<u>Twice iterated tent map</u> : find corresponding map $M^2(x)$

1) Note there are points where the map $M(x)$ goes to zero or 1.

$M(x)$ $\quad M\left(\frac{1}{2}\right)=1 \quad M(0)=0 \quad M(1)=0$

also, $\quad M\left(\frac{1}{4}\right)=\frac{1}{2} \quad M\left(\frac{3}{4}\right)=\frac{1}{2}.$

2) There are points where $M^2(x)$ goes to zero or 1.

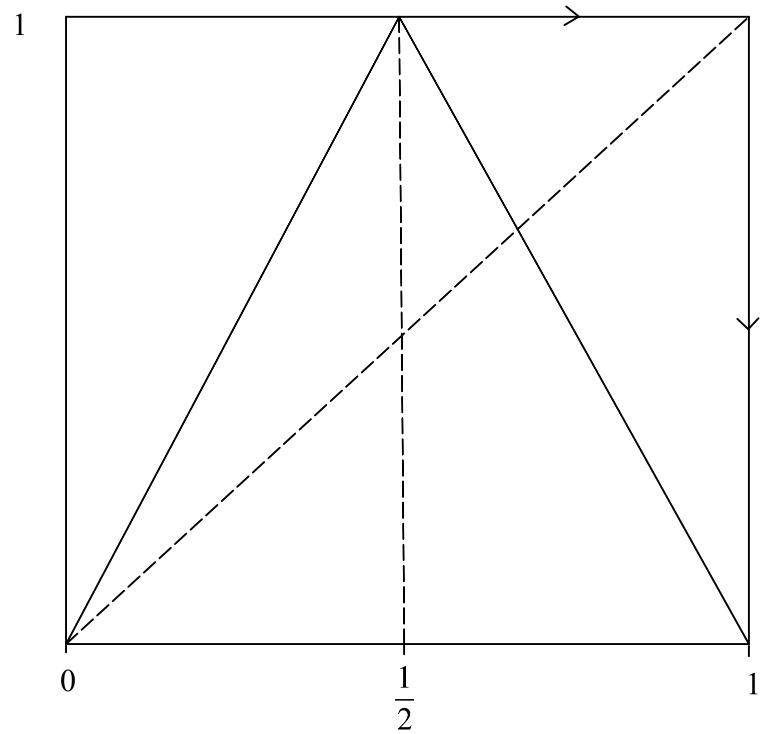
Clearly,

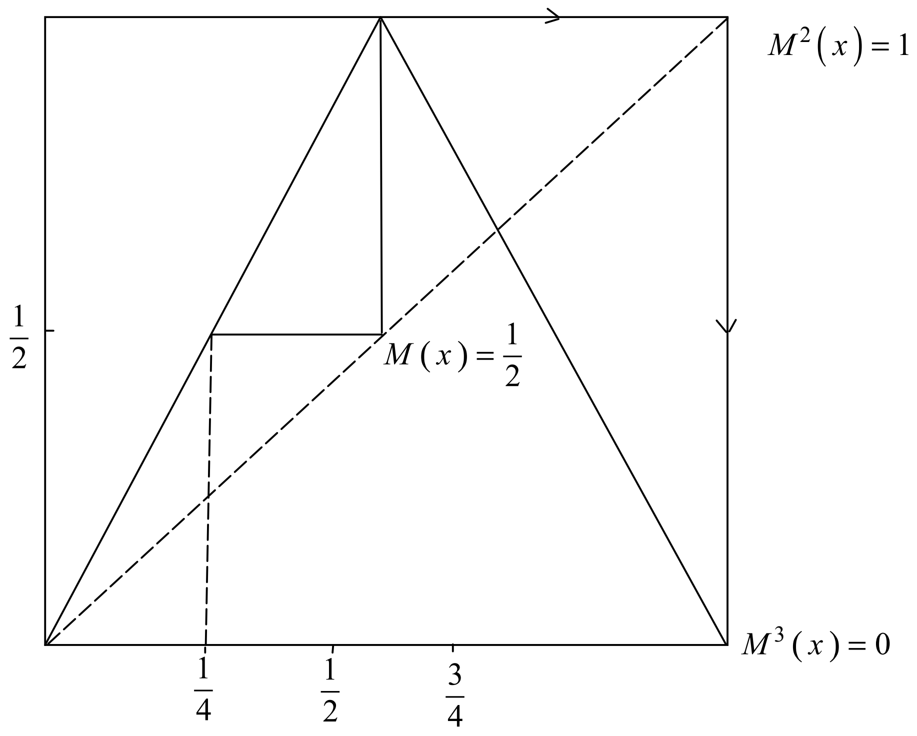
$$M^2\left(\tfrac{1}{4}\right)=M\left(M\left(\tfrac{1}{4}\right)\right)=M\left(\tfrac{1}{2}\right)=1$$
$$M^2\left(\tfrac{3}{4}\right)=M\left(M\left(\tfrac{3}{4}\right)\right)=M\left(\tfrac{1}{2}\right)=1$$

