Scaling, complex systems and all that…

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Notes for MPAGS MM1 Time Series Analysis

- **SCALING: Some generic concepts: universality, Pi theorem, turbulence, and other systems that show scaling (Self Organized Criticality) and order- disorder transitions (flocking)**
- **Fractal measures-'BURST' MEASURES- waiting times, avalanche distributions**
- **Nonlinear correlation- Mutual information and information entropy**

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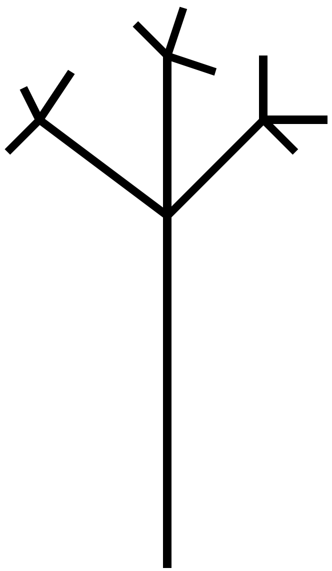
Scaling

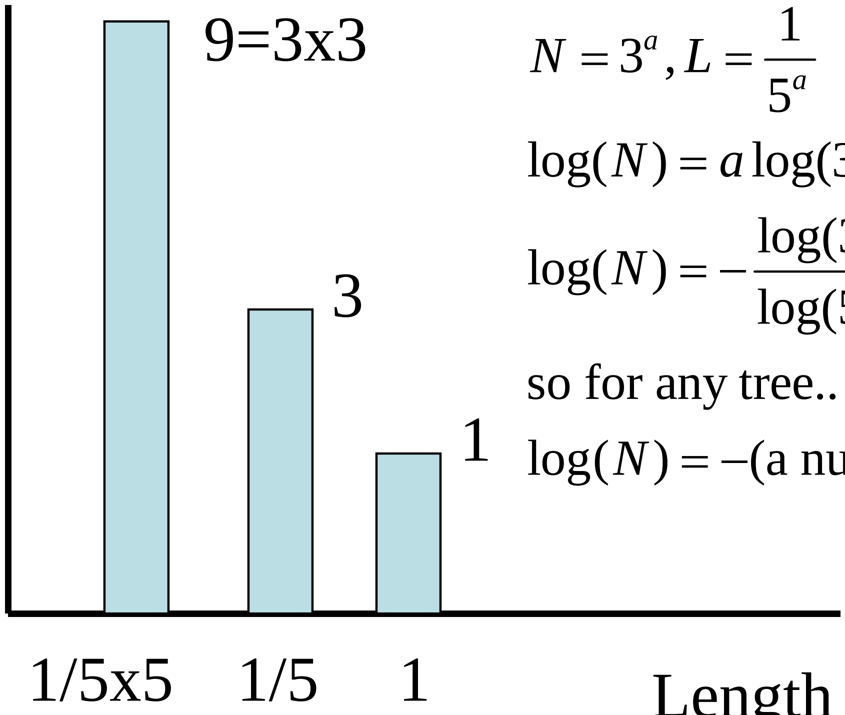
Some more ideas and examples

Scaling and universality-Branches on a self-similar tree

Each branch grows 3 new branches, 1/5 as long as itself..

Number N of branches of length L





$$N = 3^a, L = \frac{1}{5^a}$$

$$\log(N) = a\log(3), \log(L) = -a\log(5)$$

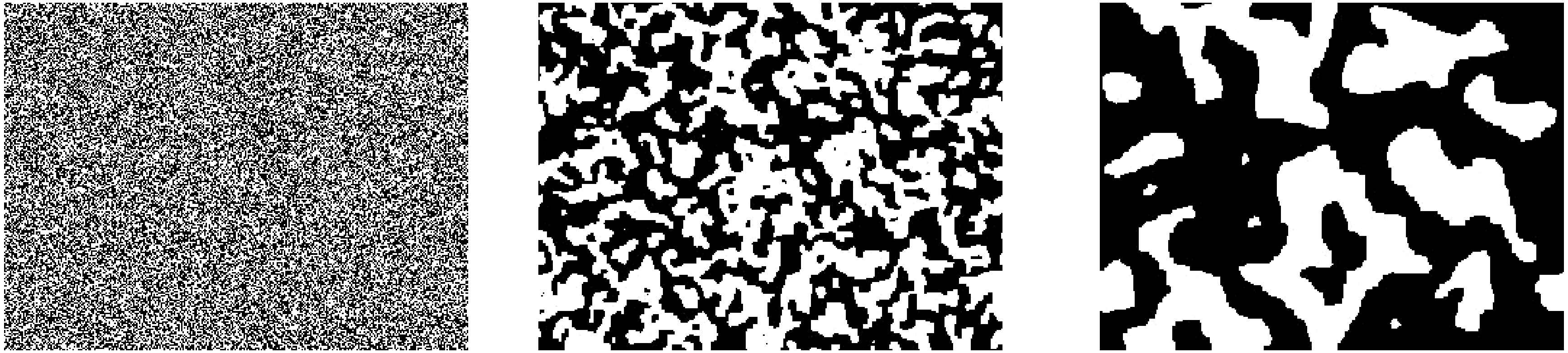
$$\log(N) = -\frac{\log(3)}{\log(5)}\log(L)$$

so for any tree..

$$\log(N) = -(\text{a number})\log(L)$$

Segregation/coarsening- a selfsimilar dynamics

Rules: each square changes to be like the majority of its neighbours
Coarsening, segregation, selfsimilarity



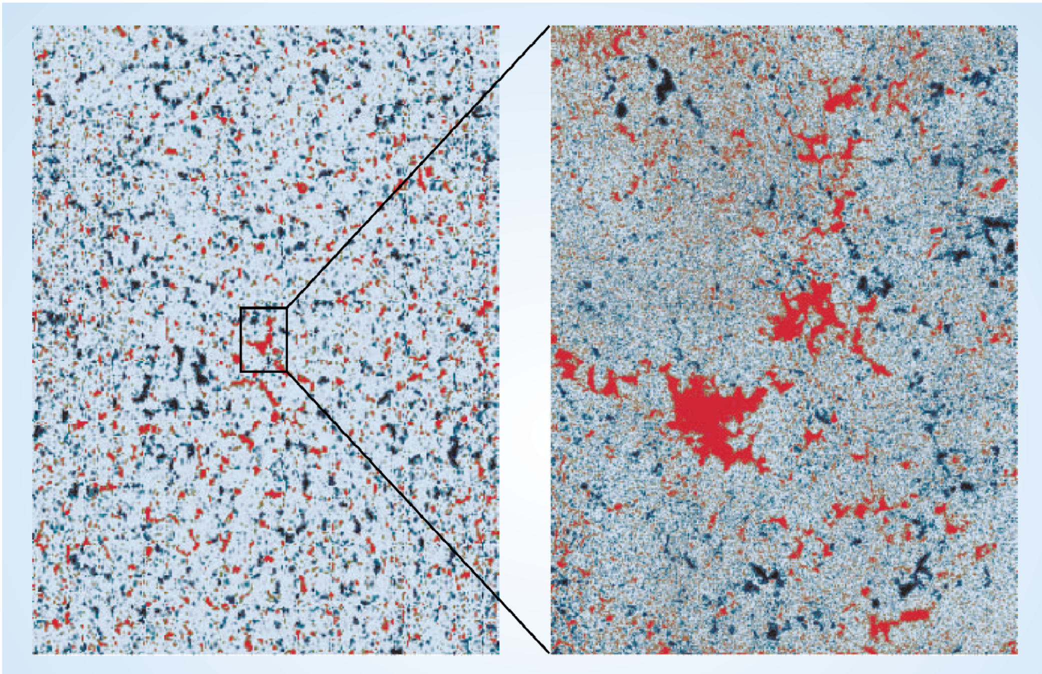
Courtesy P. Sethna

‘Fractal –like’ patches of magnetic polarity on the quiet sun

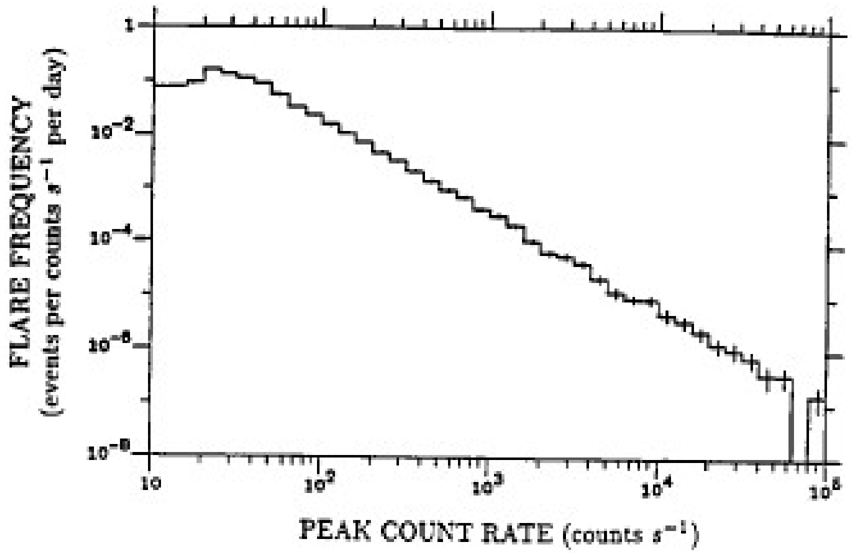
Patches of opposing polarity –

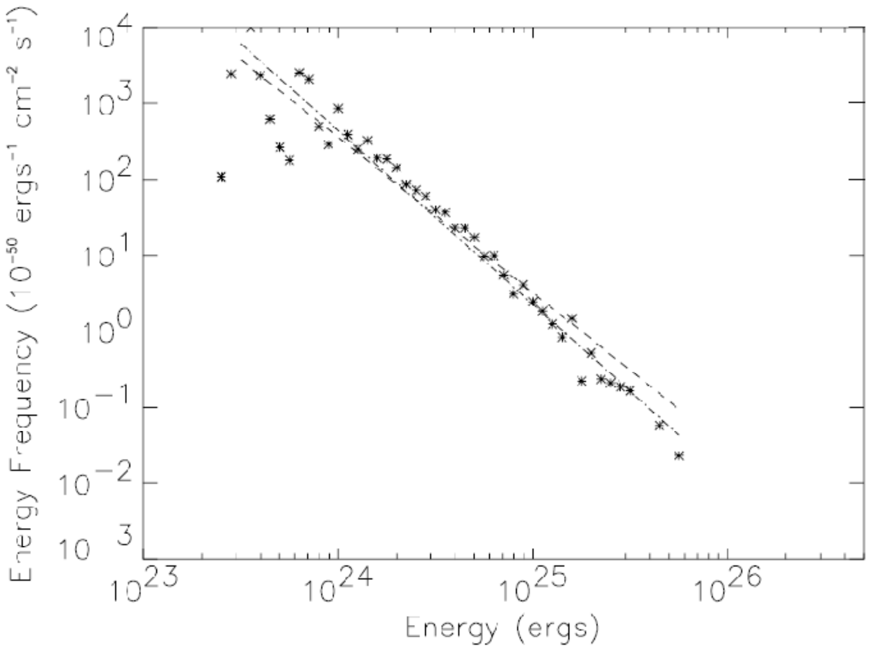
Zeeman effect photosphere, quiet sun,

(Stenflo, Nature 2004, See eg Janssen et al A&A 2003, Bueno et al Nature 2004+..) - ***spatial***



