Landau theory of Ferromagnets

Consider system which follows the minima of the free energy *F(M):*

$$F = F_0 + F_1 M + F_2 M^2 + F_3 M^3 + F_4 M^4$$

which can always be written as

$$F = F_0' + F_2'(M - M_0)^2 + F_3'(M - M_0)^3 + F_4'(M - M_0)^4$$

since both are general polynomials up to degree 4 ($M' = M - M_0$ is the reqd. transformation).

Case 1: no applied field. For symmetry $F_3 = 0$

We then have (dropping primes) $F(M) = F_0 + \alpha(T - T_c)M^2 + \beta M^4$

Extrema given by $\frac{dF}{dM} = 2\alpha(T - T_c)M + 4\beta M^3 = 2M(\alpha(T - T_c) + 2\beta M^2)$

i.e. at $M = 0$ or $M^2 = \frac{\alpha(T_c - T)}{2\beta}$

But *M* is real so $M = \pm\sqrt{\frac{\alpha(T_c - T)}{2\beta}}$ is an extremum for $T < T_c$

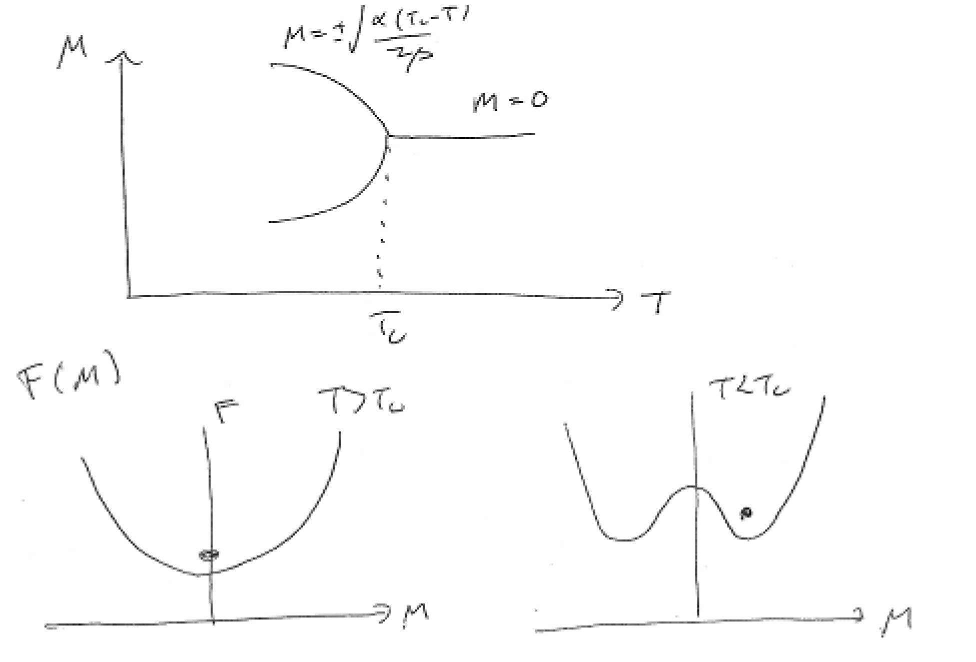
Look for minima

$$\frac{d^2F}{dM^2} = 2\alpha(T - T_c) + 12\beta M^2$$

$M = 0$: Min for $T > T_c$; Max for $T < T_c$

$M = \pm\sqrt{\frac{\alpha(T_c - T)}{2\beta}}$ $\frac{d^2F}{dM^2} = 2\alpha(T - T_c) + 12\beta.\frac{\alpha(T_c - T)}{2\beta} = -4\alpha(T - T_c)$

Min for $T < T_c$, Max for $T > T_c$
We have a pitchfork bifurcation at $T = T_c$ - see plot of $M(T)$



As we go from $T > T_c$ to $T < T_c$ the system “falls” into one of the potential wells

- which one is determined by fluctuations at $T = T_c$.