and

$$M^2\left(\tfrac{1}{2}\right)=M\left(M\left(\tfrac{1}{2}\right)\right)=M(1)=0$$
$$M^2(0)=M\left(M(0)\right)=M(0)=0$$

Folding points – another method.





$\frac{1}{4}$ iterates as $M\left(\frac{1}{4}\right) = \frac{1}{2}$ $\quad M^2\left(\frac{1}{4}\right) = 1$ $\quad M^3\left(\frac{1}{4}\right) = 0$

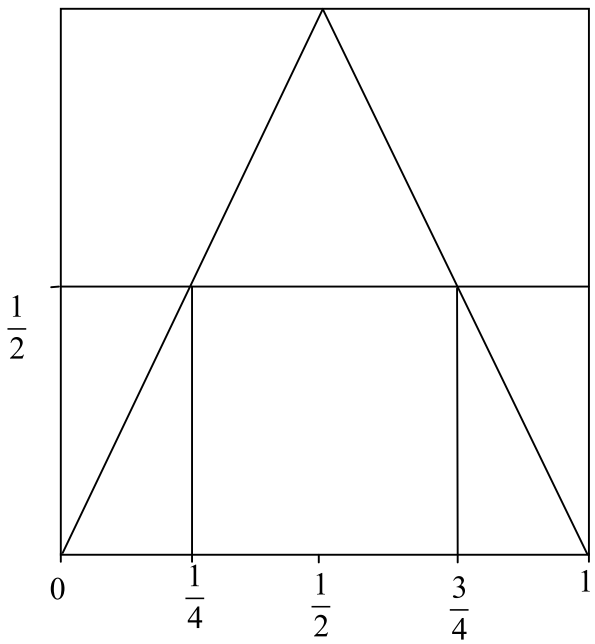
$\frac{1}{8}$ iterates as $M\left(\frac{1}{8}\right) = \frac{1}{4}$ and $M\left(\frac{1}{4}\right) = \frac{1}{2}$ $\quad M^2\left(\frac{1}{4}\right) = 1$ so $M^4\left(\frac{1}{8}\right) = 0$

$\frac{1}{16}$ iterates as $M\left(\frac{1}{16}\right) = \frac{1}{4}$ so similarly $M^5\left(\frac{1}{16}\right) = 0$

Now consider all the segments between, ie

$$0 - \frac{1}{4}, \quad \frac{1}{4} - \frac{1}{2}, \quad \frac{1}{2} - \frac{3}{4}, \quad \frac{3}{4} - 1.$$

Notation



Write $M(x)$ as

$M_<(x) = 2x$ "left half"

$M_>(x) = 2(1-x)$ "right half"

Clearly, $\left[0-\frac{1}{4}\right] \rightarrow \left[0-\frac{1}{2}\right]$ $M_<(x)$

and $\left[0-\frac{1}{2}\right] \rightarrow [0-1]$ $M_<(x)$

only need $M_<(x)$ here

$0 \leq x \leq \frac{1}{4}$ $M^2(x) = M_<^2(x)$

Now

$$\left[\frac{1}{4}-\frac{1}{2}\right] \rightarrow \left[\frac{1}{2}-1\right] \quad M_<(x)$$

