Improving spectral estimates (Fourier Transforms)

Our $1...N$ DFT is a finite sized sample therefore suffers from finite sized effects.

Some practical methods for improving spectral estimates.

i) Summed spectrogram

Sub divide x_k into intervals $1...p,\ p+1...\ 2p...$, obtain DFT of each interval then sum. Spectral accuracy at the expense of frequency resolution (NB one can consider overlapping intervals- depends on application).

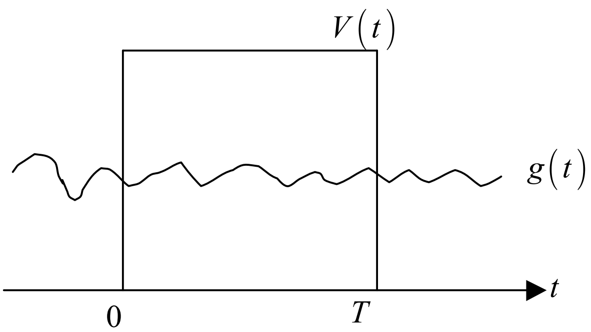
Spectral variability estimate – provided by $\langle S_m^2 \rangle$ over the interval for which there is one value of S_m for each sub-interval.

ii) Variables chosen to reflect data, eg: power law spectra – re sum over log bins in *f*.
iii) Window in time domain to reduce spectral leakage (next)
iv) Use a different transform (eg: wavelets).

Spectral estimates for a finite data interval (T)

We perform a DFT over *N* datapoints and $T = N\Delta t$. How does this affect the power spectrum?

Consider a process $g(t)$ observed from $0 \le t \le T$.



we observe
$x(t) = g(t).V(t)$

recall the convolution theorem:

$$g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau) h(t-\tau)\, d\tau$$

and

$$\int_{-\infty}^{\infty} g(t) * h(t) e^{-2\pi i f t}\, dt = G(f).H(f)$$

also holds that

$$G(f)*H(f)=\int_{-\infty}^{\infty}G(f')H(f-f')df'$$

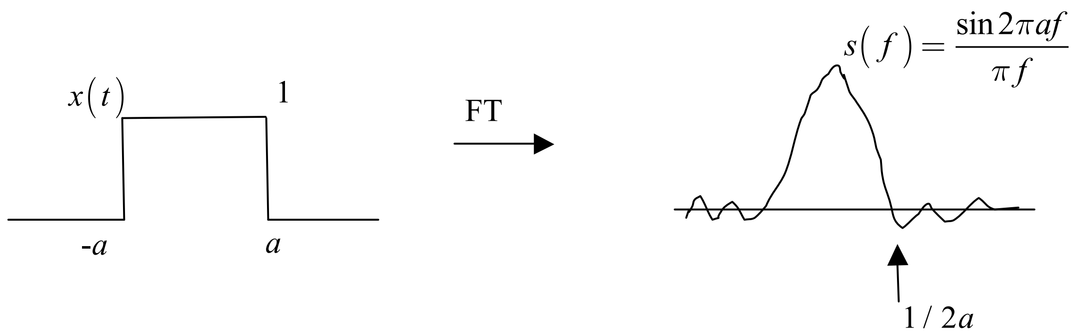
and

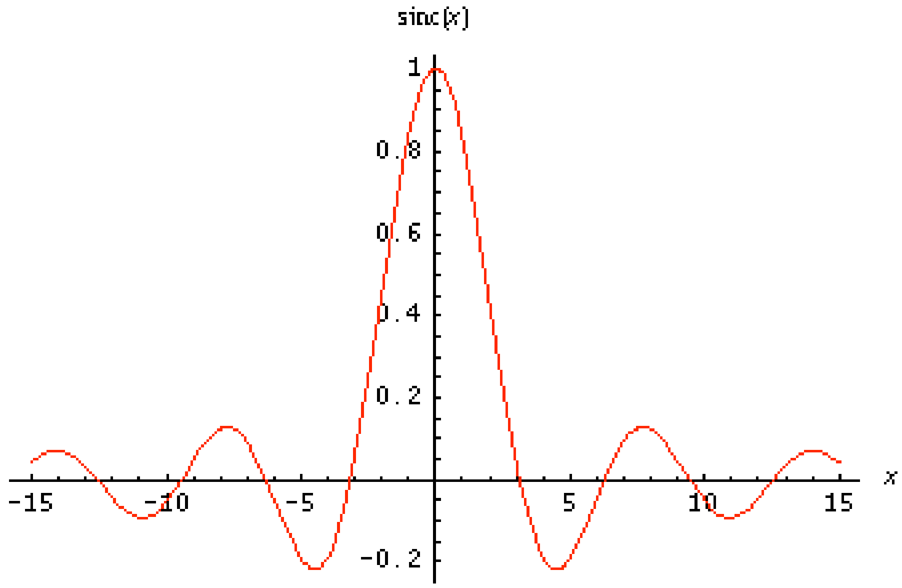
$$\int_{-\infty}^{\infty}G(f)*H(f)e^{2\pi ift}\,df = g(t).h(t)$$

Then FT of $x(t)$ sampled over interval T will be a <u>convolution – in *f* space</u> of the FT of $g(t)$ with FT of $V(t)$, this "spreads out" the local value of power spectrum.