Power law statistics of flares





Peak flare count rate *Lu&Hamilton ApJ 1991*

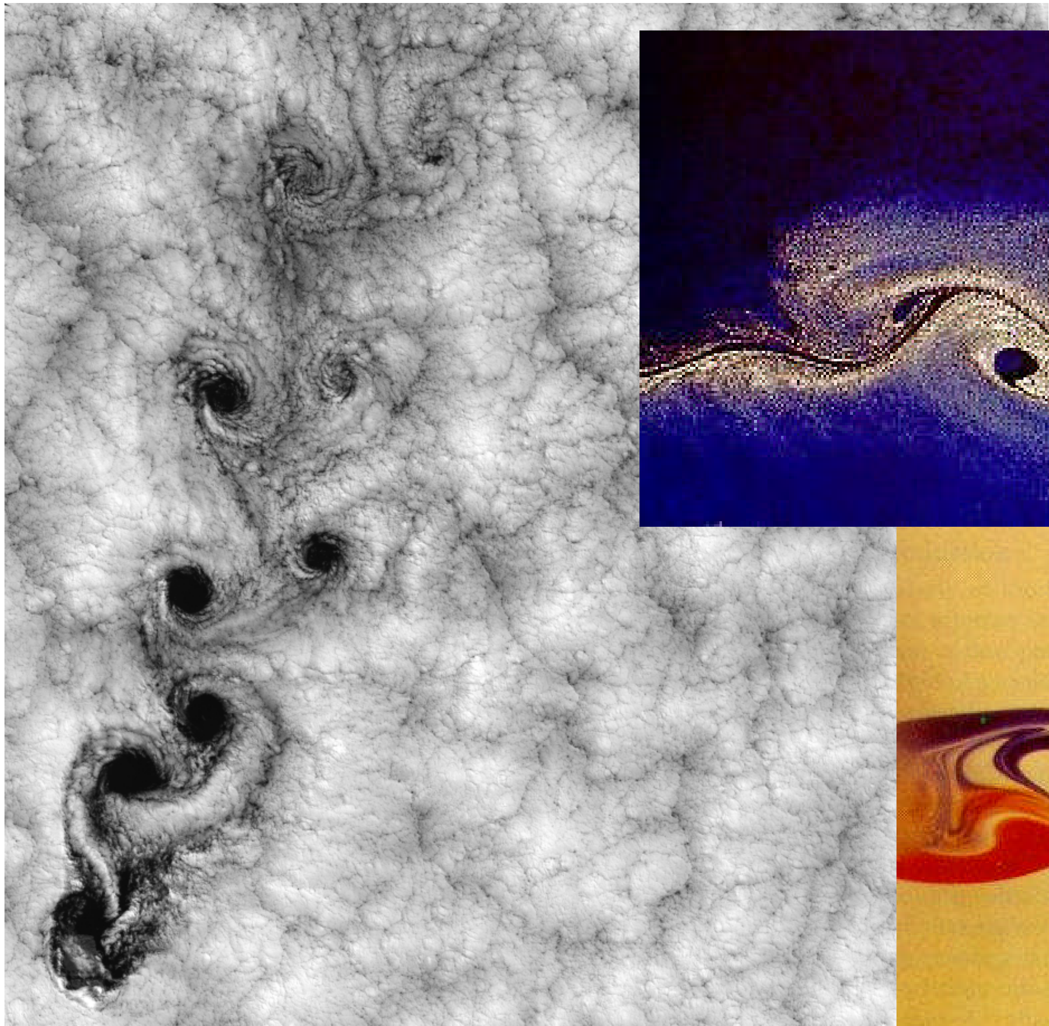
TRACE nanoflare events *Parnell&Judd ApJ 2000*

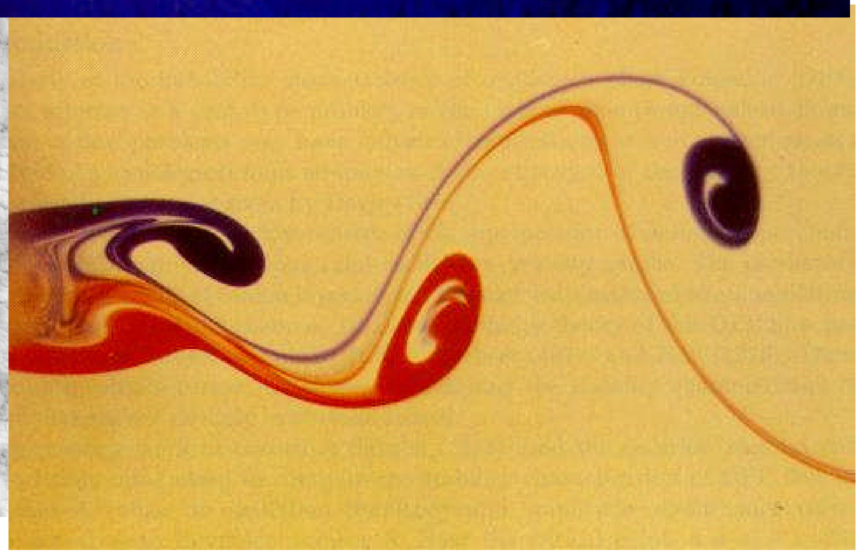
-temporal

Scaling and similarity

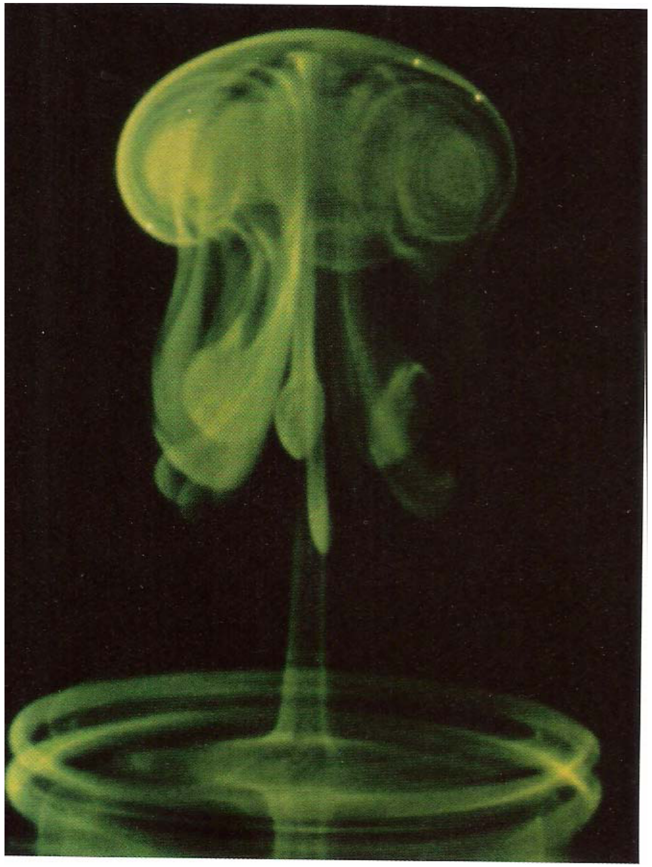
Buckingham PI theorem ('dimensional analysis') of systems that show scaling

Similarity in action…





Similarity in action…



Peck and Sigurdson, A Gallery of Fluid Motion, CUP(2003)

Universality- 1 d.o.f.

Pendulum

$$F = mg, F_t = mg\sin\theta, a_t = l\frac{d^2\theta}{dt^2}$$

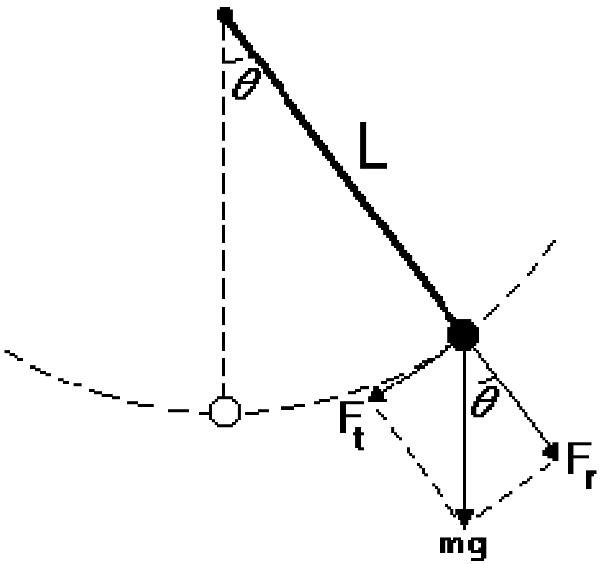
$$F_t = ma_t; \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\omega^2\frac{\partial V}{\partial\theta}$$

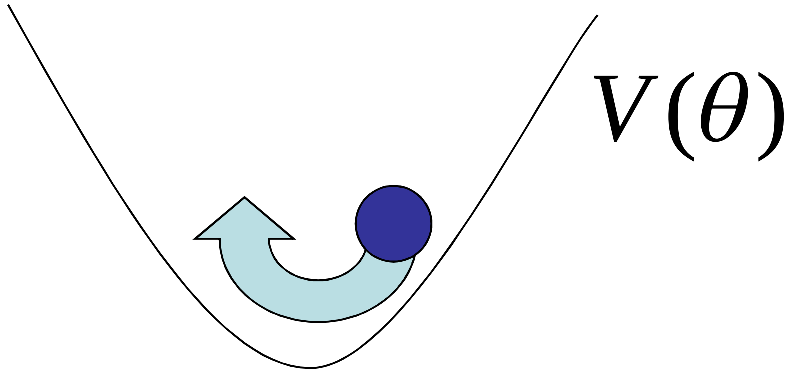
$$V(\theta) = 1 - \cos(\theta) \sim \frac{\theta^2}{2} + \ldots$$

same behaviour at

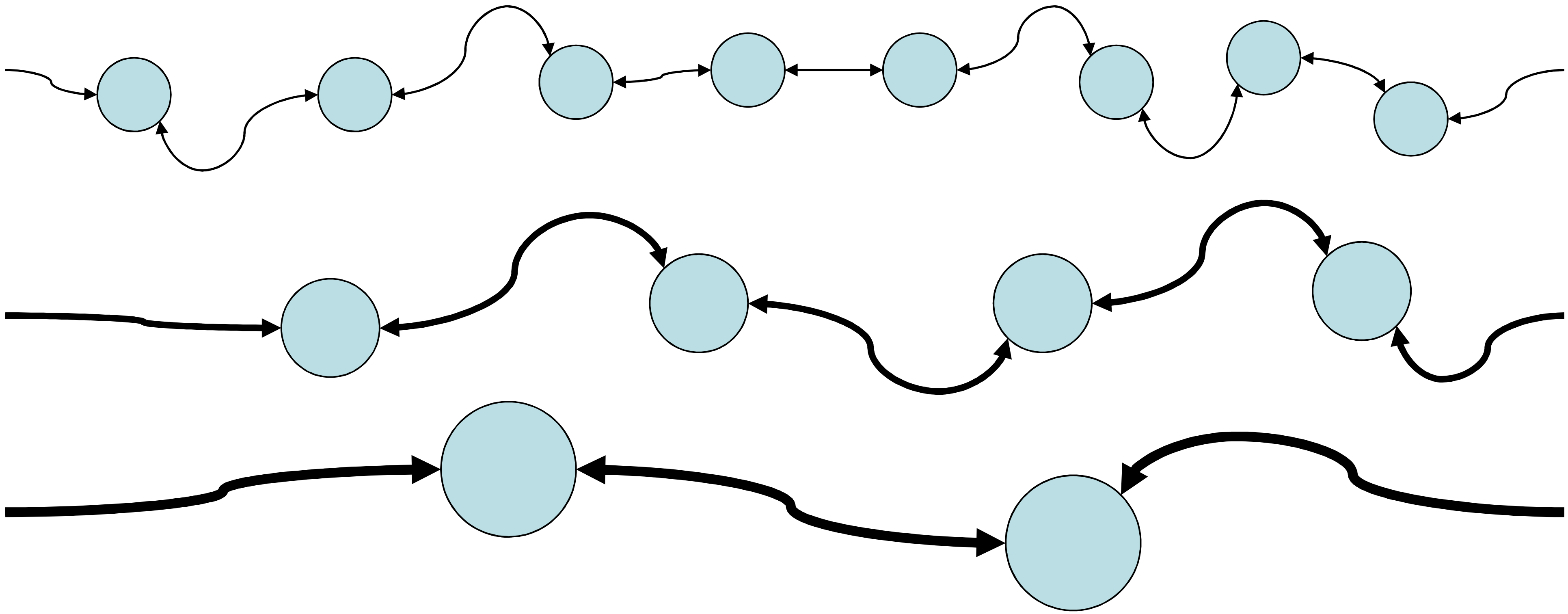
any local minimum in $V(\theta)$

(insensetive to details)





Universality- many d.o.f.



