Wavelets

Recall: we can choose $\Phi_f(t)$ as basis on which we expand, ie:

$$y(t) = \sum_f y_f(t) = \sum_f G_f \Phi_f(t)$$

Φ_f may be orthogonal – chosen for "appropriate" properties.

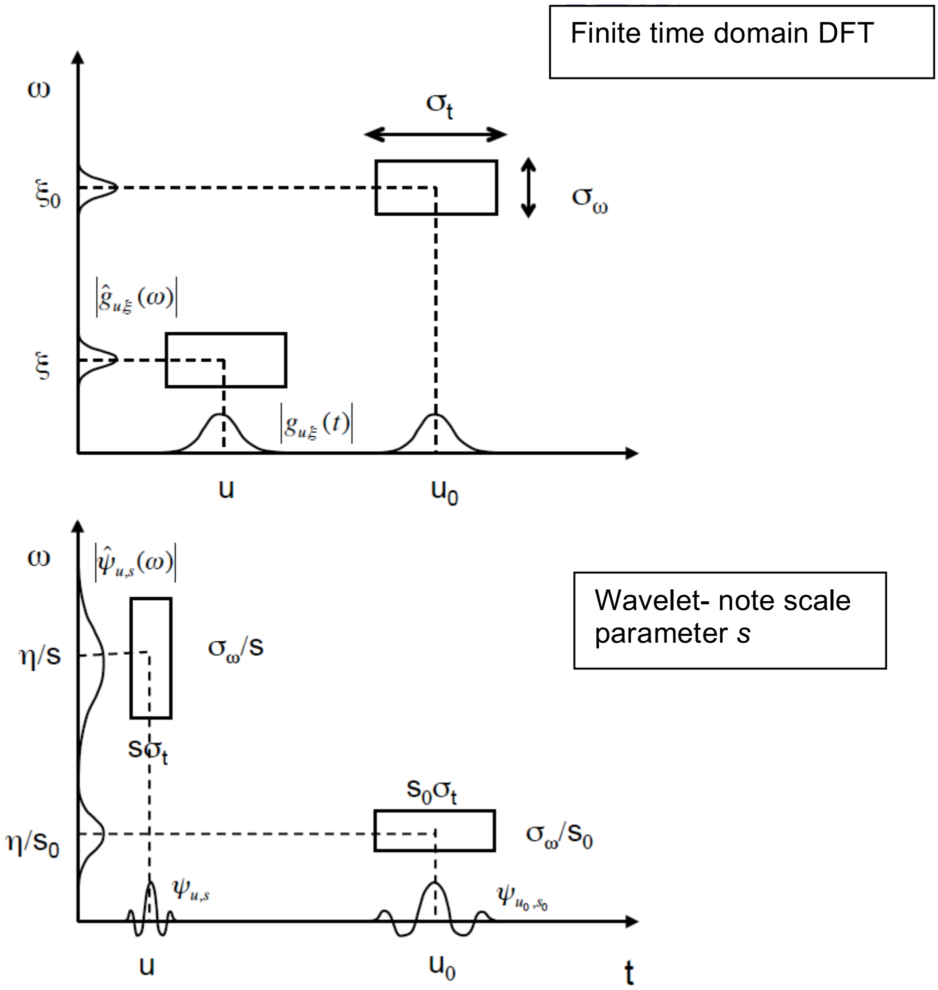
This is equivalent to the transform: $y(t) = \int_{-\infty}^{\infty} G(f)\Phi(f,t)\,df$

We have discussed $\Phi(f,t) = e^{2\pi i f t}$ for the Fourier transform. Now choose different Kernel- in particular to achieve space-time localization.

Main advantage- offers *complete* space-time localization (which may deal with issues of non-stationarity) whilst retaining scale invariant property of the Φ.

First, why not just use (windowed) short time DFT to achieve space-time localization?

Wavelets- we can optimize i.e. have a short time interval at high frequencies, and a long time interval at low frequencies; i.e. simple Wavelet can in principle be constructed as a band- pass Fourier process. A subset of wavelets are orthogonal (energy preserving c.f Parseval theorem) and have inverse transforms.



So at its simplest, a wavelet transform is simply a collection of windowed band pass filters applied to the Fourier transform- and this is how wavelet transforms are often computed (as in Matlab). However we will want to impose some desirable properties, invertability (orthogonality) and completeness.