We can see this as follows. Here $V(t)=1$ for $0 \le t \le T$ and zero elsewhere.

We have already seen that:





(reminder: a nice plot of sinc(x))

[We can always "shift" the time base here by $T/2$, answer differs by phase $e^{-i\pi fT}$ and amplitude spectrum is the same.]

Then $$V(f) = \frac{\sin(\pi f T)}{\pi f}$$

and our observed power spectrum will be

$$S(f) = V(f) * G(f)$$

where $G(f)$ is the 'original' Power spectrum if sampled over arbitrarily long *T*- spectral "leakage" over $f \simeq \frac{1}{T}$.

<u>No problem if</u>

1) $T \to \infty$ since $\text{sinc}(\pi f T) \to \delta(f)$.
2) Signal is periodic fundamental period $T_s = 2T$.

<u>Otherwise</u> what to do – reduce leakage in *f*, by applying a time domain window to make signal "look like" case (2)

There are many of these! (see Matlab, in particular signal processing toolbox).

Most common:

a) Hanning $(\cos^2)$ window

$$U_H(t) = \cos^2\left(\frac{\pi t}{T}\right) \quad -T/2 \le t \le T/2$$
$$= 0 \qquad \text{otherwise}$$

it's IFT is:

$$U_H(f) = T/2 \cdot \frac{\text{sinc}(\pi f T)}{1 - T^2 f^2}$$
$$= \frac{T}{2}\left[\text{sinc}(\pi f T) + \frac{1}{2}\text{sinc}(\pi f T - \pi) + \frac{1}{2}\text{sinc}(\pi f T + \pi)\right]$$

this is "adding" power at the zeros of $V(f)$, ie: at $f = \frac{1}{T}$ and <u>suppressing</u> at $f > \frac{1}{T}$.

We can then correct for the Hanning window as follows:

- consider a DFT – discrete $f_m = \frac{m}{N\Delta t} = \frac{m}{T}$

$$U_H(f_m) = \frac{T}{2}\left[\text{sinc}(m\pi) + \frac{1}{2}\text{sinc}((m-1)\pi) + \frac{1}{2}\text{sinc}((m+1)\pi)\right]$$

or, with $Tf_m = m$ we have:

$$U_H(f_m) = \frac{T}{2}\left[\operatorname{sinc}(\pi f_m T) + \frac{1}{2}\operatorname{sinc}(\pi f_{m-1} T) + \frac{1}{2}\operatorname{sinc}(\pi f_{m+1} T)\right]$$

If we apply this to a white noise process S

- all the $S(f_m)$ are uncorrelated
- the power spectrum is uniform

then
$$\langle S_m^2 \rangle = \left(\frac{1}{2}\right)^2 \left[1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] \cdot G_m^2$$
$$= \frac{3}{8} G_m^2$$

then the "correct" $S_m \rightarrow \sqrt{\frac{8}{3}} \times S_m$ windowed.

<u>Finally, the procedure is</u>

1) divide data into sub intervals T_0 in time domain; p of them $p = \frac{T}{T_0}$

   these can be <u>overlapped (Welch method)</u>

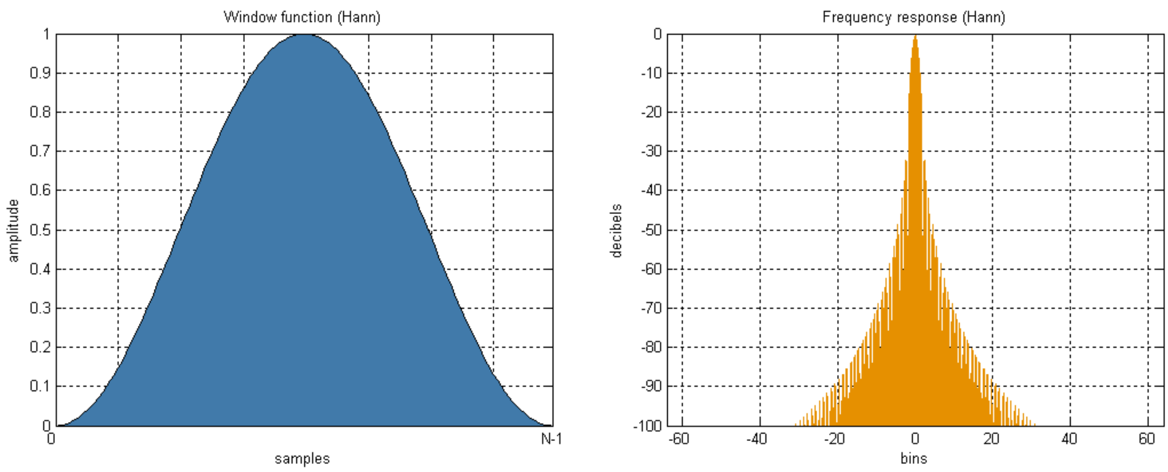
2) Hanning window over T_0.