Keep coarsegraining-
rescaled system 'looks the same' (selfsimilar), insensitive to details

Similarity and universality

- Different systems, same physical model
- The same function (suitably normalized) can describe them
- This function is universal (the details do not matter)
- The values of the normalizing parameters are not universal
- How can we find the physical model (solution)?
- Particularly useful in nonlinear systems which are ‘hard’ to solve – i.e. turbulence!
- ‘Classical’ inertial range turbulence- self similarity, intermittency…
- Leads to *order/control parameters*

Buckingham π theorem

System described by $F(Q_1 ... Q_p)$ where $Q_{1..p}$ are the relevant macroscopic variables

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.)

there are $M = P - R$ distinct dimensionless groups.

Then $F(\pi_{1..M}) = C$ is the general solution for this universality class.

To proceed further we need to make some intelligent guesses for $F(\pi_{1..M})$

See e.g. *Barenblatt, Scaling, self - similarity and intermediate asymptotics, CUP, [1996]*

also *Longair, Theoretical concepts in physics, Chap 8, CUP [2003]*

Example: simple (nonlinear) pendulum

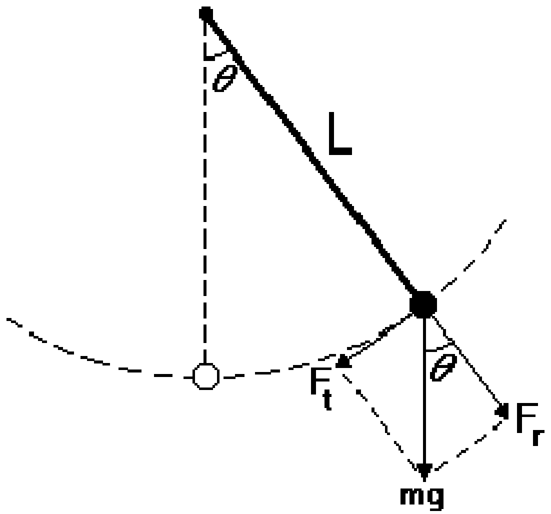
System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant macroscopic variables:

variable	dimension	description
θ_0	$-$	angle of release
m	$[M]$	mass of bob
τ	$[T]$	period of pendulum
g	$[L][T]^{-2}$	gravitational acceleration
l	$[L]$	length of pendulum



Step 2: form dimensionless groups: $P = 5, R = 3$ so $M = 2$

$\pi_1 = \theta_0, \pi_2 = \dfrac{\tau^2 l}{g}$ and no dimensionless group can contain m

then solution is $F(\theta_0, \tau^2 l / g) = C$

Step 3: make some simplifying assumption: $f(\pi_1) = \pi_2$ then the period: $\tau = f(\theta_0)\sqrt{l/g}$

NB $f(\theta_0)$ is universal ie same for all pendula-

we can find it knowing some other property eg conservation of energy..

Example: fluid turbulence, the Kolmogorov '5/3 power spectrum'

System described by $F(Q_1 ... Q_p)$ where Q_k is a macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Step 1: write down the relevant variables (incompressible so energy/mass):

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

Step 2: form dimensionless groups: $P = 3, R = 2,$ so $M = 1$

$$\pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}$$

Step 3: make some simplifying assumption:

$F(\pi_1) = \pi_1 = C$ where C is a non universal constant, then: $E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$

Buchingham π theorem (similarity analysis)

universal scaling, anomalous scaling

System described by $F(Q_1 ... Q_p)$ where Q_k is a relevant macroscopic variable

F must be a function of dimensionless groups $\pi_{1..M}(Q_{1..p})$

if there are R physical dimensions (mass, length, time etc.) there are $M = P - R$ dimensionless groups

Turbulence:

variable	dimension	description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber

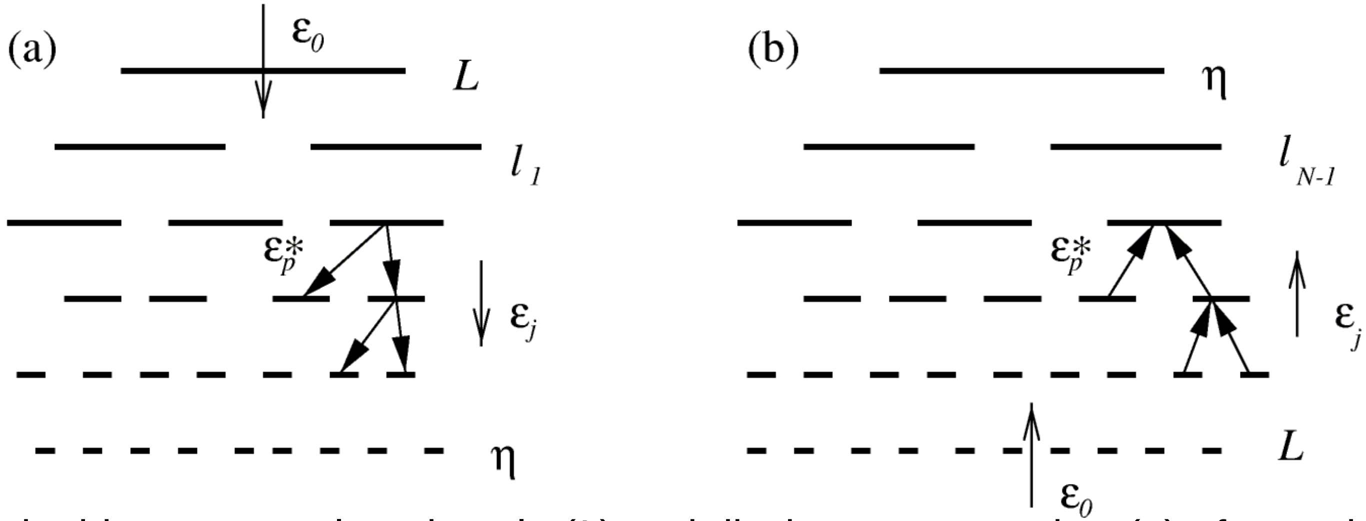
$$M = 1, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, E(k) \sim \varepsilon_0^{2/3} k^{-5/3}$$

introduce another characteristic speed....

variable	dimension	*description*
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave no.
ε_0	$[L]^2[T]^{-3}$	rate of energy input
k	$[L]^{-1}$	wavenumber
v	$[L][T]^{-1}$	characteristic speed

$$M = 2, \pi_1 = \frac{E^3(k)k^5}{\varepsilon_0^2}, \pi_2 = \frac{v^2}{Ek} \text{ let } \pi_1 \sim \pi_2^{\alpha}, E(k) \sim k^{-(5+\alpha)/(3+\alpha)}$$

Turbulence and ‘degrees of freedom’



- System is driven on one lengthscale (L) and dissipates on another (η) –forward cascade
- Inverse cascade- same thing, just the other way around
- System has many degrees of freedom i.e. structures on many lengthscales (eddies here)
- System is scaling- structures, processes can be rescaled to ‘look the same on all scales’
- These structures transmit some dynamical quantity from one lengthscale to another that is, over all the d.o.f.
- There is conservation of flux of the dynamical quantity- here energy transfer rate
- Steady state (not equilibrium) means energy injection rate balances energy dissipation rate on the average

Homogeneous Isotropic Turbulence and Reynolds Number

Step 1: write down the relevant variables:

variable	dimension	description
L_0	$[L]$	driving scale
η	$[L]$	dissipation scale
U	$[L][T]^{-1}$	bulk (driving) flow speed
ν	$[L]^2[T]^{-1}$	viscosity

Step 2: form dimensionless groups: $P=4, R=2,$ so $M=2$

$\pi_1 = \frac{UL_0}{\nu} = R_E, \pi_2 = \frac{L_0}{\eta}$ and importantly $\frac{L_0}{\eta} = f(N),$ where N is no. of d.o.f

Step 3: d.o.f from scaling ie $f(N) \sim N^{\alpha}$ here $\frac{L_0}{\eta} \sim N^3$, or $N^{3\beta}$ or $\frac{L_0}{\eta} \sim \lambda^{N/3}$ or ...

Step 4: assume steady state and conservation of the dynamical quantity, here energy...

transfer rate $\varepsilon_r \sim \frac{u_r^3}{r}$, injection rate $\varepsilon_{inj} \sim \frac{U^3}{L_0}$, dissipation rate $\varepsilon_{diss} \sim \frac{\nu^3}{\eta^4}$ - gives $\varepsilon_{inj} \sim \varepsilon_r \sim \varepsilon_{diss}$

this relates π_1 to π_2 giving: $R_E = \frac{UL_0}{\nu} \sim \left(\frac{L_0}{\eta}\right)^{4/3} \sim N^{\alpha}, \alpha > 0$ thus N grows with R_E

Statistics of ‘bursts’

Avalanche distributions, waiting times

Avalanching systems and scaling behaviour

Avalanche models: add grains slowly, redistribute only if local gradients exceeds a critical value

Suggested as a model for bursty transport and energy release in plasmas- solar corona, magnetotail, edge turbulence in tokamaks (L-H), accretion disks

Avalanching systems

- Threshold for avalanching
- Avalanches are much faster than feeding rate
- Avalanches on all sizes, no characteristic size
- Feeding rate=outflow rate on average only
- System moves through many metastable states- rather than toward an equilibrium

Measures of 'burstiness'

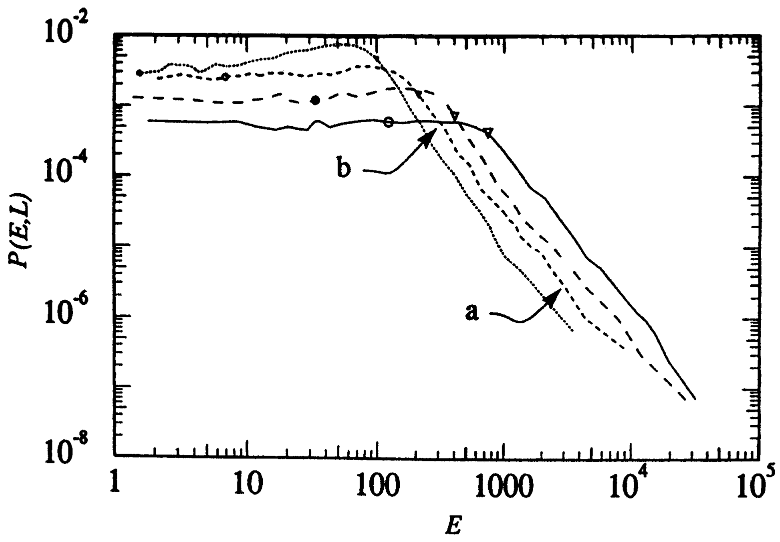
Statistics of:

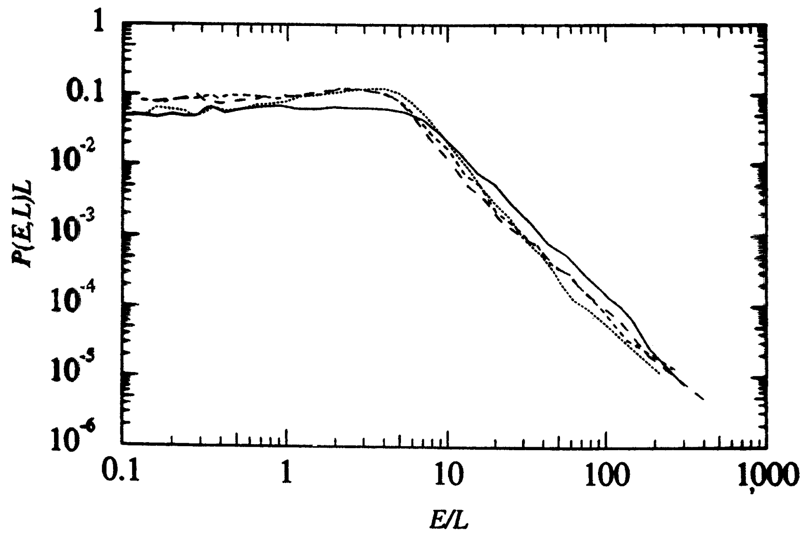
- Waiting time between events
- Energy dissipated
- Peak size
- Duration

Questions:

- Scaling? PDF, CDF, rank order plots etc
- Finite size scaling?

Statistics of avalanches (rice)





Shown: Statistics of energy dissipated per avalanche

➢Power law- no characteristic event size: scaling

➢'finite size scaling'- Normalize to the size of the box

Frette et al, Nature (1996)

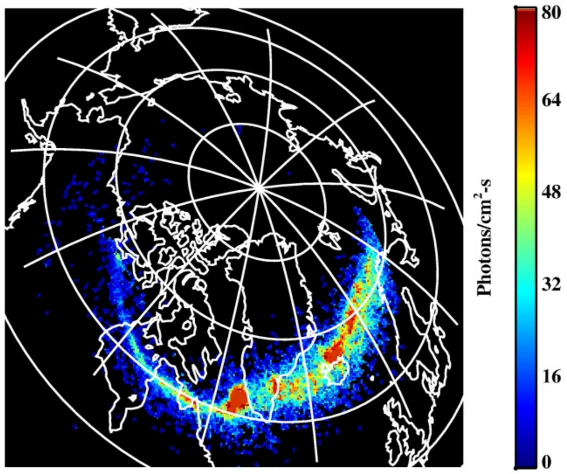
➢Dynamical quantity- rice

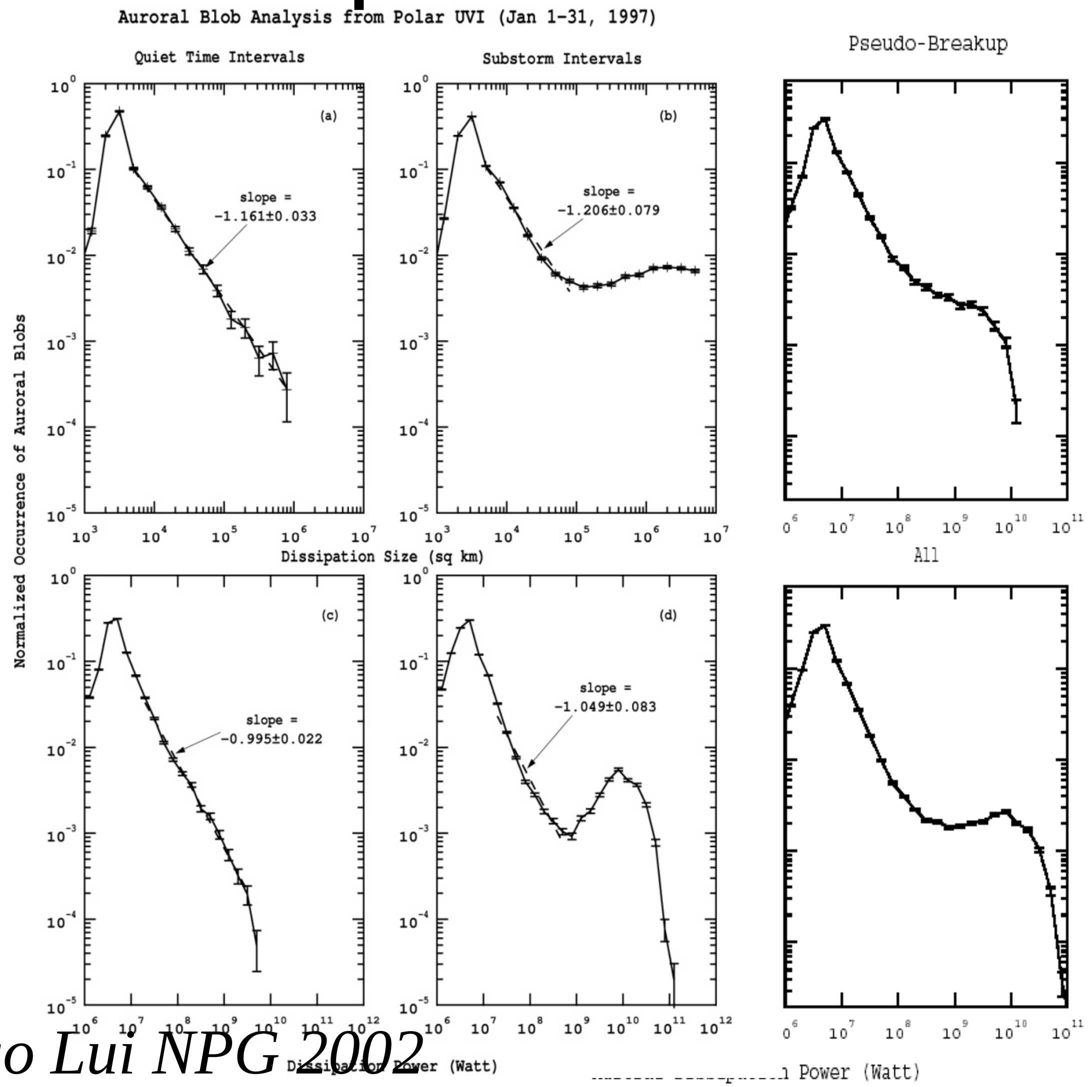
➢Flux is conserved

➢d.o.f. are the possible avalanche (sizes/topplings)

Counting auroral snapshot ‘blobs’

- 1 month of POLAR UVI data=200,000 ‘blobs’
- Quiet and active times
- Robust power law(?)
- +substorms





Lui et al GRL, 2000, see also Lui NPG 2002

Blob statistics-
Edwards Wilkinson- dynamics

A *linear* model
Shown: 100^2 grid D=0.3
Solves:

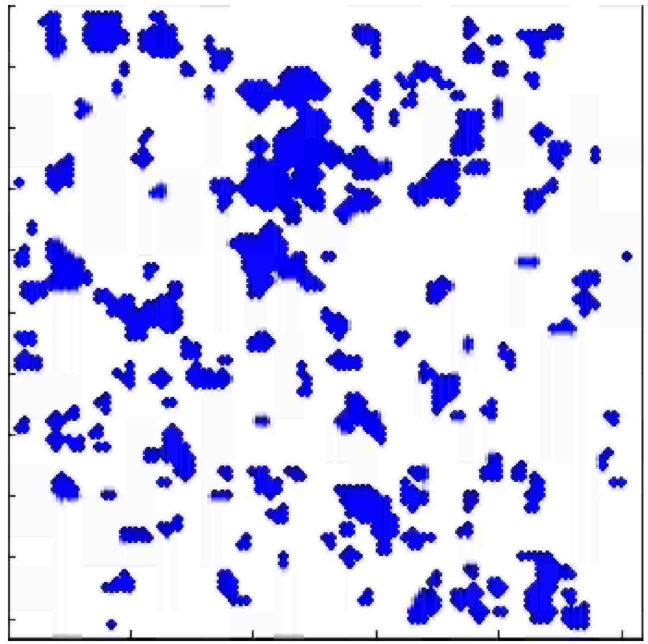
$$\frac{\partial \bar{h}}{\partial t} = D\nabla^2 \bar{h} + \eta$$

where η is iid 'white'

random source of grains

'height' $\bar{h} = h - \langle h \rangle$

blue patches are $\bar{h} > h_0$



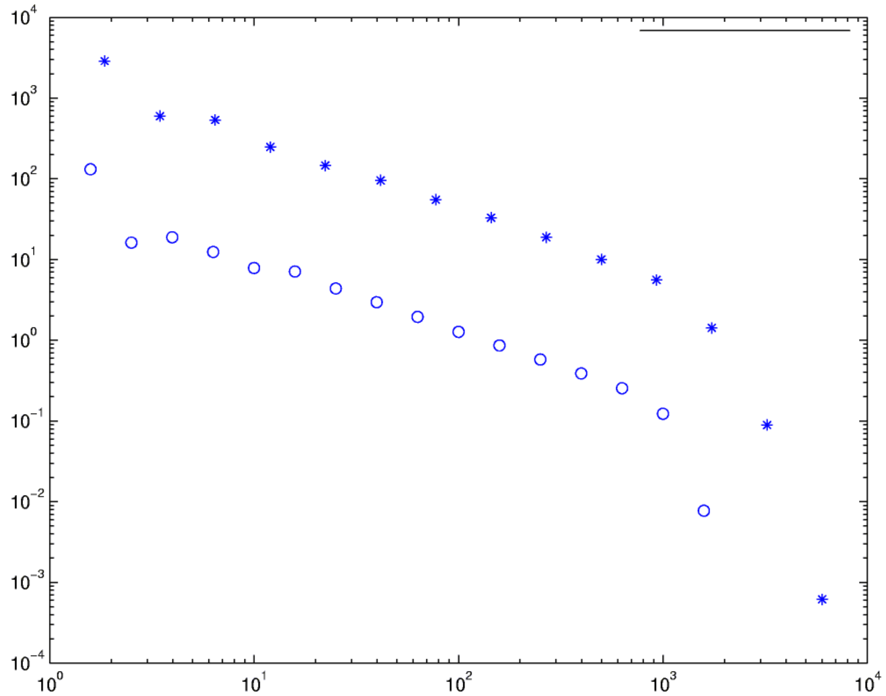
Chapman et al PPCF 2004

Edwards Wilkinson- statistics

Statistics of instantaneous patch size are power law

Linear model- driver (random rain of particles) has inherent fractal scaling (Brownian surface) +selfsimilar diffusion which preserves scaling

•No robustness- scaling exponent *depends* on drive.
•No transport of patches



Chapman et al PPCF 2004

Power laws and blobs?