$$\left[\frac{1}{2}-1\right] \rightarrow [1-0] \quad M_>(x)$$

so,

$$\frac{1}{4} \leq x \leq \frac{1}{2} \quad M^2(x) = M_>(M_<(x))$$

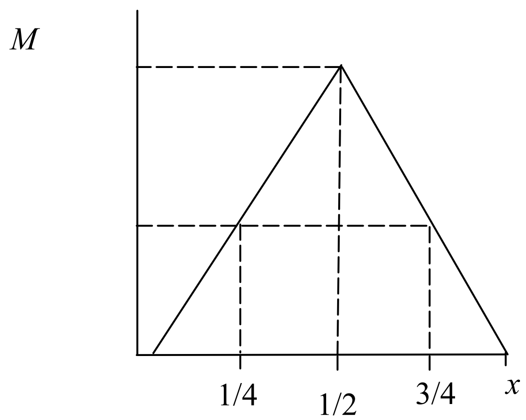
similarly

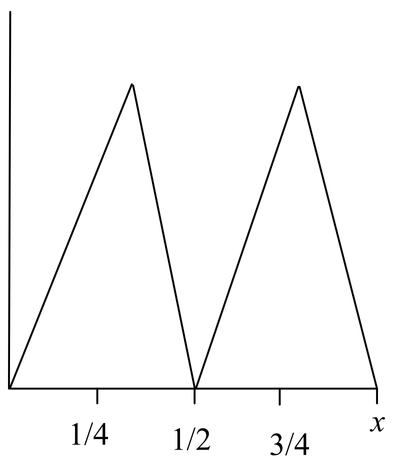
$$\frac{1}{2} \le x \le \frac{3}{4} \quad M^2(x) = M_>^2(x)$$
$$\frac{3}{4} \le x \le 1 \quad M^2(x) = M_<(M_>(x))$$

Then plug in $M_<, M_>$ to get the result.

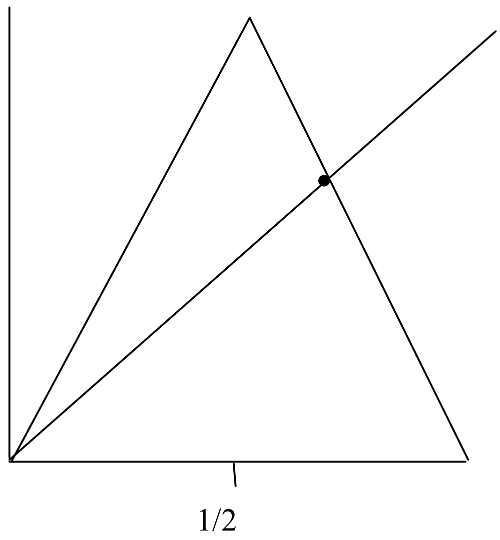
$$0 \le x \le \frac{1}{4} \quad M^2(x) = 2[2x] = 4x$$
$$\frac{1}{4} \le x \le \frac{1}{2} \quad M^2(x) = 2(1-[2x]) = 2-4x$$
$$\frac{1}{2} \le x \le \frac{3}{4} \quad M^2(x) = 2(1-[2(1-x)])$$
$$= 2(1-2+2x)$$
$$= 4x-2$$
$$\frac{3}{4} \le x \le 1 \quad M^2(x) = 2[2(1-x)]$$
$$= 4-4x$$