Case 2: Now consider applied B field - asymmetric now $F_3 = \gamma \neq 0$

$$\frac{dF}{dM} = 2\alpha(T - T_c)M + 3\gamma M^2 + 4\beta M^3$$

extrema now given by $\frac{dF}{dM} = 0 = M\{2\alpha(T - T_c) + 3\gamma M + 4\beta M^2\}$

that is, at $M = 0$, $M = \frac{-3\gamma \pm \sqrt{(9\gamma^2 - 4.2\alpha(T - T_c).4\beta)}}{2.4\beta}$

There are 2 real values of M when $9\gamma^2 > 32\alpha\beta(T - T_c)$

Now write M as $M = \frac{-3\gamma \pm 3\sqrt{\gamma^2 - {\gamma_c}^2}}{8\beta}$

Look for minima:

$$\frac{d^2F}{dM^2} = 2\alpha(T - T_c) + 6\gamma M + 12\beta M^2$$

$M = 0$ is a min at $T > T_c$

the $M \neq 0$ extrema is given by: $2\alpha(T - T_c) + 3\gamma M + 4\beta M^2 = 0$

which gives: $\dfrac{d^2F}{dM^2} = 3\gamma M + 8\beta M^2$, or $\dfrac{d^2F}{dM^2} = M\left(\pm 3\sqrt{\gamma^2 - \gamma_c^{\,2}}\right)$

Then in addition to the $M = 0$ extremum we have:

For $\gamma^2 > \gamma_c^{\,2}$ 2 real $M \neq 0$ roots, one max, one min

For $\gamma^2 = \gamma_c^{\,2}$ $M = \dfrac{-3\gamma}{8\beta}$ and $\gamma_c^{\,2} = \dfrac{32\alpha\beta(T - T_c)}{9}$; this is at $T > T_c$

For $\gamma^2 < \gamma_c^{\,2}$ - *M* is imaginary, no max/min

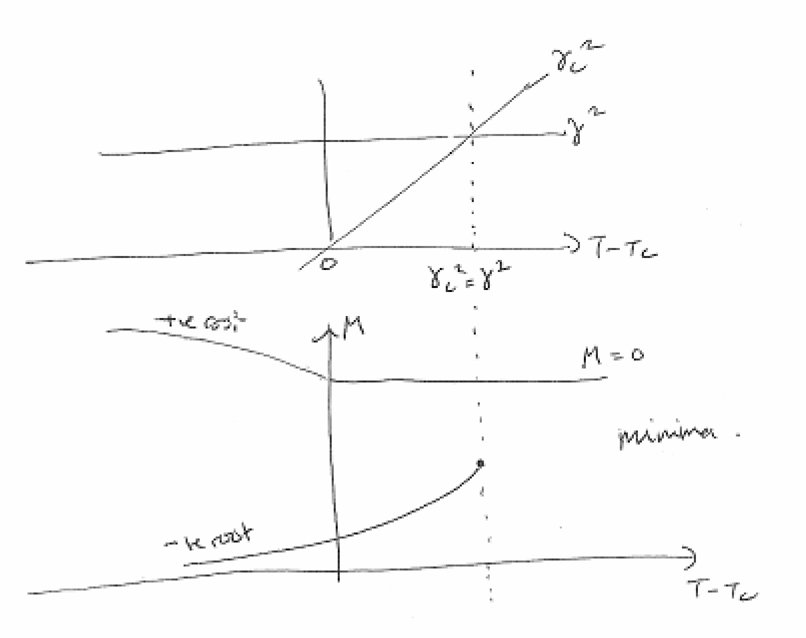
also at $\gamma_c^{\,2} = 0$ $T = T_c$ $M = \dfrac{-3\gamma \pm 3\gamma}{8\beta}$ i.e. $M = 0$ (a)