- Linear systems e.g. EW model give ‘blobs’ with power law statistics
- Missing element is ‘bursty’ (intermittent) *transport* via avalanches. Requires threshold (nonlinear diffusion)- breaks symmetry
- It matters what the exponent is

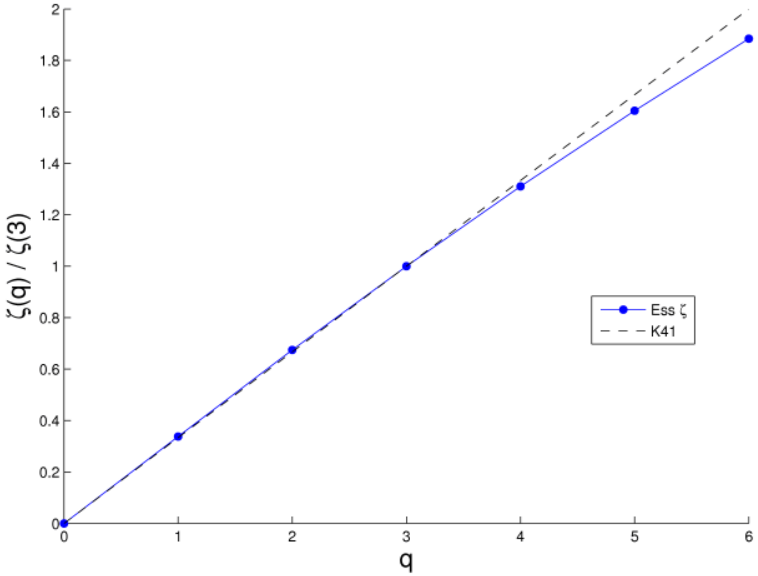
$$\frac{\partial \bar{h}}{\partial t} = D(\bar{h})\nabla^2 \bar{h} + \eta$$

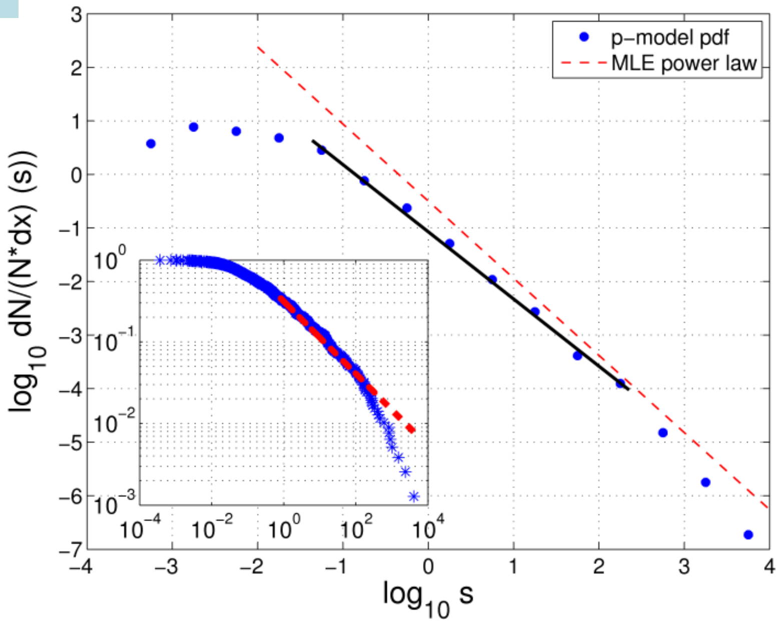
$$D(\bar{h}) \propto \mathrm{H}(\nabla\bar{h} - \bar{h}_0) \text{ - avalanche models}$$

$$D(\bar{h}) \propto (\nabla\bar{h})^2 \text{ KPZ - transforms to Burgers eqn.}$$

p-model for intermittent turbulence- shows finite range power law avalanches

p-model timeseries shows multifractal behaviour in structure functions as expected





Thresholding the timeseries to form
an avalanche distribution- finite range power law
Watkins, SCC et al, PRL, 2009, SCC et al, POP 2009

Recurrence, Information Entropy and Correlation

Recurrence and Mutual Information- principles and practice

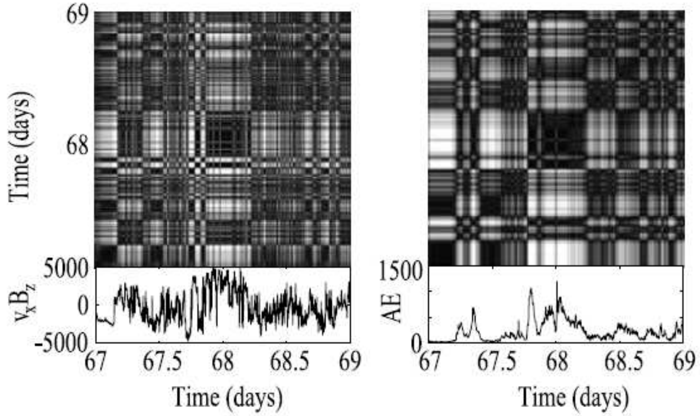
Recurrence measures

R is a recurrence matrix

$\{\boldsymbol{x}_i\}_{i=1}^{N}$, with $\boldsymbol{x}_i \in \mathcal{R}^n$ of a dynamical system and are based on the matrix

$$R_{i,j}^{(\varepsilon)} = \Theta(\varepsilon - ||\boldsymbol{x}_i - \boldsymbol{x}_j||), \quad i,j = 1,\ldots,N, \tag{1}$$

where ε is a predefined threshold and $\Theta(\cdot)$ is the Heaviside function. Then the value "1" is coded as a black dot and the value "0" as a white dot in the plot. Hence, one obtains an $N \times N$ matrix which provides a visual impression of the system behavior.



Solar wind driving of space weather- March, SCC et al, (2005)

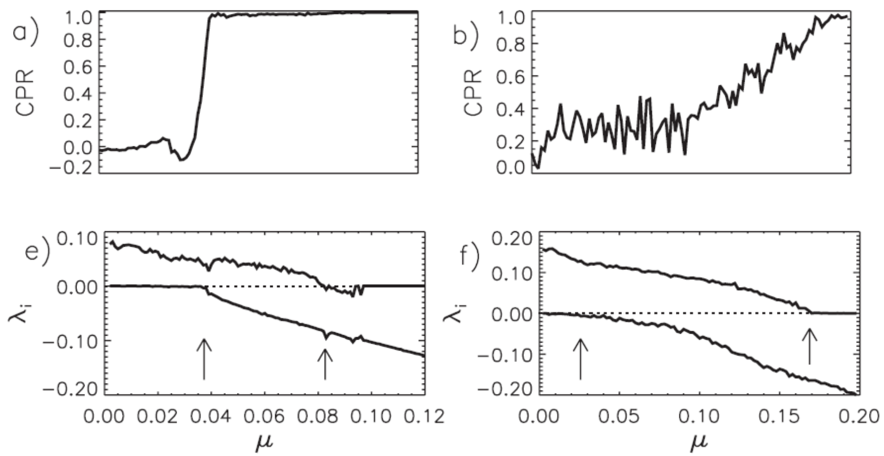
$$\hat{P}^{(\varepsilon)}(\tau) = \frac{\sum_{i=1}^{N-\tau} \Theta(\varepsilon - ||\boldsymbol{x}_i - \boldsymbol{x}_{i+\tau}||)}{N-\tau} = \frac{\sum_{i=1}^{N-\tau} R_{i,i+\tau}^{(\varepsilon)}}{N-\tau}.$$

Normalize..

$$CPR = \langle \bar{P}_1(\tau)\bar{P}_2(\tau)\rangle / (\sigma_1\sigma_2),$$

2 coupled nonlinear oscillators (left) plus noise (right)

After Romano et al Eur Lett (2005)



Information and Mutual Information

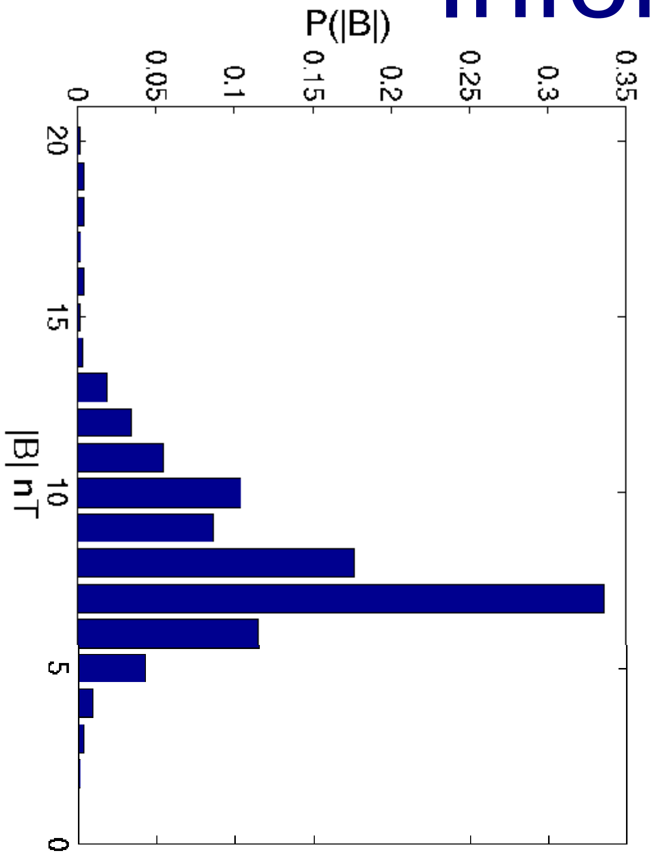
- A given signal can be thought of as a sequence of symbols that form an alphabet.

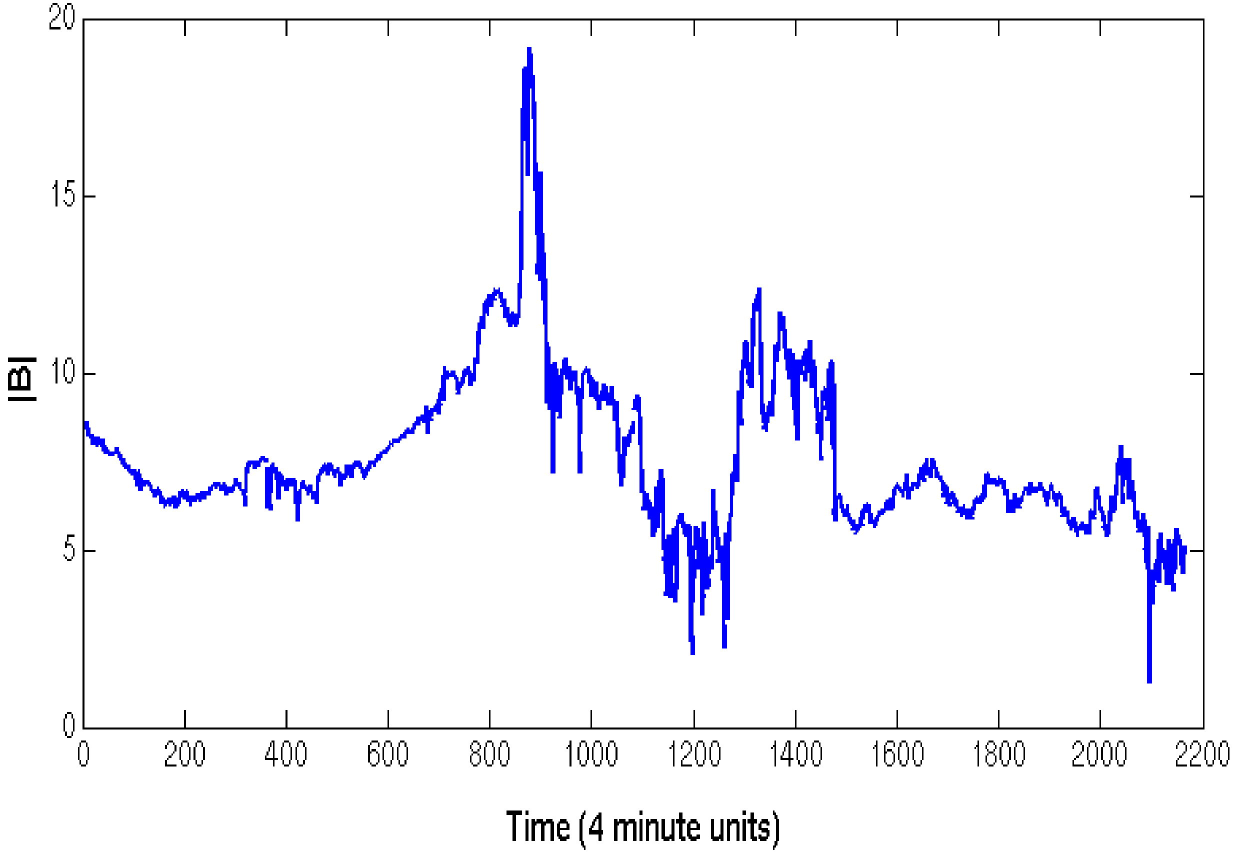
- Signal has alphabet $X = \{x_1, x_2 \ldots x_i\}$