Sketch

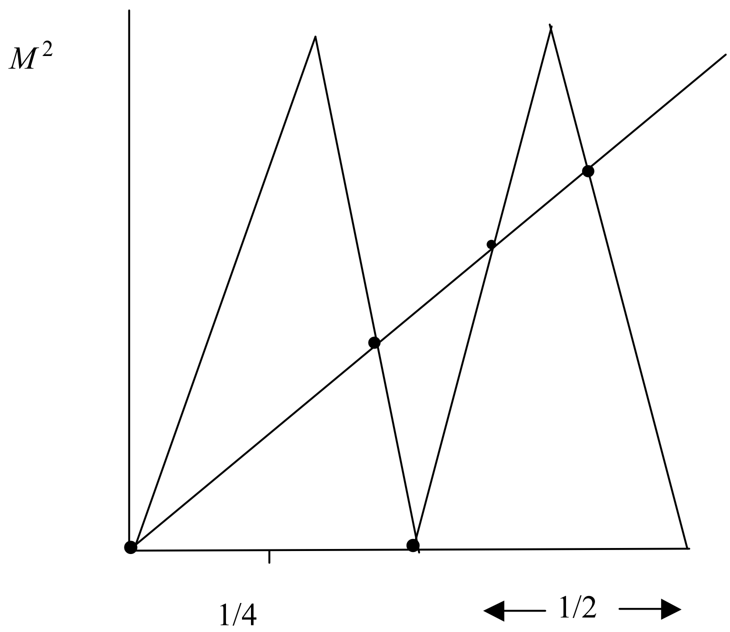




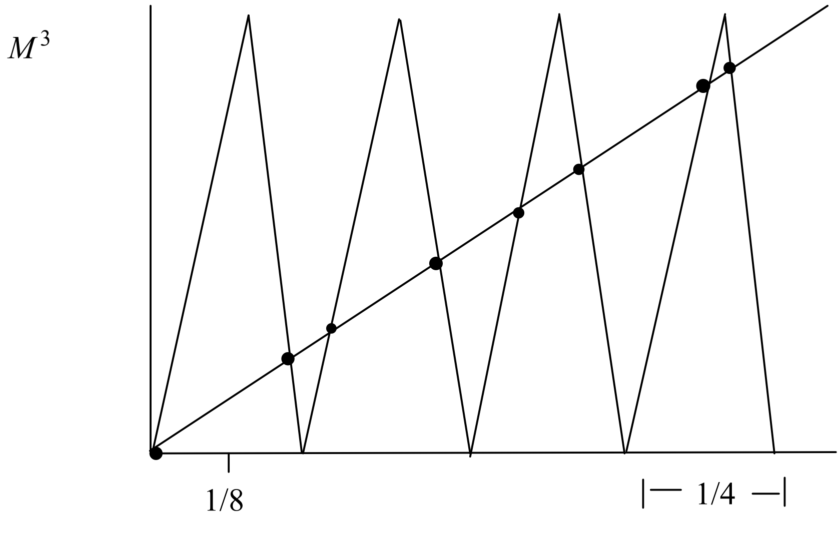
Note – neighbouring points on x move apart as length of $M(x)$ line increases as we iterate – stretch/fold.



1 triangle
2 fixed points

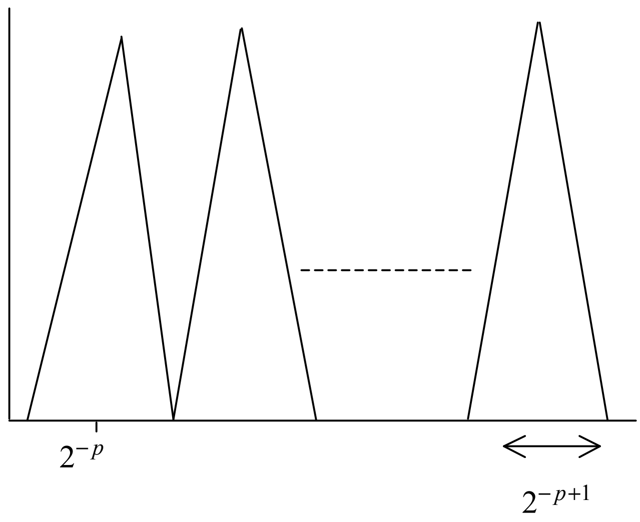


2 triangles
4 fixed points



4 triangles
8 fixed points

p^{th} iterate of map



2^{p-1} triangles

2^{p} fixed points

- base length

Therefore, any points in x separated by more than 2^{-p} can be <u>anywhere</u> in $[0,1]$ after p iterates.

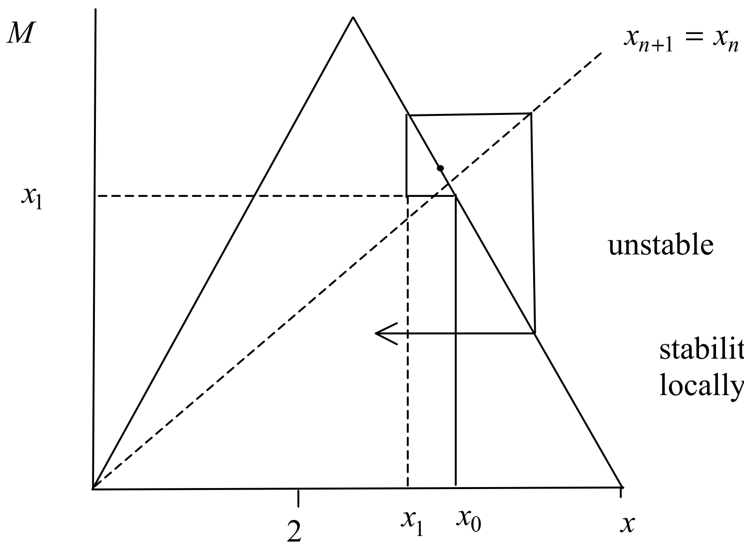
- chaos
- follows from inevitability of map.