Continuous Fourier transform:

$$x(t) = \sum_{m=-\infty}^{\infty} S_m e^{2\pi i f_m t}, \quad f_m = \frac{m}{T} \qquad S_m = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-2\pi i f_m t}\, dt$$

with orthogonality: $\int_{-\pi}^{\pi} e^{i(n-m)x} dx = 2\pi\delta_{mn}$

continuous Fourier transform pair:

$$x(t) = \int_{-\infty}^{\infty} S(f) e^{2\pi i f t}\, df$$

$$S(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t}\, dt$$

Continuous Wavelet transform:

$$W(\tau,a) = \int_{-\infty}^{\infty} x(t)\psi^*_{\tau,a}(t)dt$$

$$x(t) = \frac{1}{C_\psi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} W(\tau,a)\tilde{\psi}_{\tau,a}\, d\tau\right]\frac{da}{a^2}$$

Where the *mother wavelet* is $\psi_{\tau,a}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-\tau}{a}\right)$ where τ is the shift parameter and a is the scale (dilation) parameter (we can generalize to have a scaling function *a(t)*).

Here, ψ^* is the complex conjugate and $\tilde{\psi}$ is the dual of the mother wavelet. For the transform pair to 'work' we need an orthogonality condition (recall Fourier transform pair):

$$\int_0^{\infty}\int_{-\infty}^{\infty}\frac{1}{a^3}\psi\left(\frac{t'-\tau}{a}\right)\tilde{\psi}\left(\frac{t-\tau}{a}\right)d\tau da = \delta(t-t')$$

This defines the dual. A subset of 'admissible' (information/energy preserving) wavelets are:

$$\tilde{\psi} = C_\psi^{-1}\psi$$

with

$C_\psi = \int_0^{\infty}\frac{|\Psi|^2}{f}df$ where Ψ is the FT of ψ

is a positive constant that depends on the chosen wavelet.

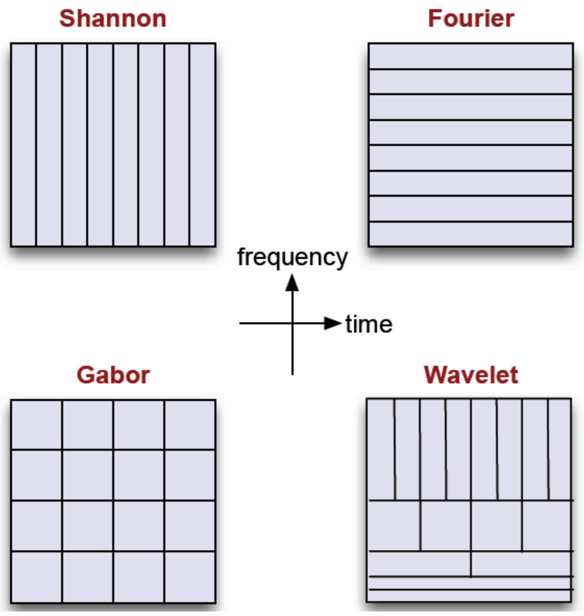
Properties of (all) wavelets:

$$\int_{-\infty}^{\infty} \psi(u)\,du = 0$$

vanishes at infinity and integrates to 1 – energy preserving

$$\int_{-\infty}^{\infty} \psi^2(u)\,du = 1$$

Choice of the dilation and shift parameter- we want to 'tile' the frequency, time domain in a complete way (again, this partitioning can be considered as a filter/convolution process).



choose the dilation and translation parameters to completely cover the domain and have self-similar property:

$$a_p = 2^p, \tau_{pq} = 2^p q$$

so that $$\psi_{pq}(t) = \frac{1}{\sqrt{2^p}}\psi\left(\frac{t-2^p q}{2^p}\right)$$

(actually any $a_p = a_0^p$ can be used, general practice is $a_0 = 2$; also one can have $a(t)$).