- Each symbol in the alphabet has a probability of occurrence

$$P(x_i) = \frac{n_{x_i}}{N}$$

Information entropy





Information and entropy

- A signal (X) carries a certain amount of information expressed as an entropy H(X) in the order of its symbols $\{x_i\}$

$$H(X) = -\sum_i P(x_i)\log_2\left(P(x_i)\right)$$

- Log_2 => binary units
- We assume the relation $$0\times \log_2 0 = 0$$

Mutual Information

- Entropy can also be defined for joint probability distributions

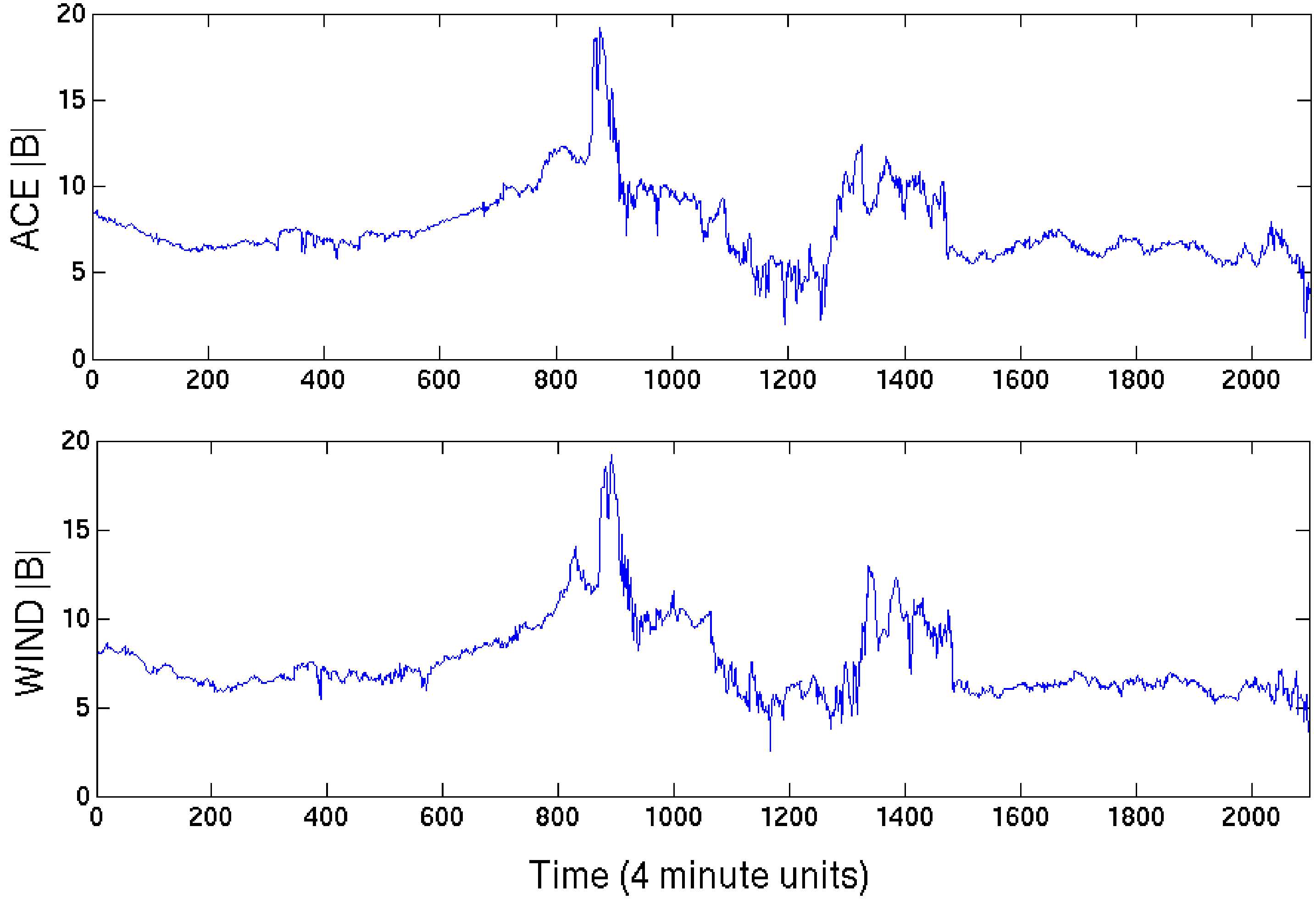
$$H(X,Y) = -\sum_{ij} P(x_i, y_j) log_2 \left(P(x_i, y_j) \right)$$

- Mutual Information compares the information content of two signals

$$I(X;Y) = \sum_{ij} P(x_i, y_j) log_2 \left[P(x_i, y_j) / P(x_i) P(y_j) \right]$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

Timeseries



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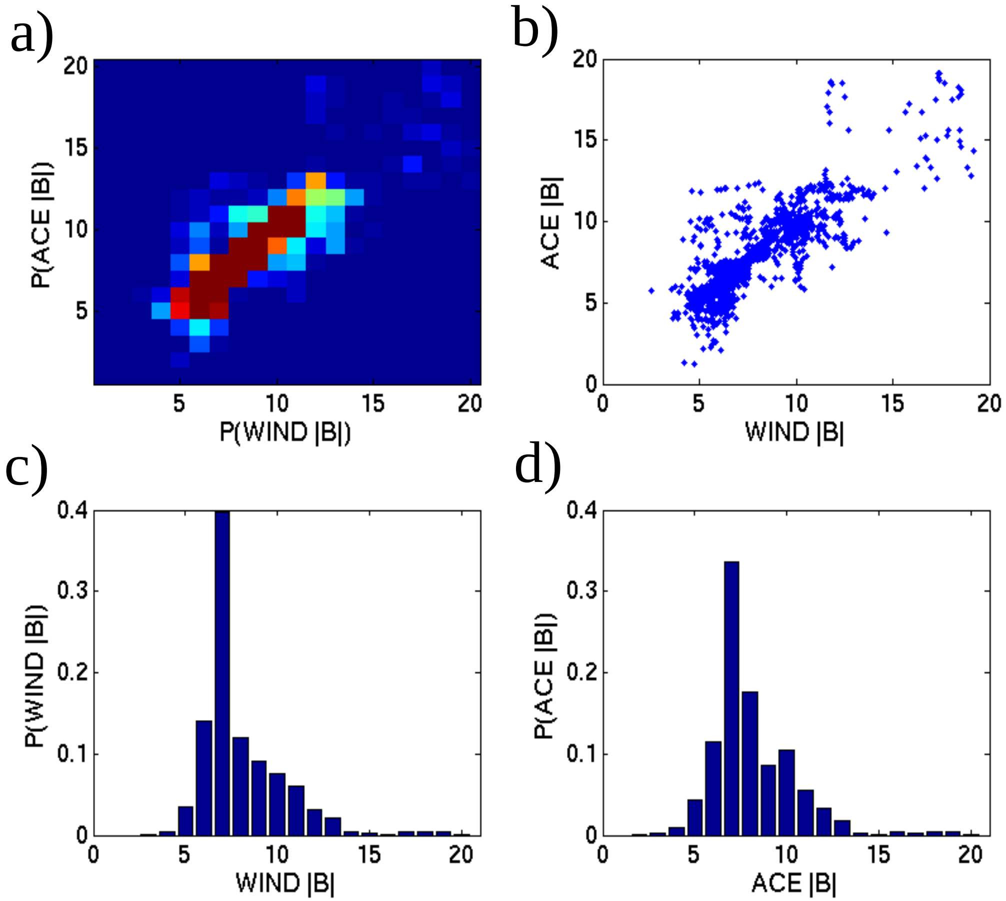
Mutual Information



a) P(WIND |B|, ACE |B|)

b) Raw data WIND |B| vs ACE |B|

c) P(WIND |B|)

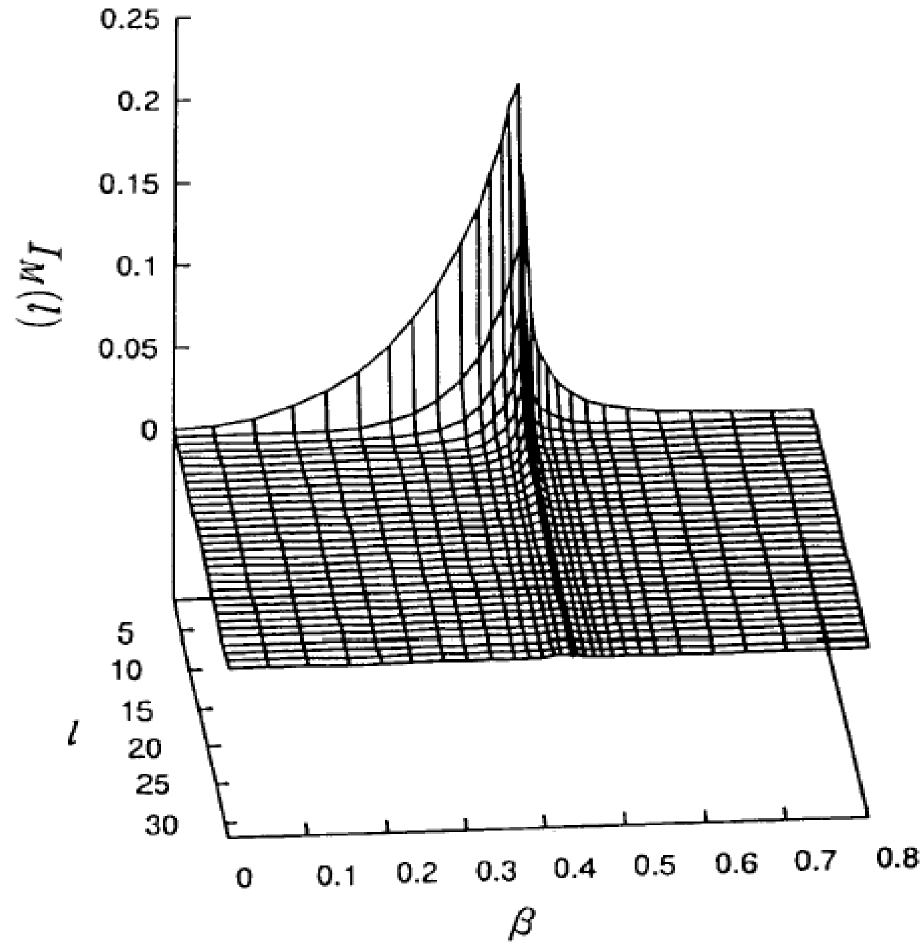
d) P(ACE |B|)