Note, gradient is always $2^{p} > 1$
therefore, always globally unstable.

local gradient and stability

Note that this shows if a fixed point is unstable.

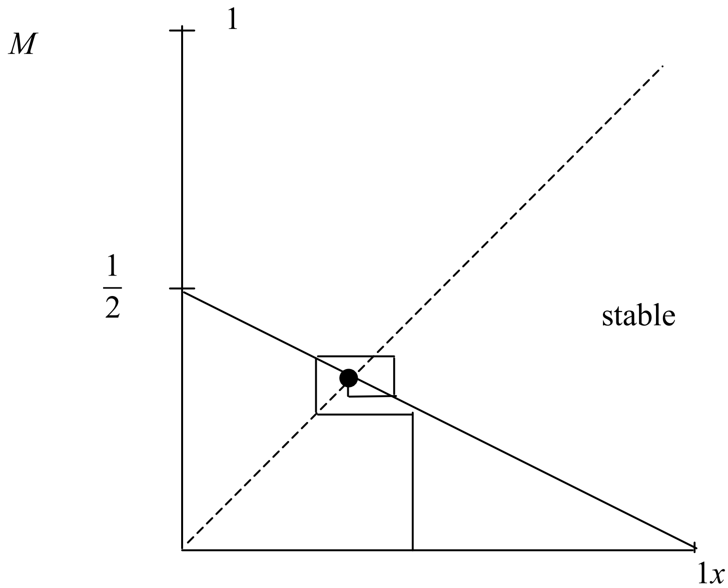
eg: tent map $x_{n+1} = 2x_n,\ x_{n+1} = 2(1 - x_n)$

- 

unstable

stability just depends on gradient locally

here |gradient|>|



here |gradient|<|

stable

Mathematically, for some map $M(x)$ consider small displacement δx_n from fixed point $\bar{x}$ and Taylor expand $x_{n+1} = M(x_n)$ at $x_n = \bar{x} + \delta x_n$.

$$x_{n+1} = M(\bar{x} + \delta x_n) \simeq M(\bar{x}) + \delta x_n \frac{dM}{dx}(\bar{x}) + 0(\delta x^2) \simeq \bar{x} + \delta x_n \frac{dM}{dx}(\bar{x})$$

but $\quad x_{n+1} = \bar{x} + \delta x_{n+1}$

so $\quad \bar{x} + \delta x_{n+1} = \bar{x} + \delta x_n \frac{dM}{dx}(\bar{x}) \qquad$ ie: $\frac{\delta x_{n+1}}{\delta x_n} = \frac{dM}{dx}(\bar{x})$.

So, for stability $\quad \left| \frac{dM}{dx}(\bar{x}) \right| < 1$.

Must hold for all iterates.. Lyapunov exponents

<u>Measure of divergence of trajectories</u>.

Lyapunov exponents (or Lyapunov number)

Consider a general map

$$x_{n+1} = f(x_n)$$

how fast are two initially neighbouring points converging/diverging?

Have initial condition x_0.

This then has iterates $x_1, x_2 ... x_n$

$$x_1 = f(x_0), \quad x_2 = f(x_1), \text{ etc.}$$

Consider an initially neighbouring point $\tilde{x}_0 = x_0 + \varepsilon_0$ separated from x_0 by $\varepsilon_0 \ll 1$.

After we iterate $\tilde{x}_1 = f(\tilde{x}_0) = f(x_0 + \varepsilon_0) = f(x_0) + \varepsilon_0 \frac{df}{dx}(x_0) + ...$, by Taylor expansion.

But, after one iterate $\tilde{x}_1 = x_1 + \varepsilon_1 \qquad$ Now two points are separated by ε_1

then $\quad \tilde{x}_1 = x_1 + \varepsilon_1 = f(x_0 + \varepsilon_0) = f(x_0) + \varepsilon_0 \frac{df}{dx}(x_0) + ...$

ie: $\quad \varepsilon_1 = \varepsilon_0 f'(x_0)$ to first order in ε_0.

So, generally for j^{th} iterate have

$$\tilde{x}_j = x_j + \varepsilon_j \qquad \varepsilon_j = \varepsilon_{j-1} f'(x_{j-1})$$

provided $\varepsilon_j \ll 1 \quad 0 < j < n$.

