Logistic map Feigenbaum 1970's

- period doubling – bifurcation route to chaos
- a universal behaviour

$$x_{N+1} = \lambda x_N \left(1 - x_N\right) \qquad \lambda > 0$$

May 1976 - insect population dynamics

λ - birth rate

$-\lambda x_N^2$ - saturation/competition

1) Why not use corresponding differential equation instead?

- construct analogous continuous 1st order DE and investigate the dynamics

assume that x_N has a continuous limit $x(t)$

time $t = N\Delta$ $N = 0,1,2....$

so $x_N = x(N\Delta)$ $x_{N+1} = x(N\Delta + \Delta)$, etc.

Taylor expand $-\Delta$ is small – for continuous $x(t)$:

$$x(N\Delta + \Delta) = x(N\Delta) + \frac{d}{dt} x(N\Delta).\Delta + ... \qquad \text{(A)}$$

but this is also coincident with x_{N+1} etc., so for map:

$$\begin{aligned} x(N\Delta + \Delta) &= M\left(x(N\Delta)\right). \\ &= \lambda x(N\Delta)\left(1 - x(N\Delta)\right) \end{aligned} \qquad \text{(B)}$$

$\left(\text{A}\right) \equiv \left(\text{B}\right)$ and write $x(N\Delta) = x(t) = x$

$$\Delta \frac{dx}{dt} = (\lambda - 1) x - \lambda x^2$$

this is the continuous 1st order DE analogue to $M\left(x\right)$.

Look at fixed points, classify for continuous case.

Fixed points

$$\begin{aligned} (\lambda - 1)\bar{x} - \lambda \bar{x}^2 &= 0 \\ \bar{x}\left[(\lambda - 1) - \lambda \bar{x}\right] &= 0 \end{aligned}$$

so $\quad \bar{x} = 0, \quad \bar{x} = \dfrac{\lambda - 1}{\lambda}.$

Classify by sketching the phase plane

$$H(x) = \Delta \frac{dx}{dt} = (\lambda - 1)x - \lambda x^2$$

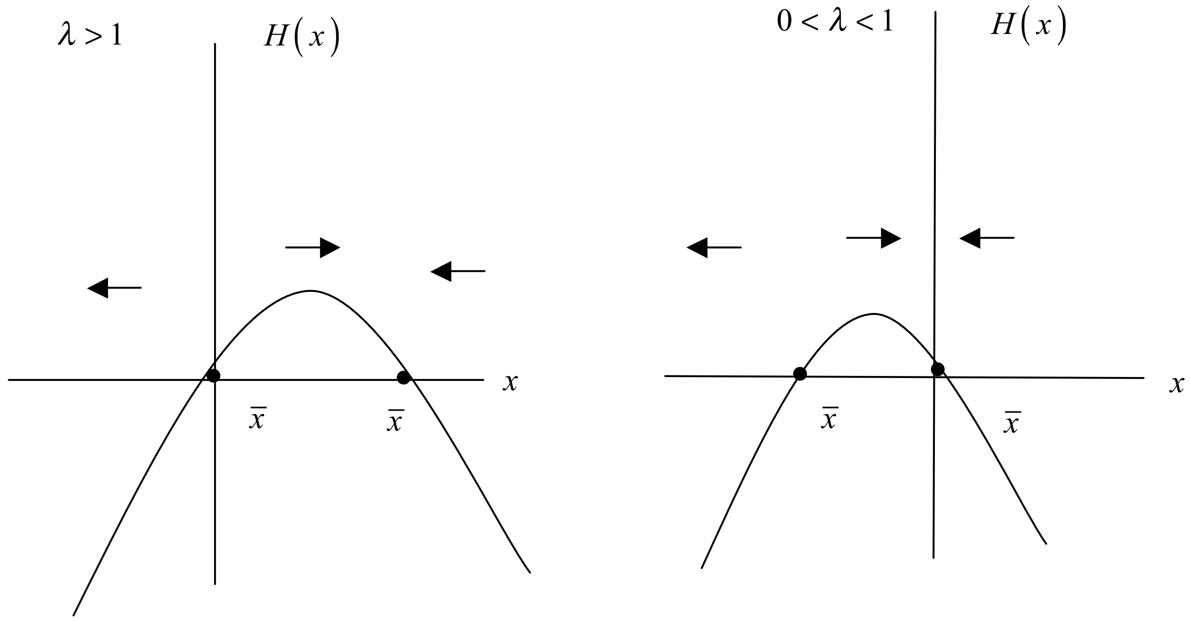
sketch $\quad H(x)$ *vz* $x \quad \Delta$ does not matter

$\lambda > 0 \quad$ so $H(x) \to -\infty \quad x \to \pm\infty$

t.p at $\quad \dfrac{dH}{dx} = 0 = (\lambda - 1) - 2\lambda x$

ie: at $x = \dfrac{\lambda - 1}{2\lambda} \quad$ or $\quad x = \dfrac{\bar{x}}{2}$

<u>Sketch of phase plane</u>



Then for the continuous DE:

$\lambda > 1$	$\bar{x} = 0$	repellor
	$\bar{x} = \dfrac{\lambda - 1}{\lambda}$	attractor
$0 < \lambda < 1$	$\bar{x} = 0$	attractor

$$\bar{x} = \frac{\lambda - 1}{\lambda} \qquad \text{repellor}$$

Now look at the difference equation (map)

$$x_{N+1} = \lambda x_N \left(1 - x_N\right)$$

fixed points at $\quad x_{N+1} = x_N = \bar{x}$

$$\bar{x} = \lambda \bar{x}(1 - \bar{x})$$

ie: $\quad \left(\lambda - 1\right)\bar{x} - \lambda \bar{x}^2 = 0 \quad$ - $\quad$ same expression as continuous DE.