Making time discrete:

Discrete Fourier Transform:

$$x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} S_m e^{2\pi imk/N}$$

$$S_m = \Delta t \sum_{k=0}^{N-1} x_k e^{-2\pi ikm/N}$$

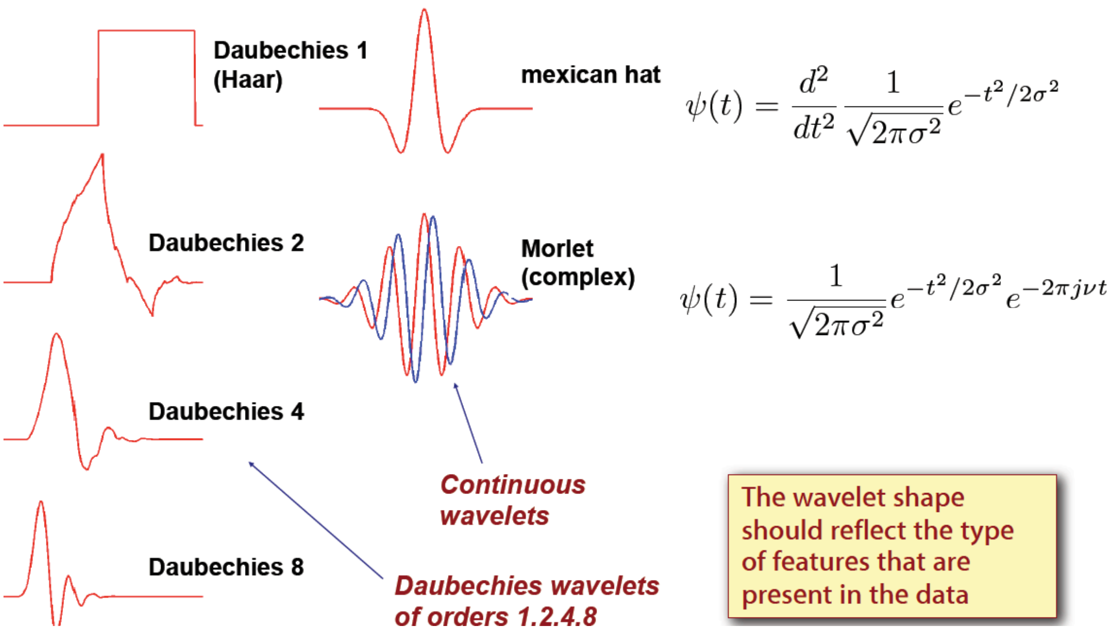
Discrete Wavelet Transform (DWT):

$$W_{m,j} = \frac{\Delta t}{a_0}\sum_{k=0}^{N} x_k \psi^*\left(\frac{k-m}{a_j}\right)$$

$$a_j = a_0^j \qquad f_j = \frac{a_0}{a_j}\frac{1}{\Delta t}$$

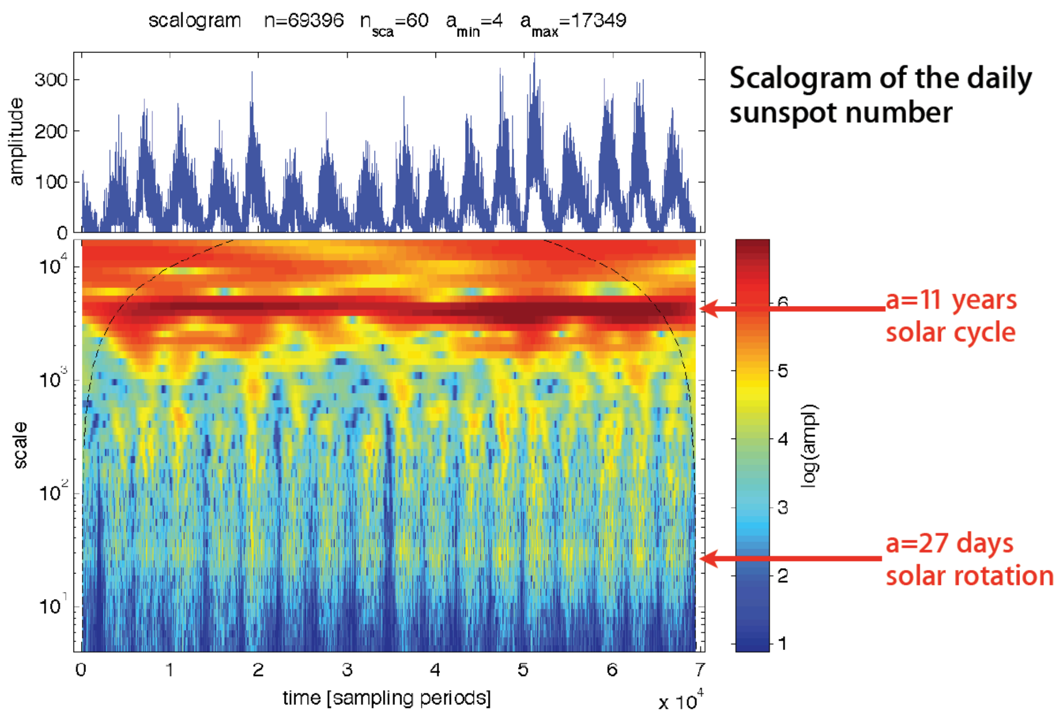
As above, this is a convolution- therefore is realized as a set of (Fourier) band pass filters.