Then
$$\begin{aligned}\tilde{x}_n = x_n + \varepsilon_n &= x_n + \varepsilon_{n-1} f'(x_{n-1}) \\ &= x_n + \varepsilon_{n-2} f'(x_{n-2}) f'(x_{n-1}) \\ &= x_n + \varepsilon_0 f'(x_0) f'(x_1) ... f'(x_{n-1})\end{aligned}$$

$$\tilde{x}_{n+1} = x_{n+1} + \varepsilon_0 f'(x_0)...f'(x_n)$$

$$\tilde{x}_{n+1} = x_{n+1} + \varepsilon_0 \prod_{j=0}^{n} f'(x_j)$$

Now use a trick (handy to turn a product into a sum)

$$f'(x_j) = e^{\ln f'(n_j)}.$$

In addition, we do not know the signs of all $f'(x_j)$ (and we don't need to know – just interested in whether the product is getting bigger or not in magnitude).

So, write

$$\tilde{x}_n = x_n + \varepsilon_0 \exp\left(\sum_{j=0}^{n-1} \ln\left| f'(x_j) \right| \right),$$

and hence, define Lyapunov number (exponent)

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln\left| f'(x_j) \right|$$

NB: after written with
$n - 1 \simeq n$
$n \to \infty$

which is a measure of exponential divergence of trajectories

$$\tilde{x}_n - x_n = \varepsilon_0 e^{n\lambda}.$$

What about approximation $\varepsilon_j \ll 1$?

Can always take ε_0 (and thus ε_j) infinitesimally small – still same dynamics.

So $\lambda < 0$ - x is at an attractor – trajectories converge.
$\lambda > 0$ - x is at a repellor → chaos
trajectories diverge.

Note, there is a λ for each degree of freedom (coordinate) here one (x).

Note that generally if all $\left| f'(x_j) \right| > 1$ all iterates then all $\ln\left|f'\left(x_j\right)\right| > 0$

then $\sum \ln\left|f'\left(x_j\right)\right| > 0$ and $\lambda > 0$

chaotic

Similarly, if <u>all</u> $\left|f'\left(x_j\right)\right| < 1$

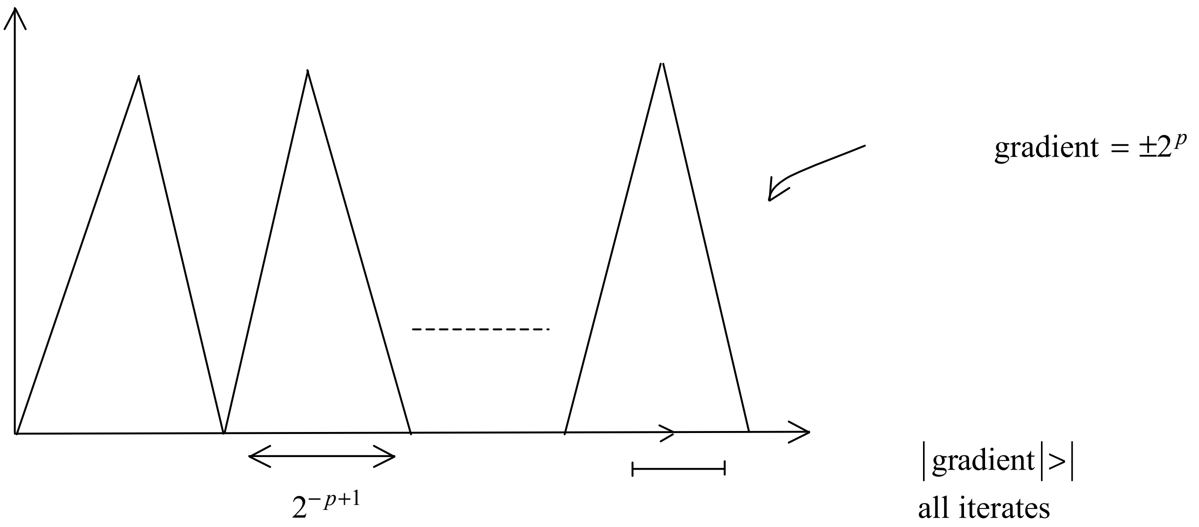
then all $\ln\left|f'\left(x_j\right)\right| < 0$

and $\lambda < 0$ - stable attractor

- if our graphical result that for each iterate if $|\text{gradient}| > 1$ unstable

* - now generalised this for all iterates of the map – characterised the dynamics. *

Tent map after p iterates:



* Therefore, tent map globally chaotic *

Next, consider polynomial maps. More complicated $f'\left(x\right)$