MI = 1.09 bits

Ratio of MI to H = 0.39

The Ising Model- phase transition

- Matsuda *et al (1996)*:
- MI peaks at the phase transition and is robust to coarse graining



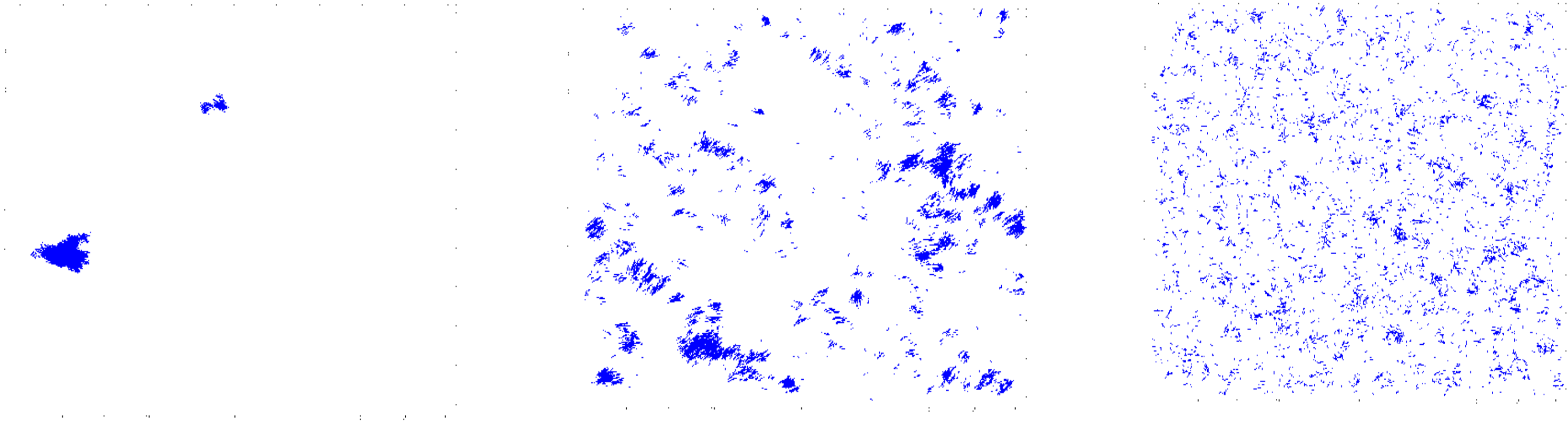
Competition between order and disorder

Rules: random fluctuation plus 'following the neighbours'

$$\mathbf{x}_{n+1}^k = \mathbf{x}_n^k + \mathbf{v}_n^k dt, \quad \left|\mathbf{v}_n^k\right| \text{ constant}$$

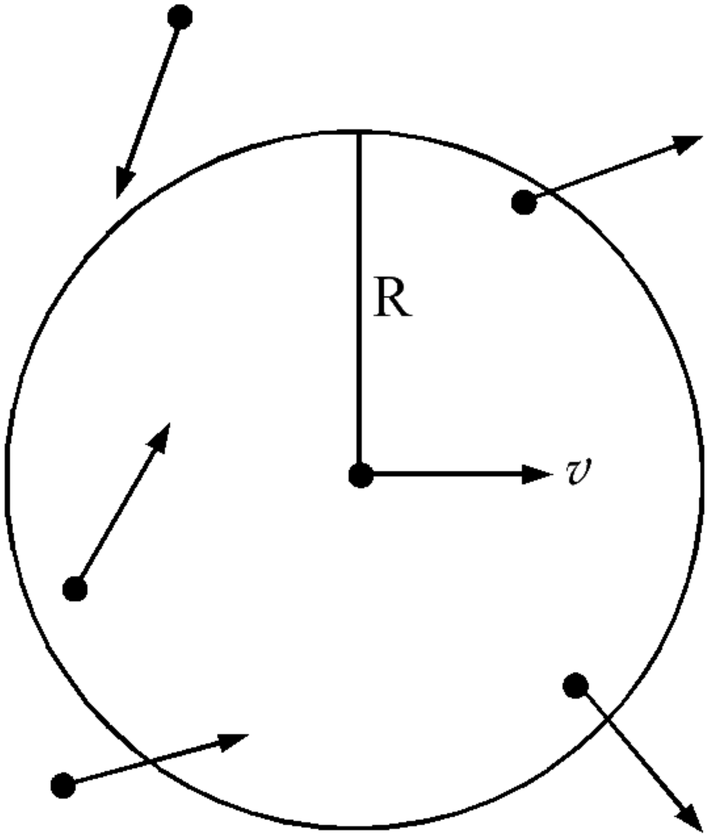
$$\theta_{n+1}^k = \left\langle \theta_n^k \right\rangle_{k \cap R} + \delta\theta, \;\; \delta\theta = [-\eta, \eta] \text{ iid random variable}$$

order parameter: total speed $\frac{1}{N}\left|\sum_{i=1}^{N} \mathbf{v}_i\right|$



Vicsek bird model

The Vicsek Model



Dynamical rules for each particle:

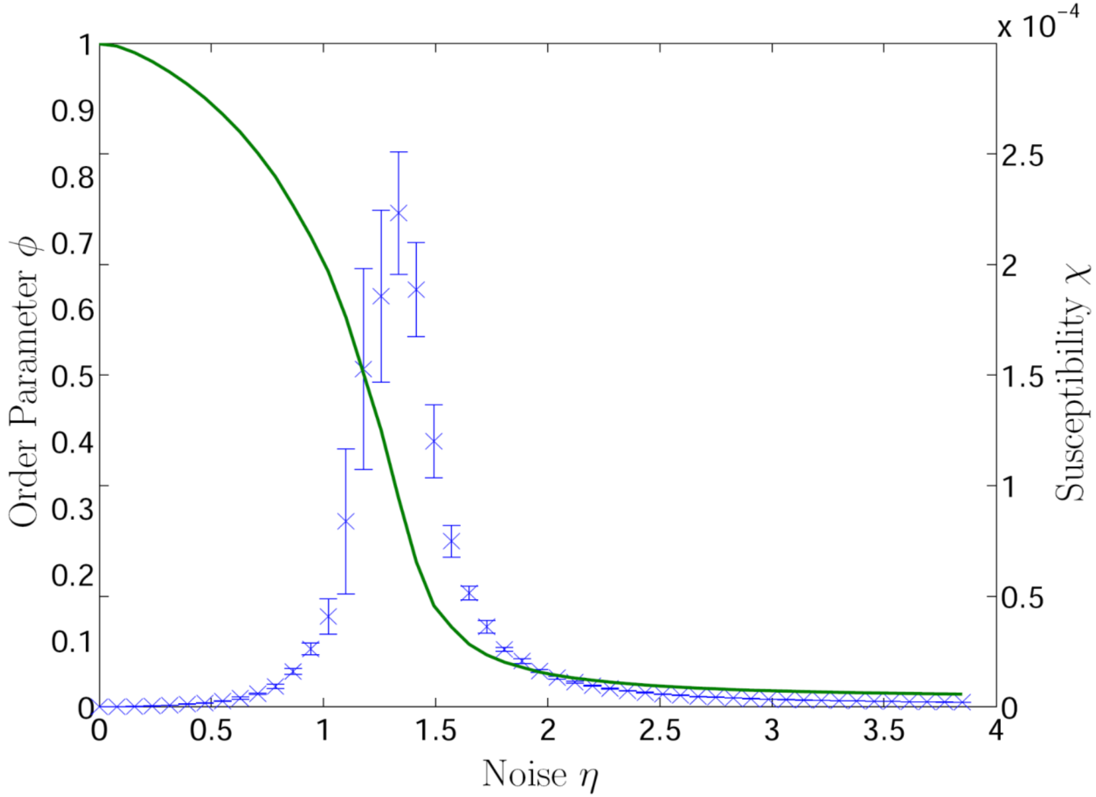
$$
\begin{aligned}
x_{n+1} &= x_n + \vec{v}\,\delta t \\
\theta_{n+1} &= \langle \theta_n \rangle_R + \delta\theta_n
\end{aligned}
$$

Order parameter and susceptibility:

$$\phi = \frac{1}{N v_0}\left|\sum_{i=1}^{N} \underline{v}_i\right|$$

$$\chi = \sigma^2(\phi) = \frac{1}{N}\left(\langle\phi^2\rangle - \langle\phi\rangle^2\right)$$

The Vicsek Model



The Vicsek Model

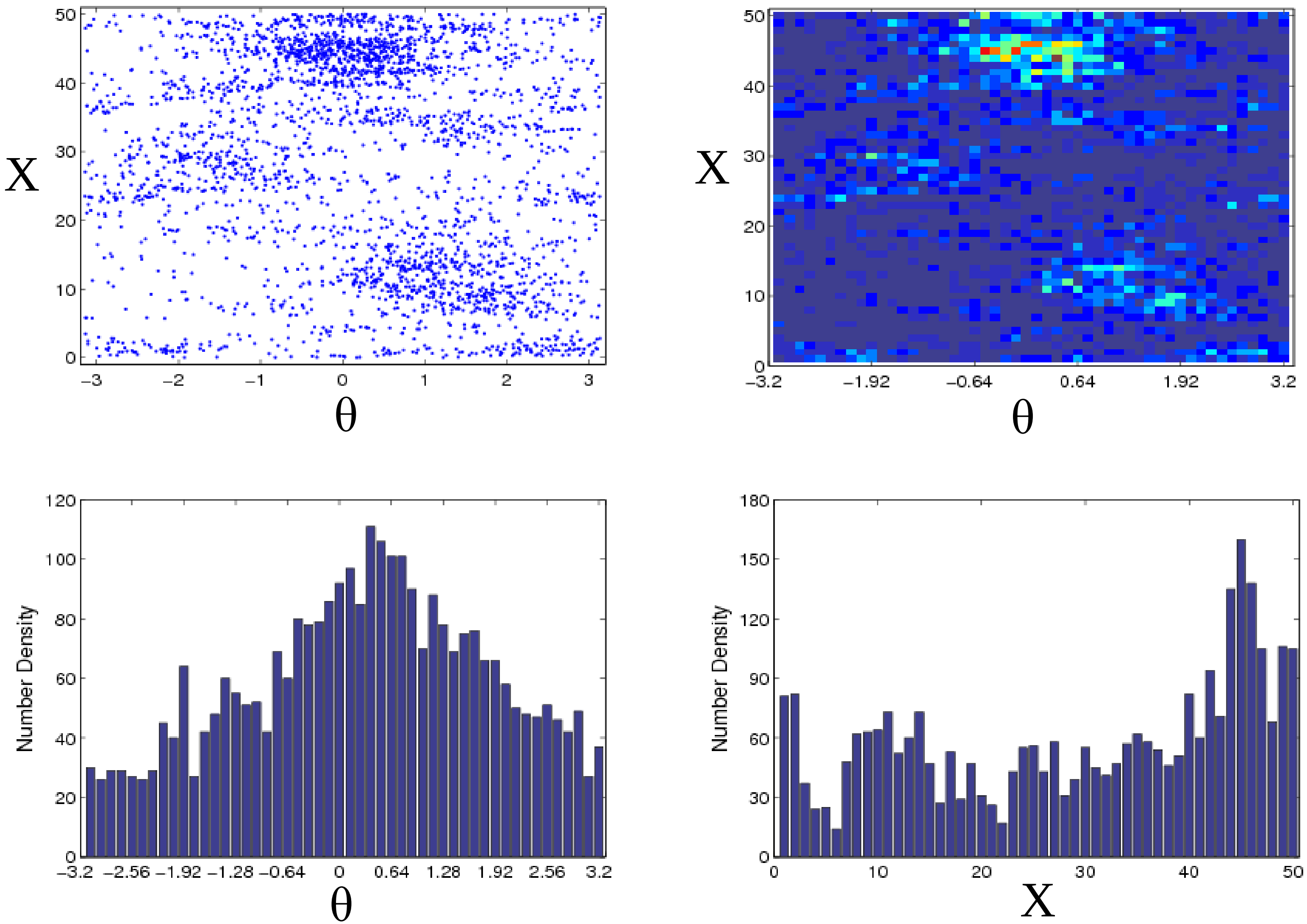
- Mutual information is calculated between position and angle of motion for a snapshot.
- MI for each dimension is the averaged to give total.
- This is done for 50 realisations of the model.

