Some Generalizations of Fourier Theory

see also Dudok de Wit review

Higher Order Spectra (summary)

Consider the nonlinear system:

$$\frac{\partial u(x,t)}{\partial x} = f(u(x,t))$$
$$decompose -$$
$$\frac{\partial u(x,t)}{\partial x} = \int g(\tau_1)u(x,t-\tau_1)d\tau_1$$
$$+\iint g(\tau_1,\tau_2)u(x,t-\tau_1)u(x,t-\tau_2)\,d\tau_1\,d\tau_2$$
$$+\iiint g(\tau_1,\tau_2,\tau_3)u(x,t-\tau_1)u(x,t-\tau_2)u(x,t-\tau_3)\,d\tau_1\,d\tau_2 d\tau_3$$
$$+...$$

Taking the DFT we obtain the Volterra series:

$$\frac{\partial u_p}{\partial x} = \Gamma_p u_p + \sum_{k,l}\Gamma_{kl}u_k u_l \delta_{k+l,p} + \sum_{k,l,m}\Gamma_{klm}u_k u_l u_m \delta_{k+l+m,p} + ...$$
$$with$$
$$u_p = u(x,\omega_p)$$

The leading term is a linear (cf Fourier) decomposition. The rest are mode coupling. Ensemble average over x and consider a homogeneous medium:

$$\text{setting } \frac{\partial}{\partial x} = 0$$
$$\Gamma_p \left\langle u_p^* u_p \right\rangle + \sum_{k+l=p}\Gamma_{kl}\left\langle u_k u_l u_{k+l}^* \right\rangle + \sum_{k+l+m=p}\Gamma_{klm}\left\langle u_k u_l u_m u_{k+l+m}^* \right\rangle + .. = 0$$

Now recall convolution:

$$g_k * h_k = \sum_{u=0}^{N-1} g_u h_{k-u}$$

The DFT is $G_m H_m$ where G_m is the DFT of g_k etc.,

This relates to cross correlation: $C_\tau = \sum_{k=0}^{N-1} g_k h_{k+\tau}$ DFT is $G_m^* H_m$

auto correlation: $R_\tau = \sum_{k=0}^{N-1} x_k x_{k+\tau}$ DFT is $S_m^* S_m$ (the power spectrum)

generalise these to:

bispectrum $B_{kl} = S_k S_l S^*_{k+l}$

trispectrum $T_{klm} = S_k S_l S_m S^*_{k+l+m}$

There are normalized versions, eg:

bicoherence = $\dfrac{|B_{kl}|^2}{|S_k S_l|^2 |S_{k+l}|^2} = b_{kl}$

One can obtain averaged bispectra in the same way as averaged power spectra – average over M consecutive intervals.

Alternatively, use the 2nd order autocorrelation: $M_{\tau_1\tau_2} = \sum_{k=0}^{N-1} x_k x_{k+\tau_1} x_{k+\tau_2}$

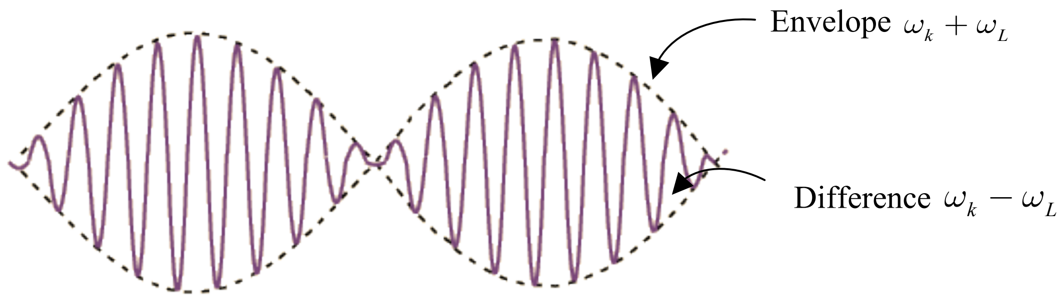
The bispectrum is the twice applied DFT on M (cf convolution theorem to prove).