Some examples of mother wavelets:
(Note- Daubechies family of mother wavelets and dilation (scaling) functions- property that it is always zero outside a fixed time doman)

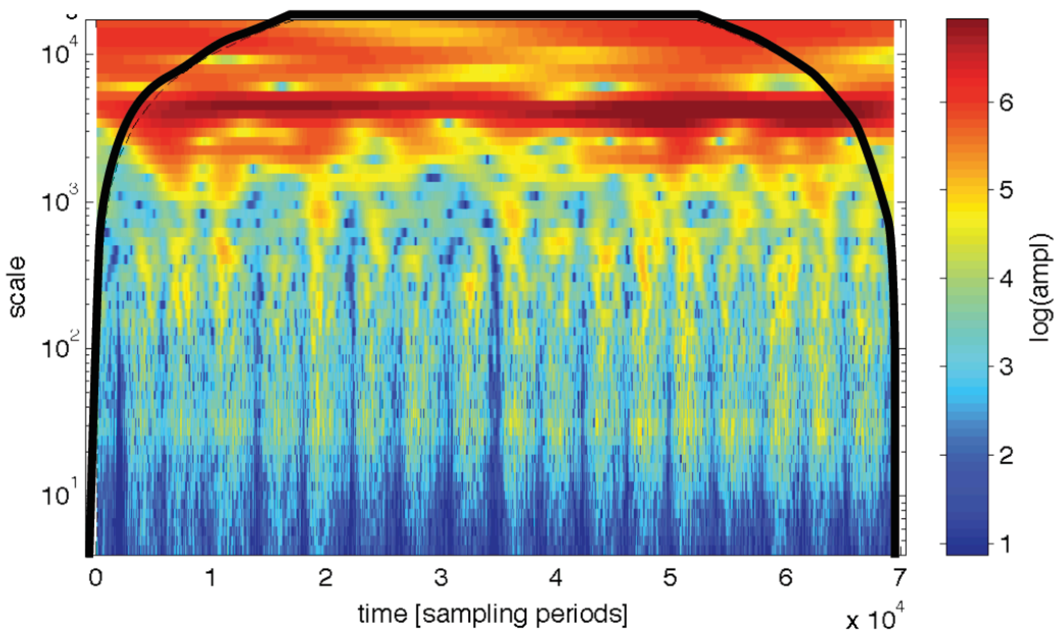


'order' of a filter refers to the degree of the approximating polynomial in frequency space- must exceed that of the signal.

Power spectrum estimation:

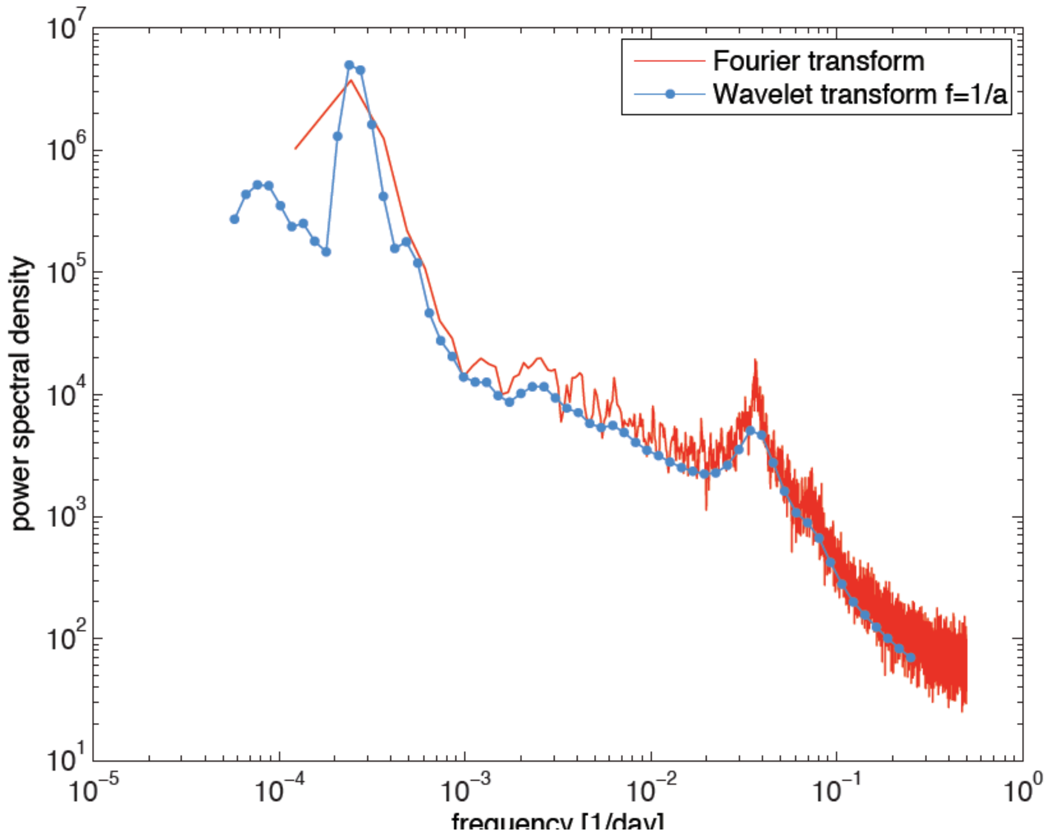


Cone of influence: defined as the *e*-folding time for the autocorrelation of wavelet power at each scale.

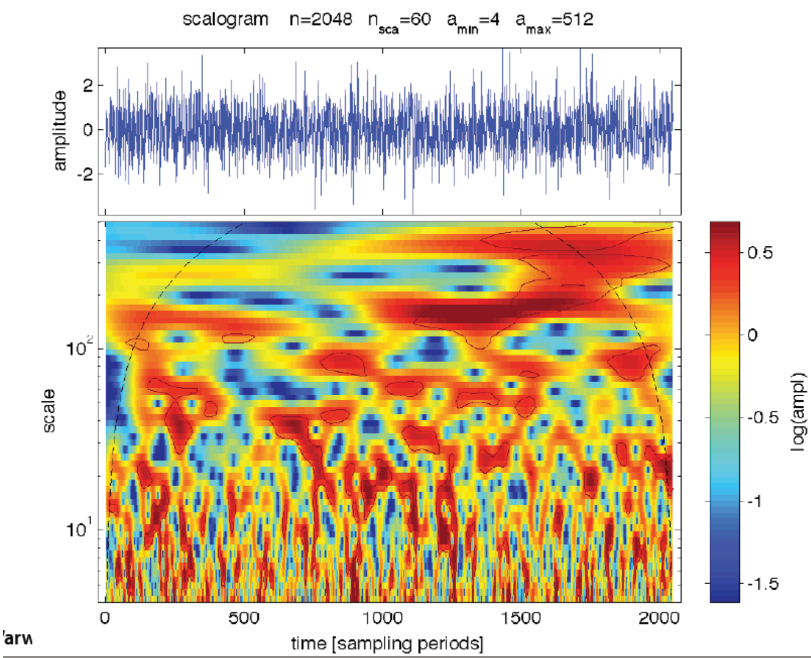


Estimate of Power Spectral Density (PSD): in the same manner as integrating a Fourier spectrogram across time to obtain an averaged PSD one can integrate the wavelet scaleogram across time:

Mortlet wavelet PSD of the sunspot number, with (normalized) frequency=1/scale



Note the self- similar nature of the scaling/dilation of the wavelet transform results in uniform binning in log space of frequency (Fourier has uniform binning in linear frequency space).



Scalogram of a white noise process

Are the peaks really that significant ?

We can define a level of significance (confidence level) w.r.t. a white noise process - details in Torrence and Compo (1998).