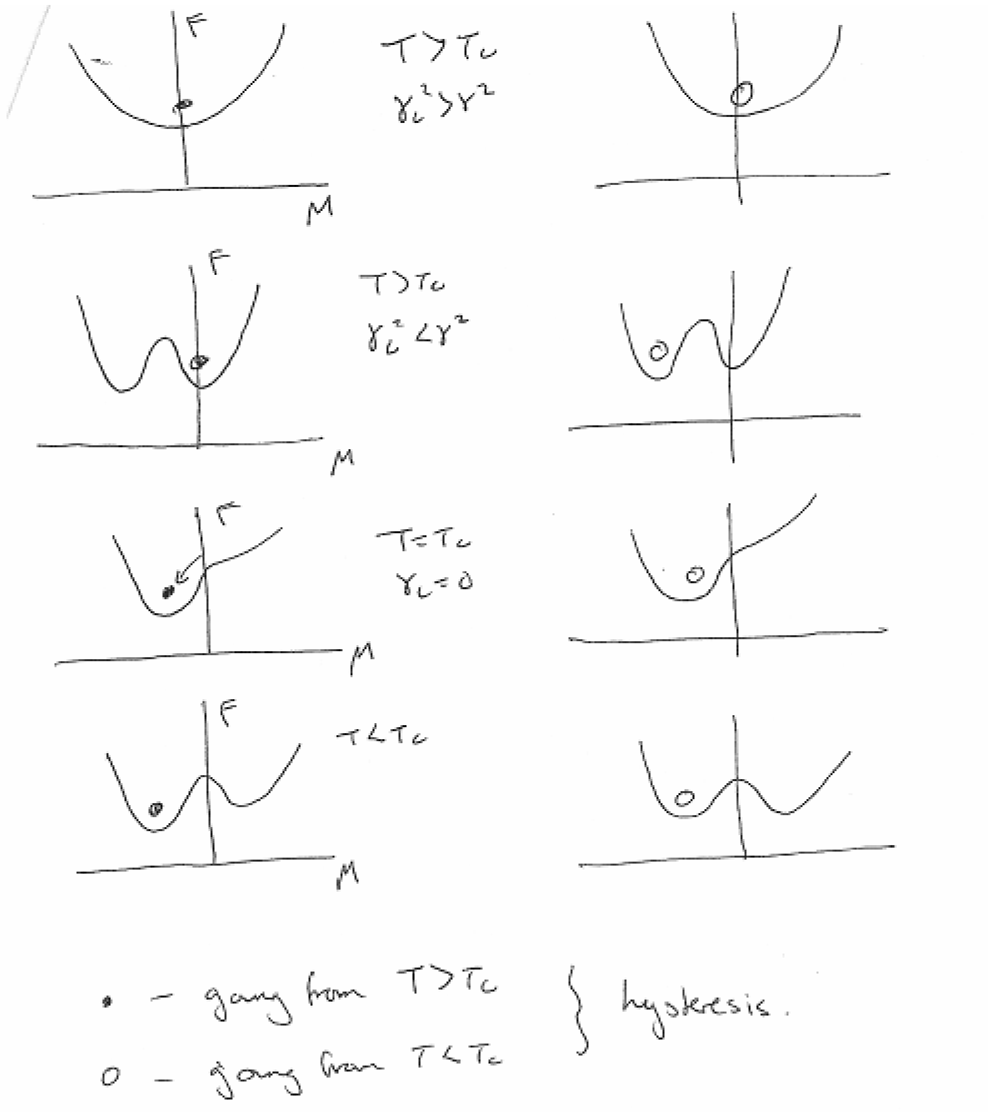
or $M = \dfrac{-6\gamma}{8\beta}$ (b)

(a) is (-)ve root hence $\dfrac{d^2F}{dM^2} > 0$ -this is a min.

(b) is an inflexion.

Finally, for $\gamma_c^{\,2} < 0$ - we have 2 real $M \neq 0$ roots, both minima, and $M = 0$ which is a maximum. Graphically:





Now fluctuations are unimportant.