs

- Alternatively, percolation can be defined on an infinite graph G (e.g. $\mathbb{Z}^d$) with

$$\textbf{percolation probability} \quad \theta(p) = \mathbb{P}[|C_0| = \infty] \begin{cases} = 0\,, & \text{for } p < p_c \\ > 0\,, & \text{for } p > p_c \end{cases} ,$$

  changing behaviour at a **critical value** $p_c \in [0,1]$.
- In **site percolation** vertices and their adjacent edges are deleted.
- E-R random graphs $\mathcal{G}_{N,p}$ have the same distribution as open subgraphs $(G^o, E^o) \subseteq K_N$ under **percolation on the complete graph** K_N with parameter p.

Theorem 6.1 (Giant component for E-R graphs)

Consider $G_{N,p} \sim \mathcal{G}_{N,p}$ with $p = z/N$ and **maximal connected component** $\bar{C}_{N,p}$. Then

$$|\bar{C}_{N,p}| = \begin{cases} O(\log N)\,, & \text{for } z < 1 \\ O(N^{2/3})\,, & \text{for } z = 1 \\ O(N)\,, & \text{for } z > 1 \end{cases} \quad \text{and} \quad \theta(z) = \begin{cases} 0\,, & \text{for } z \le 1 \\ > 0\,, & \text{for } z > 1 \\ \to 1\,, & \text{for } z \to \infty \end{cases}$$

where $\theta(z) := \lim_{N\to\infty} |\bar{C}_{N,p}|/N$ is a continuous, monotone increasing function of z. For $z > 1$, $\bar{C}_{N,p}$ is the only **giant component**, and the second largest is $O(\log N)$.

Local trees with $1 + \mathrm{Poi}(z)$ degrees die out with probability 1 if and only if $z \le 1$.

6. Preferential attachment

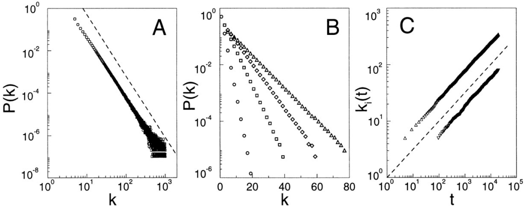
The prevalence of power-law degree distributions in real complex networks can be attributed to growth mechanisms subject to **preferential attachment**.

Definition 6.3

Starting with a complete graph (V_0, E_0) of $|V_0| = m_0$ nodes, at each time step $t = 1, \ldots, N - m_0$ a new node $j = t + m_0$ is added. It forms $m \leq m_0$ undirected edges with existing nodes $i \in V_{t-1}$ with a probability proportional to their degree $\pi_{j \leftrightarrow i} = k_i / \sum_{l \in V_t} k_l$ (preferential attachment).
The resulting, undirected graph with N nodes and $K = m_0(m_0 - 1)/2 + m(N - m_0)$ is called a **Barabási-Albert** random graph, denoted by $\mathcal{G}^{\mathrm{BA}}_{N,K}$.

- As $N \to \infty$, the average degree is $\langle k \rangle = 2m$ and the degree distribution $p_N(k)$ converges to a distribution $p(k)$ with **power law tail**, i.e. $p(k) = Ck^{-\alpha}$ for large k where $\alpha = \mathbf{3}$, which is close to exponents observed for real-world networks. This is independent of the parameters m_0 and m.
- Characteristic path length and clustering coefficient typically behave like $L = O(\log N)$ and $C = O(N^{-0.75})$ for $\mathcal{G}^{\mathrm{BA}}_{N,K}$ graphs, and they are **uncorrelated**.
- They are **not homogeneous**, the expected degree of nodes increases with age.

6. Preferential attachment



(A) power law for $\gamma = 1$, $m_0 = m = 5$, $N = 200K$, (B) exponential tail for $\gamma = 0$, $m_0 = m = 1,\ 3,\ 5,\ 7$,

(C) degree increasing with time for $t_1 = 5,\ t_2 = 95$

taken from [A.-L. Barabási, R. Albert, Science **286**(5439), 509-512 (1999)]

- Variations of the model connecting to vertices i with probability proportional to $k_i + k_0$ lead to power law degree distributions with $\alpha = 3 + k_0/m$.

- For **non-linear preferentail attachment** proportional to k_i^γ we get

  $\gamma \in [0, 1)$: $p(k)$ has a **stretched exponential tail** $\exp(-Ck^{1-\gamma})$ and the graph is **assortative**

  $\gamma > 1$: all vertices connect to m **super vertices** and the graph is **disassortative**

6. Small-world networks

Definition 6.4

A sequence of connected graphs G_N with increasing size $|V_N| = N$ exhibits the **small-world property**, if the characteristic path length $L(G_N) = O(\log N)$.

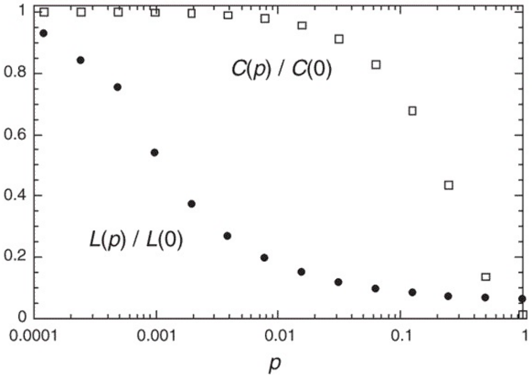
Examples include trees with degrees $k_i \geq 3$ and also the giant or largest component in E-R random graphs. In most graph models small-worldness is paired with low clustering coefficients, e.g. 0 for trees and p for $\mathcal{G}_{N,p}$ graphs. However, many real examples of small world networks exhibit also **large clustering coefficients**, such as networks of social contacts.

