Concepts

1. Landau theory – a macroscopic model for a microscopic system.
   - few relevant parameters – many dof system
   - bifurcation and hysteresis - phase transitions

2. 1st order nonlinear DEs
   fixed points, stability (get dynamics without actually solving).

3. 2nd order nonlinear DEs - phase plane analysis.

4. Limit cycles – predator prey, etc.

5. Difference equations – chaos in 1*D* maps
   Lyapunov exponents.
6. Bifurcation sequence to chaos and RG
   Feigenbaum nos and universality in chaos.

7. Many dof systems on computers – order-disorder transitions and self organisation – scale free behaviour.

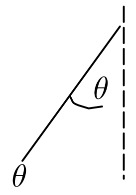
8. Buckingham Pi theorem – dimensional analysis approach to finding the few relevant parameters.

<u>Introduction</u> – universality (linear systems)

Most of what you have seen so far is <u>linear</u>:

An example: pendulum

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$



- Nonlinear since $\sin\theta$ term –

  not e.g. $\theta, \frac{d\theta}{dt}, \frac{d^2\theta}{dt^2} \left(\leftarrow \text{linear} \right)$

Immediately assume θ small than $\sin\theta \simeq \theta$ and $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$.

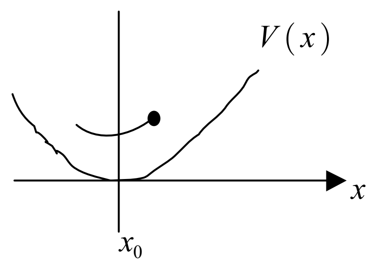
- now linear – solution of the form $\theta - A\cos\left(\omega t + \phi\right)$

  A, ϕ from initial conditions.

- this is an example of general description of particle motion in potential V

$$\frac{md^2x}{dt^2} = -\frac{\partial V\left(x\right)}{\partial x}$$

$\ldots\frac{dV}{dx}$ here - function of x only



V has a minimum – expect small oscillations about x_0 - find frequency ω of oscillations.

Lets (do properly) linearize about $x = x_0$ - Taylor expand

$x = x_0 + \delta x \qquad V\left(x_0\right)$ is a minimum,

$$\frac{dV\left(x\right)}{dx} = \frac{dV\left(x_0 + \delta x\right)}{dx} = \frac{dV}{dx} + \frac{d}{dx}\left(\frac{dV}{dx}\right)\delta x + \frac{d^2}{dx^2}\left(\frac{dV}{dx}\right)\frac{\delta x^2}{2} + \ldots$$

$r.h.s.$ evaluated at x_0

$$\frac{dV}{dx}\left(x\right) = \frac{d^2V\left(x_0\right)}{dx^2}\ \delta x + 0\left(\delta x^2\right)$$

So,

$$m\frac{d^2\delta x}{dt^2} = -\frac{d^2V(x_0)}{dx^2}\,\delta x + 0(\delta x^2) \qquad \frac{d^2V(x_0)}{dx^2} \equiv \frac{d^2}{dx^2}V(x)\Big|_{x=x_0}$$

$$= m\frac{d^2x}{dt^2}$$

'Linearize' means terms $0(\delta x^2) \ll 0(\delta x)$ etc.

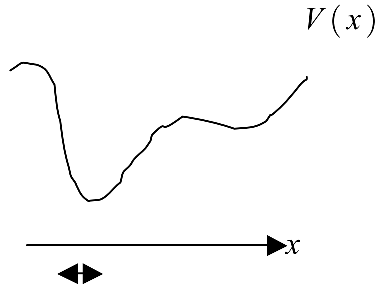
ie: all functions are linear in δx
equivalent to $\delta x \ll 1$ small displacements from x_0.

- recover linear pendulum equation with $\omega^2 = \frac{V''(x_0)}{m}, \frac{d^2\delta x}{dt^2} = -\frac{V''(x_0)\delta x}{m}$

* hence this linear equation is <u>universal</u> – works for <u>any</u> (conservative) system with a minimum in $V(x)$ provided δx is small enough. $(V'' > 0) \to \omega$ is real

[intuitively obvious – anything with potential ⇒

δx - oscillations



and know $\delta x(t)$ - oscillates about x_0
"roughly" sinusoidal]

Other advantage of <u>linear</u> equations – **principle of superposition**
ie: if you have the simplest situation – ie: one oscillator $\theta = A\cos(\omega t + \phi)$
any system can be obtained by

$$\theta(t) = \sum A_j \cos(\omega_j t + \phi_j)$$

which is why we have <u>Fourier</u> theory.

Here we have *j* linear coupled oscillators (normal modes) each one has two constants (A_j, ϕ_j) and one frequency ω_j.

Another equation you will have seen –

<u>wave equation</u> $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ *c* constant

Nonlinear of $c(\Psi)$ - dispersion, diffusion.
Linear equation 'works' for any waves, ie: water waves, light waves… universality.

Diffusion equations $\frac{\partial\psi}{\partial x^2} = \eta \frac{\partial\psi}{\partial t}$ η constant linear.

So linear equations – good news was

- <u>universality</u> – (works for anything)
- superposition – (only need simplest solution – just add them up)

- Not many *parameters*, eg:
  - oscillator – just ω
  - waves - just c
  - diffusion η
- Not many *variables* - eg: ψ, θ....

<u>Now</u> real life is <u>Nonlinear</u> - a problem!

- superposition fails (try it!) - bad news

<u>Good news:</u>
- still get universal equations (so don't need to learn many)
- only need few order parameters
- only need a few variables.

Last point $\rightarrow$ few variables – often this is because system appears to have *many* variables but relevant physics can be described by *few*

eg: Magnet – many spins – but we can understand it by just following M (total magnetisation).

Why this is so is a current hot topic in physics (self organisation, critical phenomena, complexity).

<u>Normalisation and Dimensional Analysis</u> – (more later)

Normalisation (boring but important)
Have written down universal equations eg:

wave equation $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ Normalised units!

experimentally you have displacement- x (meters)
time- t (seconds)

write $\psi = \psi^*\psi_0$ $t = t^*T$ T seconds
$x = x^*L$ L meters
ψ_0 - whatever ψ is measured in

then sub in
$$\frac{\partial^2\psi^*\psi_0}{\partial x^{*2}L^2} = \frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^{*2}}\frac{\psi_0}{T^2}$$
$$\frac{\partial^2\psi^*}{\partial x^{*2}} = \frac{\partial^2\psi^*}{\partial t^{*2}}\frac{1}{c^2}\cdot\frac{L^2}{T^2}$$

this is dimensionless if $c^2 = \frac{L^2}{T^2}$ $\quad c$ is a velocity

ψ is anything (this is why it is universal).

So, to do the maths you can work in (*) units – or to solve on a computer.
[Also found out what c meant – and if equation is correct!]