$$I(X,\Theta) = \sum_{i,j} P(X_i,\Theta_j)\log_2\left(\frac{P(X_i,\Theta_j)}{P(X_i)P(\Theta_j)}\right)$$

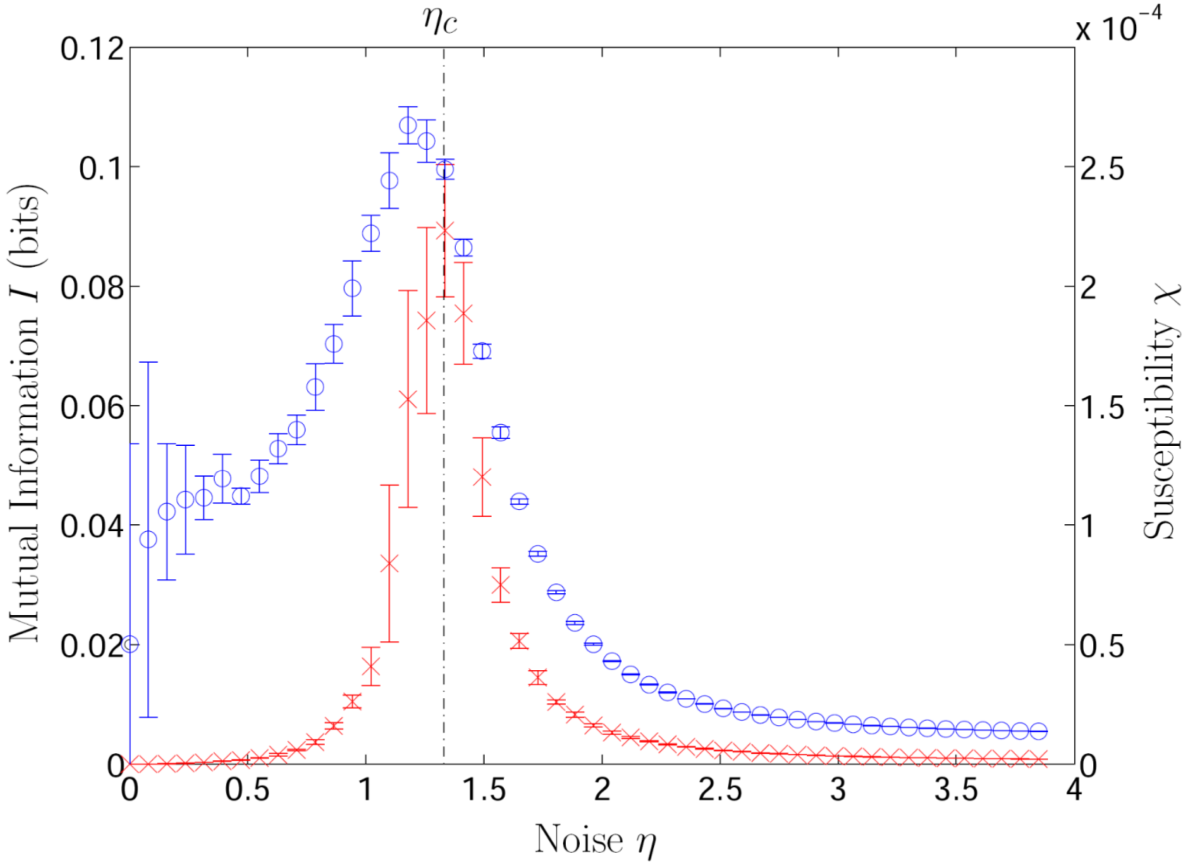
$$I(Y,\Theta) = \sum_{i,j} P(Y_i,\Theta_j)\log_2\left(\frac{P(Y_i,\Theta_j)}{P(Y_i)P(\Theta_j)}\right)$$

$$I = \frac{I(X,\Theta)+I(Y,\Theta)}{2}$$

The Vicsek Model



The Vicsek Model

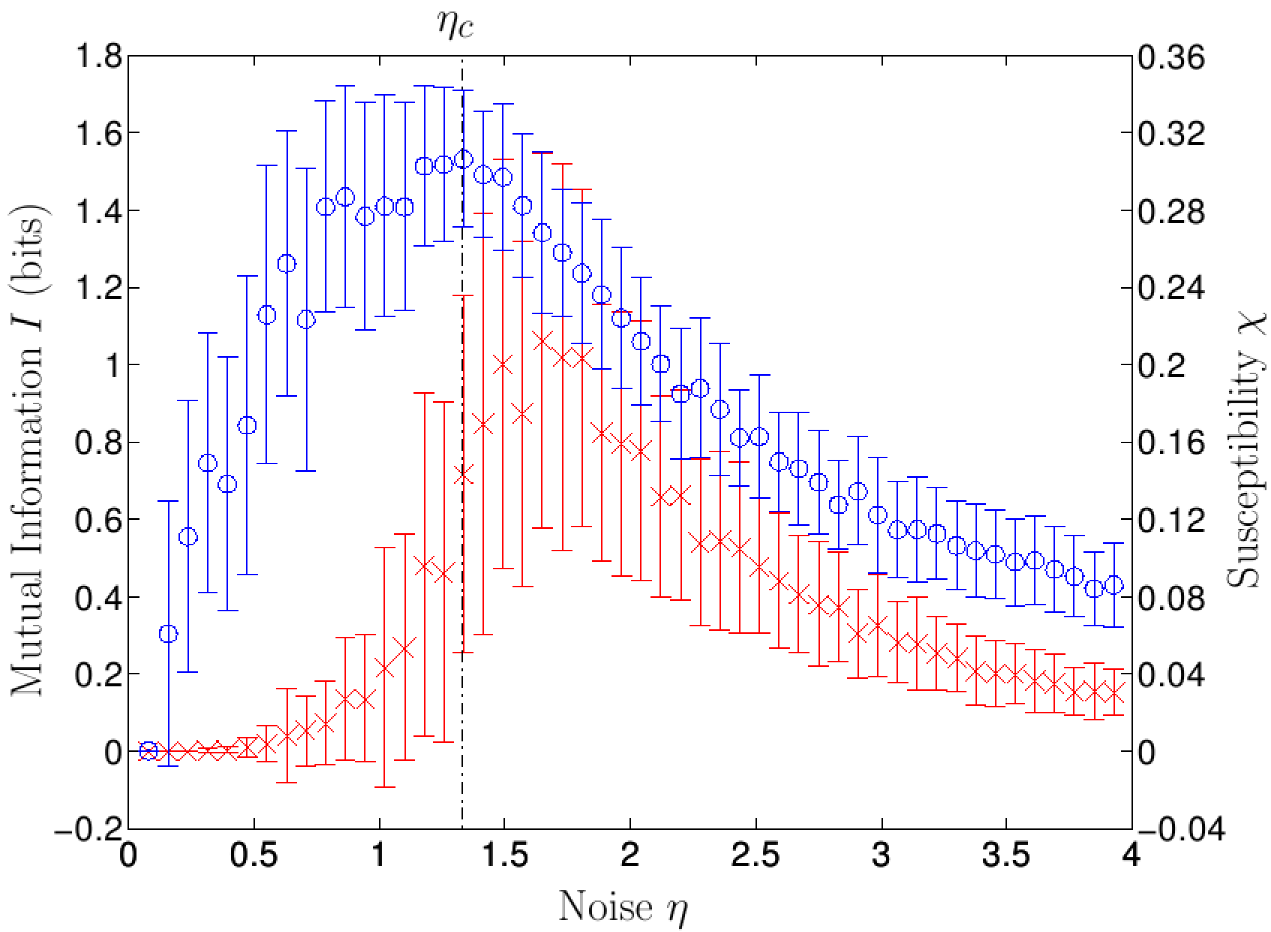


Wicks, SCC et al PRE (2007)

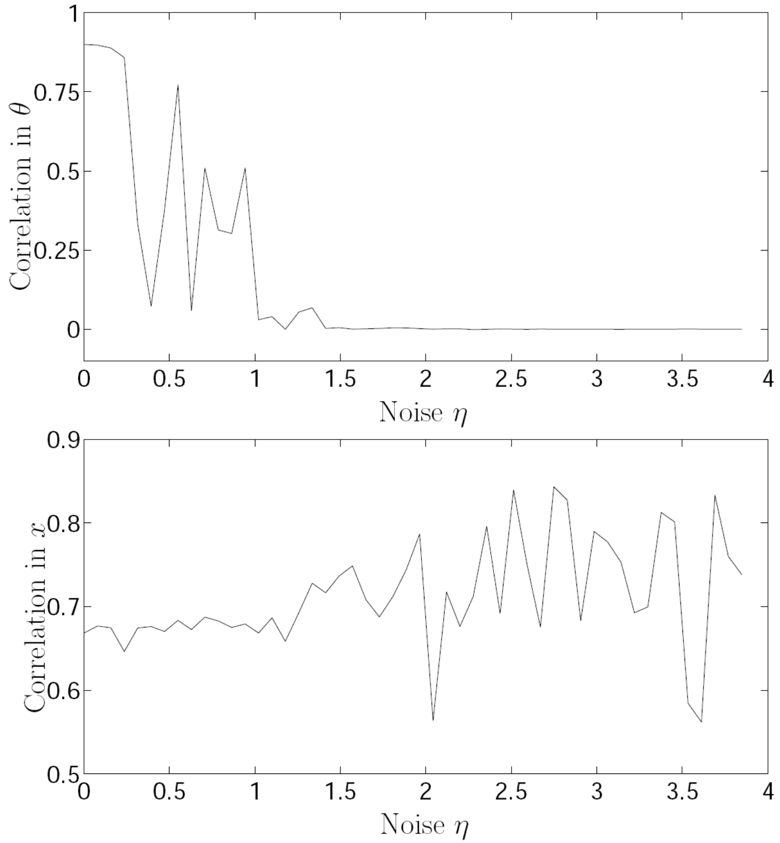
‘real world’- follow only a few particles

- 10 particles chosen at random.
- Time series of 5000 steps used.
- MI calculated between each particle's X position and X velocity for 500 step sections
- Compared to susceptibility for same sections.

(assumption: Vicsek model is ergodic)



Follow only a few particles- linear measure



- Average cross correlation between the same 10 particles.

End

See the MPAGS web site for more reading…