Physical meaning (see also Dudok de Wit review)

Recall frequency $f_m = \dfrac{m}{N\Delta t} \equiv \omega_m$

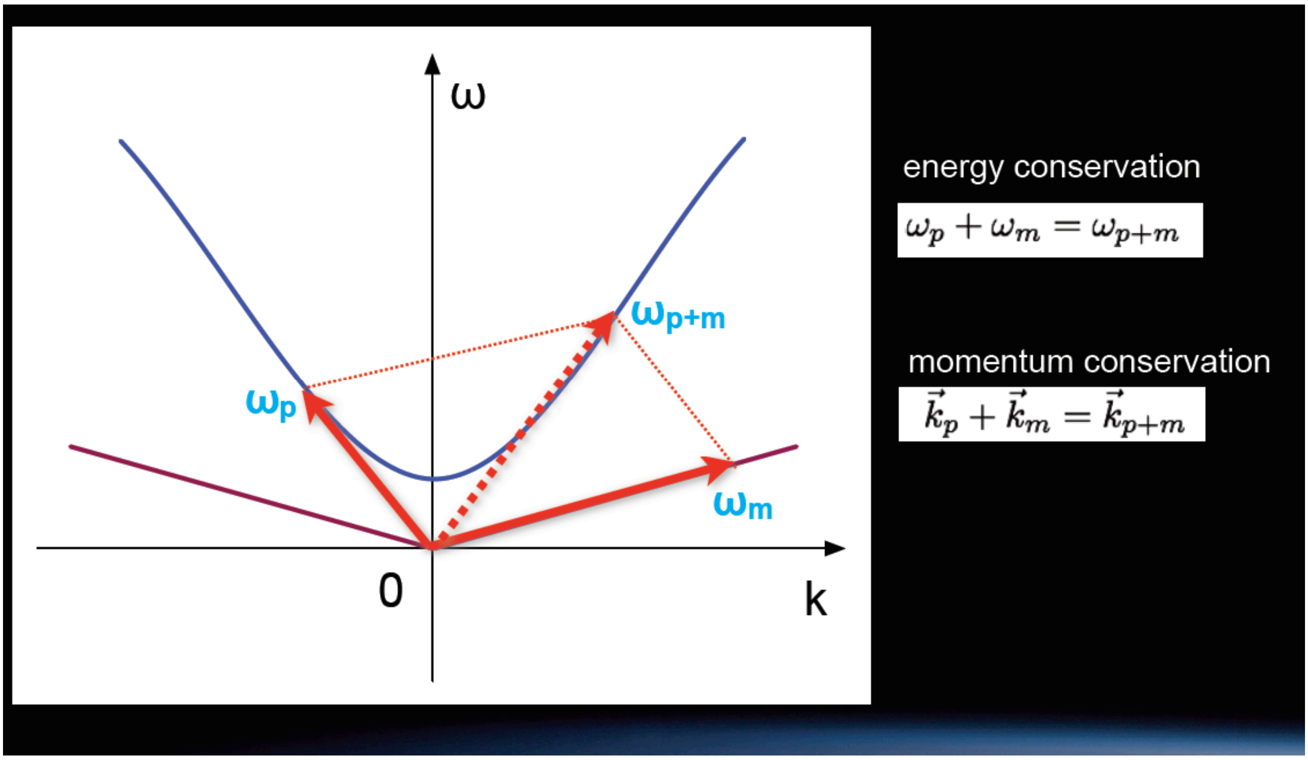
Thus : $S_k \to S(\omega_k),\ S_l \to S(\omega_l),\ S_{k+l} \to S(\omega_k + \omega_l)$

This tests for coherence between beating oscillations



where $\omega_{k,l} = \omega_k + \omega_l$, for strongest mode we see strong bicoherence.

For wave- wave coupling we need to satisfy physical constraints in both space and time:



Principal domain of bicoherence- due to symmetries:

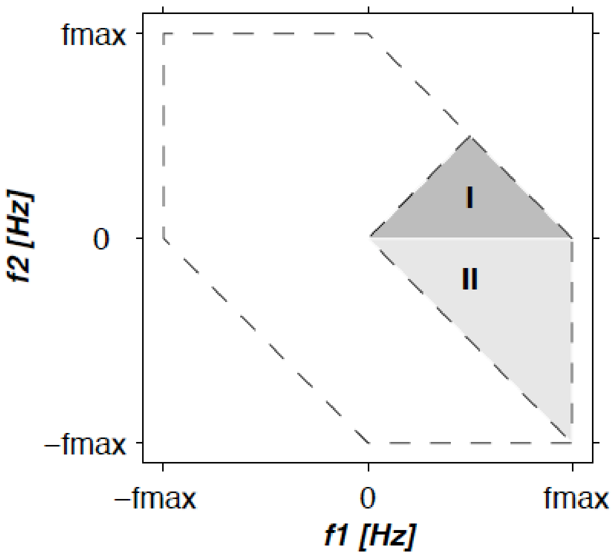
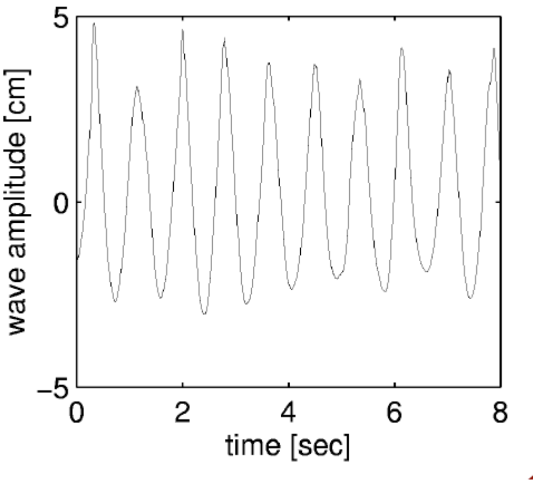
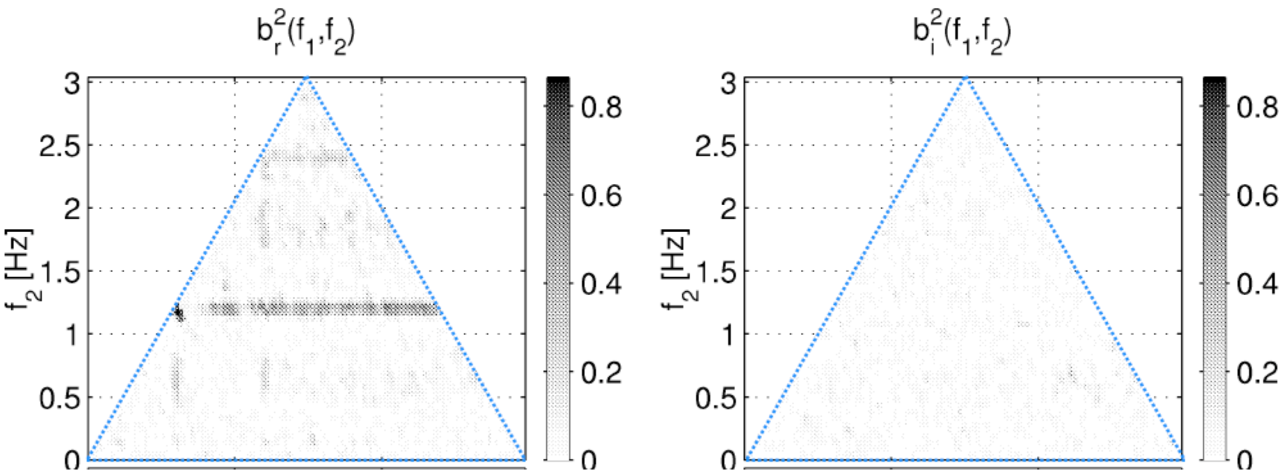


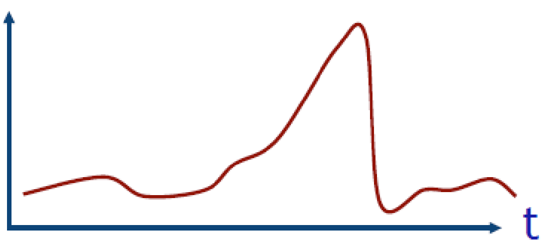
Fig. 3. Principal domain of the bicoherence. The Nyquist theorem restricts the display to the area enclosed by a dashed line. For the autobicoherence, the principal domain is I, for the cross-bicoherence it is I and II.

Example- water waves: (courtesy T. Dudok De Wit)

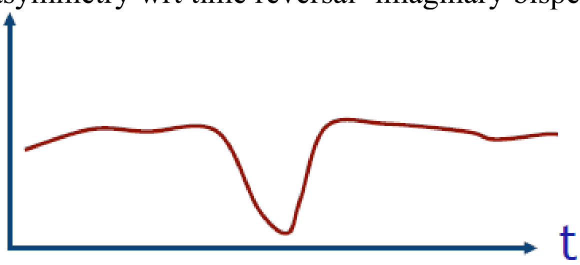


Real part and imaginary part of bicoherence





asymmetry wrt time reversal- imaginary bispectra



up- down asymmetry- real bispectra

so wave is not steepening- simply have a 'wobble' on amplitude.

Linear Time Invariant (LTI) Filters

A way to generalize the Fourier world to other transforms.

