Power Law Power Spectra and Scaling

Consider the autocorrelation function

$$R(\tau)=\int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

and its F.T. is the power spectral density $G^2(f)$

The discrete form of the ACF is $R_\tau = \sum_{t=0}^{N-1} x_t x_{t+\tau}$.

We will discuss scaling in terms of increments (here $\langle ... \rangle$ denotes ensemble average over *t*):

$$\left\langle (x(t+\tau)-x(t))^2 \right\rangle \sim \tau^{2H}$$

H – Hurst exponent

Now $\left\langle (x(t+\tau)-x(t))^2 \right\rangle = 2\left[\left\langle x^2(t)\right\rangle - R(\tau)\right]$

where $R(\tau)=\langle x(t+\tau)(x(t))\rangle$ and we insist that $\langle x^2(t+\tau)\rangle = \langle x^2(t)\rangle$ (stationary process)

so $R(\tau)\sim\tau^{2H}$

Now relate this to the power spectrum.

Scaling argument

let $G^2(f)\sim\dfrac{1}{f^\beta}$ a 'power law' power spectrum

Now $R(\tau)=\int_{-\infty}^{\infty} G^2(f)e^{2\pi i f\tau}\, df$

Rewrite in new variables: $f\to af$ and $df = d[af]/a$ $G^2(af)=\dfrac{1}{(af)^\beta}=\dfrac{G^2(f)}{a^\beta}$

then $R(\tau)=\int_{-\infty}^{\infty} a^\beta G^2(af)e^{2\pi i af(\tau/a)}\dfrac{d(af)}{a}$

so $R(\tau)=\int_{-\infty}^{\infty} a^{\beta-1}G^2(af)e^{2\pi i af(\tau/a)}\, d(af) = a^{\beta-1}R(\tau/a)$

Thus $R(\tau)=a^{\beta-1}R(\tau/a)$ or $R(a\tau)=a^{\beta-1}R(\tau)$

so $\quad R(\tau) \sim \tau^{\beta-1}$

then $\quad G^2(f) \sim \frac{1}{f^\beta} \Rightarrow R(\tau) \sim \tau^{\beta-1}$

this is general, for stationary processes.

then $\quad 2H = \beta - 1$ - Result, relates scaling exponent to power spectral exponent β.

<u>By evaluating the integral</u>

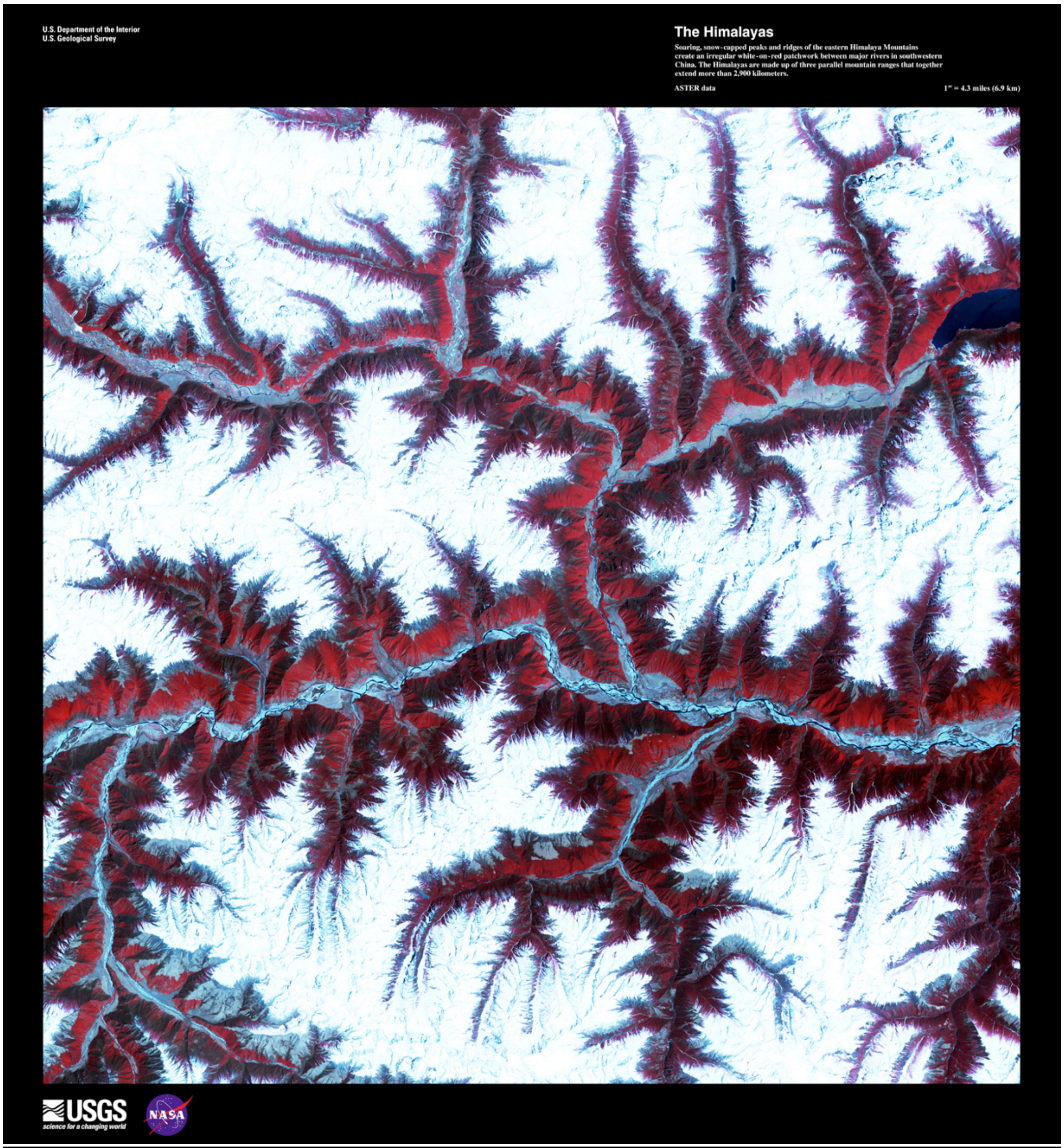
IFT is: $$\int_{-\infty}^{\infty} \frac{1}{f^\beta} e^{-2\pi i f t}\, df$$