3) correct by $\sqrt{\frac{8}{3}}$

4) average over the p spectra to obtain $\langle S_m \rangle_p$

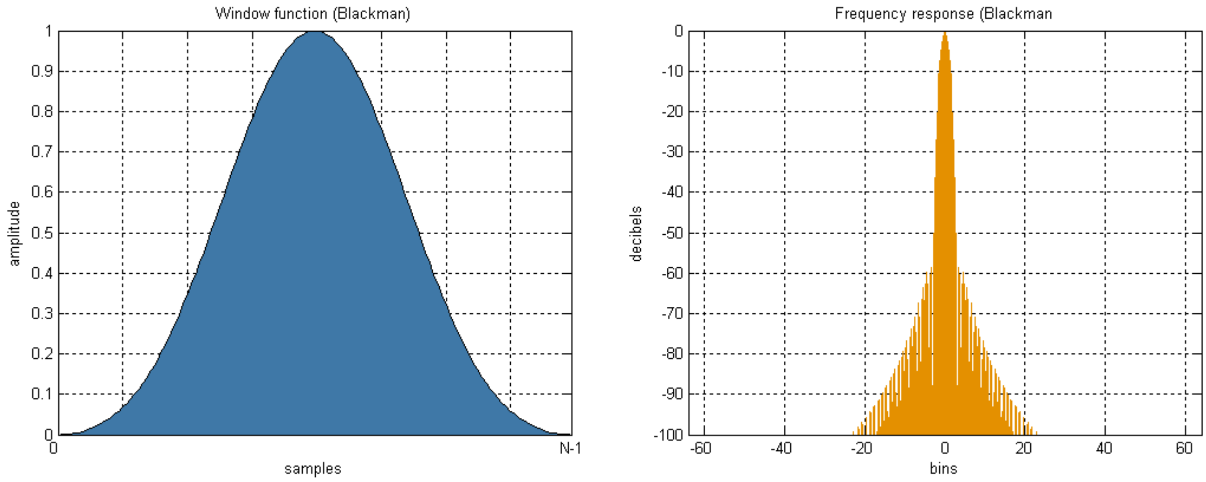
   - with uncertainty $\langle S_m^2 \rangle_p$.

<u>Other windows</u> (there is a nice list on Wikipedia-NB these are all in dB ie a logarithmic scale in power) :

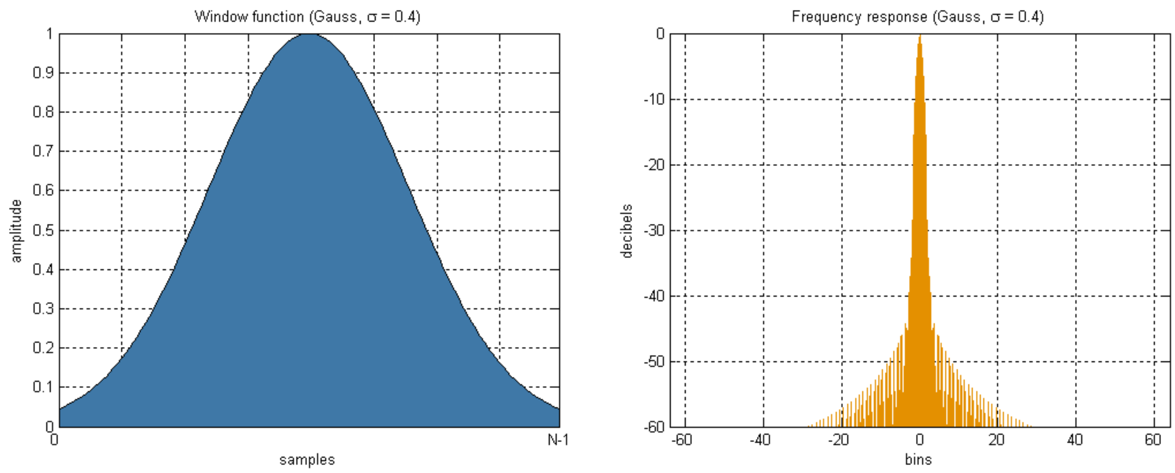


Hann(shifted cosine)

Blackman (sum of shifted cosines, period 2T,4T)



Gaussian



Flat top (sum of shifted cosines, period 2T, 4T,6T,8T)