So far – everything flowed from the idea:

$$x(t) = \sum_{m=-\infty}^{\infty} S_m e^{2\pi i f_m t} \qquad f_m = \frac{m}{T}$$

with
$$S_m = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-2\pi i f_m t} dt$$

Discrete version:

$$x_k = \frac{1}{N\Delta t} \sum_{m=0}^{N-1} S_m e^{2\pi i m k/N} \qquad f_m = \frac{m}{N\Delta t}$$
$$S_m = \Delta t \sum_{k=0}^{N-1} x_k e^{-2\pi i k m/N} \qquad t_k = k\Delta t$$

and orthogonality property of the Fourier kernel $\Phi = e^{2\pi i f t}$

A framework to consider other Φ -

Write filter as: $L[x(t)] = y(t)$

where L is a linear operator, with the properties:

1. scale preserving $\qquad L[ax] = aL[x]$
2. distributative (superposition) $\qquad L[x_1 + x_2] = L[x_1] + L[x_2]$
3. time invariant $\qquad L[x(t)] = y(t)$

$$\Rightarrow \qquad L[x(t+\tau)] = y(t+\tau)$$

Then in general: $L\left[\sum_{p=1}^{N} \alpha_p x_p(t)\right] = \sum_{p=1}^{N} \alpha_p L[x_p(t)]$

Now consider for input x to the filter, the Fourier kernel:

$$\Phi_f(t) = e^{2\pi i f t} \qquad f = \text{const}$$
$$y_f = L[\Phi_f] = y_f(t)$$

then: $y_f(t+\tau) = L[\Phi_f(t+\tau)] = L[e^{2\pi i f\tau}\Phi_f(t)] = e^{2\pi i f\tau} y_f(t)$
This is just the "shift theorem" from Fourier theory.

Now let $t = 0$.

$$y_f(\tau) = e^{2\pi i f\tau} y_f(0)$$

true for any τ so let $\tau \ldots t$

$$y_f(t) = e^{2\pi i f t} y_f(0)$$

so $y_f(0)$ is a constant – there is one value for each *f*.

We can consider a <u>spectrum</u> of values of $y_f(0)$

$$y_f(0) = G_f = A_f e^{i\theta_f}$$

Then

$$y_f(t) = G_f \cdot \Phi_f(t) = L[\Phi_f(t)] = G_f e^{2\pi i f t}$$

generally:

$$\left.\begin{array}{l} \Phi_f - \text{ eigenvalues} \\ G_f - \text{ eigenvectors} \end{array}\right\} \text{of } L$$

We can think of any transform in this way.

We need to <u>identify</u> an appropriate $\Phi_f(t)$. Two methods:

1. Choose $\Phi_f(t)$ as basis on which we expand, ie: $y(t) = \sum_f y_f(t) = \sum_f G_f \Phi_f(t)$

   Φ_f may be orthogonal – chosen for "appropriate" properties.

   This is equivalent to the transform: $y(t) = \int_{-\infty}^{\infty} G(f)\Phi(f,t)\,df$

   again, $\Phi(f,t) = e^{2\pi i f t}$ for the Fourier transform.
2. Perform an SVD (single value decomposition, or principle component analysis) on *L*, so that the Φ_f are generated by the data (beyond the scope of this course).

Application of LTI filters – coloured noises

Consider some $x(t)=\sum_{m=-\infty}^{\infty} S_m e^{2\pi i f_m t}$

The discrete version of this is

$$x_k=\frac{1}{N\Delta t}\sum_{m=0}^{N-1} S_m e^{2\pi imk/N} \qquad f_m=\frac{m}{N\Delta t}$$

We now have:

$$y(t)=L\left[x(t)\right]=\sum_{m=-\infty}^{\infty} G_m S_m e^{2\pi i f_m t}$$

$$y_k=\frac{1}{N\Delta t}\sum_{m=0}^{N-1} G_m S_m e^{2\pi imk/N}.$$

Now for a stochastic process we have

$$x_k=\frac{1}{N\Delta t}\sum_{m=0}^{N-1} D_m e^{2\pi imk/N}$$

where

$D_m=S_m e^{i\phi_m}$ $\quad D_m,\phi_m$ are random *iid* processes (these are stationary)

then as above

$y_f(t)=G_f x_f(t)$

or $\quad y_{k,m}=G_m x_{k,m}$

where $x_{k,m}=S_m e^{2\pi imk/N}$

The G_f are just constants.

Then for process x_k stochastic, there will be a process y_k also stochastic, with spectral components $G_f D_f$.

The x_k, y_k should share the statistical properties .

More formally, since the R-S integral gives, for stochastic dz_x (see notes on stationarity)

$$x(t)=\int_{-1/2}^{1/2} e^{2\pi ift} dz_x(f)$$

then

$$y(t)=\int_{-1/2}^{1/2} e^{2\pi ift} G(f) dz_x(f)$$

$$=\int_{-1/2}^{1/2} e^{2\pi ift} dz_y(f)$$

ie $\quad \left\langle \left|dz_y(f)\right|^2\right\rangle = G^2(f)\left\langle \left|dz_x(f)\right|^2\right\rangle$ so this filter generates "coloured" noises if $G\sim\frac{1}{f^{\beta}}$