substitute $\quad ft = x$

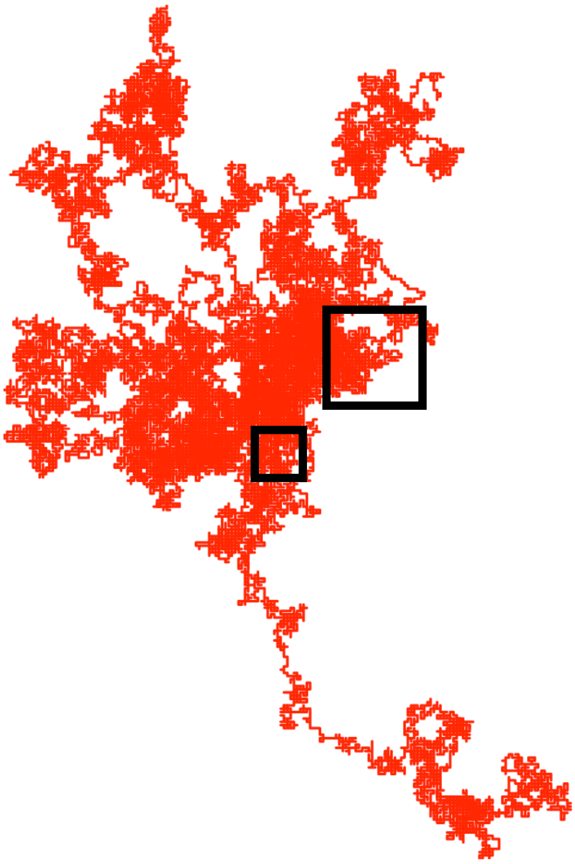
giving $$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x}}{x^\beta} \frac{dx}{t} t^\beta$$
$$= t^{\beta-1}\left(\int_{-\infty}^{+\infty} e^{-2\pi i x} \frac{dx}{x^\beta} \right) = A t^{\beta-1}$$

Again, implies $R(\tau) \sim \tau^{\beta-1}$ so that $2H = \beta - 1$.

<u>Notes on Fractal Dimension</u>



Satellite image of the Himalayas- a natural fractal (courtesy USGS).



Brownian walk on a grid in 2D

Consider $P(m,r)$ - probability of finding *m* points in square side *r*

then $$\sum_{m=1}^{N} P(m,r) = 1$$

Defns: Mass dimension D given by: $M(r) = \sum_{m=1}^{N} mP(m,r) \sim r^D$

Box counting dimension: $D_c = -\lim_{r \to 0^+} \frac{\ln(N)}{\ln(r)}$

where *N(r)* boxes of side *r* are required to just cover the surface.