Definition 6.5

Consider a $2m$-regular ring graph with adjacency matrix $a_{ij} = \begin{cases} 1\,, & |i-j| \leq m \\ 0\,, & \text{otherwise} \end{cases}$
of size N with a total number of $K = mN$ undirected edges.
For all i, each edge (i,j) with a clockwise neighbour with $j > i$ is **rewired** with probability $p \in [0,1]$, i.e. replaced by an edge (i,l) where l is chosen uniformly among vertices not adjacent to i. The resulting graph is a **Watts-Strogatz random graph**, denoted by $\mathcal{G}^{\mathrm{WS}}_{N,K}$.

6. Watts-Strogatz model

- W-S random graphs interpolate between a regular lattice for $p = 0$ and a $\mathcal{G}_{N,K}$ E-R random graph conditioned on the event that all vertices have degree $k_i \geq m$.
- Expected clustering coefficient $\mathbb{E}[C(p)]$ and characteristic path length $\mathbb{E}[L(p)]$ are monotone decreasing functions of p and show the following behaviour.



$N = 1000$ and $m = 5$, taken from [D.J. Watts, S.H. Strogatz, Nature **393**, 440-442 (1998)]

6. Configuration model

Definition 6.6

The **configuration model** $\mathcal{G}^{\text{conf}}_{N,D}$ is defined as the uniform distribution among all undirected graphs with N vertices with a given degree sequence $D = (k_1, \ldots, k_N)$, such that $\sum_{i \in V} k_i = 2K$.

- Not all sequences D that sum to an even number are **graphical**.
- Sampling is usually done by attaching k_i half edges to each vertex i and matching them randomly. This can lead to self loops and rejections.
- General **randomized graphs** with given degree distribution $p(k)$ can be sampled in the same way. If $k_{\max} = \max_i k_i$ is bounded, one can show that these graphs exhibit a giant (connected) component of size $O(N)$ if

$$Q := \sum_{k \geq 0} k(k-2)p(k) > 0 \; ,$$

  and if $Q < 0$ the largest component is of size $O(k_{\max}^2 \log N)$.
- For directed versions with D^{in} and D^{out} we need $\sum_{i \in V} k_i^{\text{in}} = \sum_{i \in V} k_i^{\text{out}}$.

Definition 6.7

A **planar graph** is an undirected graph that can be embedded in the plane, i.e. it can be drawn in such a way that no edges cross each other. The edges of a particular embedding partition the plane into **faces**. A connected planar graph G has a **dual graph** G^*, which has one vertex in each face of G, and a unique edge crossing each edge of G. G^* may be a multigraph with self-loops.
A **maximal planar graph** is called a **triangulation**.

- Every planar graph is 4-partite or 4-colourable.
- In a triangulation each face is bounded by three edges. By induction, every triangulation with $N>2$ nodes has $K=3N-6$ undirected edges and $2N-4$ faces.

Definition 6.8

A random countable set $\Pi \subseteq \mathbb{R}^d$ is called a **spatial point process**.
$\Pi \subseteq \mathbb{R}^d$ is called a homogeneous **Poisson point process** $\mathrm{PPP}(\lambda)$ with rate $\lambda > 0$ if

- for all $A \subseteq \mathbb{R}^d$ we have $N(A) := |\Pi \cap A| \sim \mathrm{Poi}(\lambda|A|)$,
- for all disjoint $A_1, \ldots, A_n \subseteq \mathbb{R}^d$, $N(A_1), \ldots, N(A_n)$ are independent.

6. Planar graphs and spatial point processes

- To sample from a PPP(λ) e.g. in a box $A = [0, L]^d$, pick $N(A) \sim \text{Poi}(\lambda L^d)$, then place $N(A)$ particles independently in A each with uniform distribution, i.e. pick the d coordinates uniformly in $[0, L]$.
- A Poisson process PP(λ) is equivalent to a PPP(λ) on $[0, \infty)$.

Definition 6.9

Let $\Pi = \{x_1, x_2, \ldots\}$ be a countable subset of $\mathbb{R}^d$, endowed with a distance function $d(x, y)$. A **Voronoi tesselation (or diagram)** is given by the family of **Voronoi cells** $\{A_1, A_2, \ldots\} \subseteq \mathbb{R}^d$ where

$$A_i = \left\{x \in \mathbb{R}^d : d(x, x_i) \leq d(x, x_j) \text{ for all } j \neq i\right\}$$

is the set of points closest to x_i.

Properties in 2 dimensions.

- The shape of Voronoi cells depends on the distance function, for Euclidean distance $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ they are convex polygons, and boundaries between adjacent cells are straight lines.
- The dual graph of a Voronoi diagram of a set Π is called **Delaunay triangulation**, which is not unique if 4 or more cells intersect in a point.