However a practical definition is given by: $N_B(r) = \sum_{m=1}^{N} \frac{1}{m} P(m,r) \sim \frac{1}{r^D}$

These are all just examples of moments: $M^q(r) = \sum_{m=1}^{N} m^q P(m,r)$.

For a fractal $$P(m,r) \sim f\left(\frac{m}{r^D}\right)$$

then $$\sum_{m=1}^{N} m^q P(m,r) \sim \sum_{m=1}^{N} m^q f\left(\frac{m}{r^D}\right)$$

$$m' = \frac{m}{r^D}$$

write

$$M^q(r) \sim \sum_{m'=1}^{N} m'^q r^{qD} f\left(m'\right) \sim r^{qD}$$

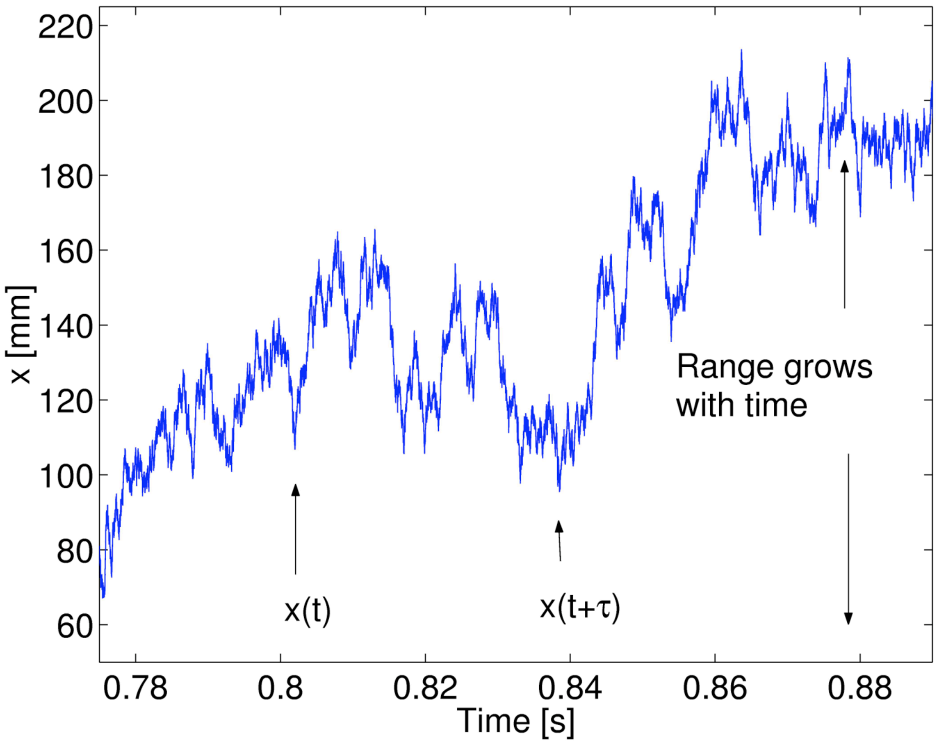
This will yield a straight line on a plot of log $\left(M^q\right)$ *vz* $\log(\,r\,)$ with exponent qD. For a fractal, a plot of exponents as a function of *q* has slope *D*.

Relationship to Hurst exponent

For *fBm* (fractional Brownian motion) – shown here in 1D

$$\left\langle \left| x\left(t+\tau\right) - x(t) \right|^2 \right\rangle \sim \tau^{2H} \qquad \text{on any } t.$$

This has a power law power spectrum as above.



How many boxes are needed to cover the line?

For an interval τ, the line on average spans $\Delta x \approx \left\langle \left| x(t+\tau) - x(t) \right|^2 \right\rangle^{\frac{1}{2}}$ and this will be covered by $\Delta x/\tau \sim \tau^{H-1}$ boxes.

Now consider the entire trace is divided in time by *N* intervals of length τ ie: $\tau \sim \dfrac{1}{N}$,

then the trace is covered by $N\dfrac{\Delta x}{\tau}$ boxes, and so

$$N(\tau) \sim N^{2-H} \sim \frac{1}{\tau^{2-H}} \sim \frac{1}{\tau^{D}}$$

so $$D = 2 - H$$

In E Euclidean dimensions $D = E + 1 - H$ and for the zero sets, $D = E - H$.