Classify: $\quad$ Linearize the map

$$x_N = \bar{x} + \delta x_N$$
$$x_{N+1} = \bar{x} + \delta x_{N+1}$$

Sub into map

$$\bar{x} + \delta x_{N+1} = \lambda(\bar{x} + \delta x_N)(1 - \bar{x} - \delta x_N)$$
$$= \lambda\left[\bar{x}(1 - \bar{x}) - \bar{x}\delta x_N + (1 - \bar{x})\delta x_N - \delta x_N^2\right]$$

but at the fixed points $\bar{x} = \lambda \bar{x}(1 - \bar{x})$

so $\quad \delta x_{N+1} = \lambda \delta x_N (1 - 2\bar{x}) \qquad +0\left(\delta x_N^2\right)$

$$= [\lambda(1 - 2\bar{x})]^{N+1}\delta x_0$$

and at $\bar{x} = 0 \qquad \delta x_{N+1} = \lambda^{N+1}\delta x_0$

$\bar{x} = \frac{\lambda - 1}{\lambda} \qquad \delta x_{N+1} = \left[\lambda\left(\frac{\lambda - 2\lambda + 2}{\lambda}\right)\right]^{N+1}\delta x_0$

$$\delta x_{N+1} = (2 - \lambda)^{N+1}\delta x_0$$

so $\quad \bar{x} = 0 \qquad \lambda > 1 \qquad \bar{x} = 0 \qquad$ repellor

$\qquad\qquad\qquad 0 < \lambda < 1 \qquad \bar{x} = 0 \qquad$ attractor

-same as continuous DE

$\bar{x} = \frac{\lambda - 1}{\lambda} \quad$ - for $\quad 0 < \lambda < 1$ is a repellor- same as continuous DE

<u>however</u>, no longer an attractor for all $\lambda > 1$

- ***different*** to continuous DE

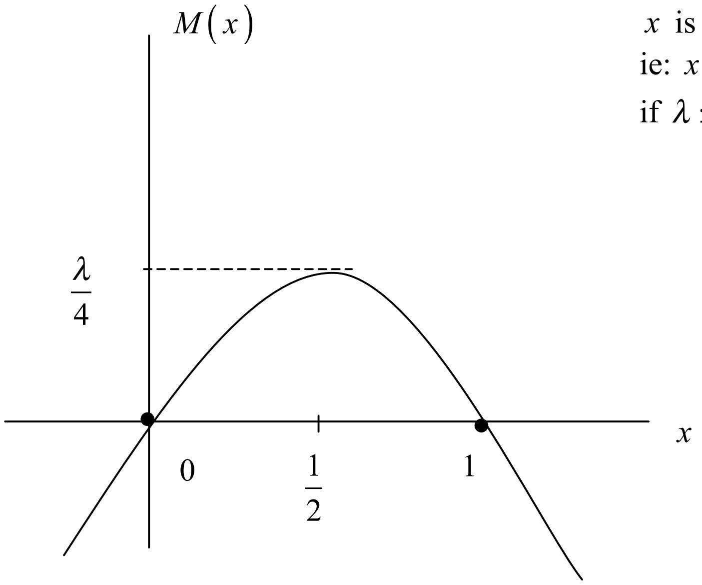
instead, for $|2-\lambda|<1$ i.e $1<\lambda<3$ <u>map</u> has attractor at $\bar{x}=\frac{\lambda-1}{\lambda}$

What happens for $\lambda>3$?

<u>Sketch of map:</u> $\lambda>1$

$M(x)=\lambda x(1-x)$
intercepts at $x=0,1$

$$\frac{dM}{dx}=\lambda-2\lambda x$$
$$=0 \text{ at } x=\frac{1}{2}$$
$$M\left(\tfrac{1}{2}\right)=\frac{\lambda}{2}\left(1-\frac{1}{2}\right)=\frac{\lambda}{4}$$



x is bounded
ie: $x\le[0,1]$
if $\lambda\le 4$

We will look in the range $3<\lambda<4$ (this is where all the interesting behaviour is).

<u>First</u> we can do $\lambda=4$

in this case $M^{p}(x)=[0,1]$ (like tent map).

Topologically, same as tent map as well – see this by change of variables

$$x=\sin^2\left(\frac{\pi y}{2}\right)=\frac{1}{2}[1-\cos(\pi y)]$$

$x=[0,1]$ as $y=[0,1]$

sub into $x_{N+1}=4x_N(1-x_N)$

$$\sin^2\left(\frac{\pi y_{N+1}}{2}\right)=2(1-\cos\pi y_N)\left(1-\frac{1}{2}+\frac{1}{2}\cos\pi y_N\right)$$

$$=(1-\cos\pi y_N)(1+\cos\pi y_N)=1-\cos^2(\pi y_N)=\sin^2(\pi y_N)$$

ie: $\sin^2\left(\frac{\pi y_{N+1}}{2}\right) = \sin^2\left(\pi y_N\right)$

so $\frac{\pi y_{N+1}}{2} = \pm\pi y_N + S\pi$ S integer

$y = \left[0,1\right]$ so can only have certain *s*.

$$\frac{y_{N+1}}{2} = \pm\, y_N + S$$

If $S = 0$ $y_{N+1} = 2y_N$ $0 \le y_N \le \frac{1}{2}$ + sign

$S = 1$ $y_{N+1} = 2\left(1 - y_N\right)$ $\frac{1}{2} \le y_N \le 1$ - sign

this is just the tent map.

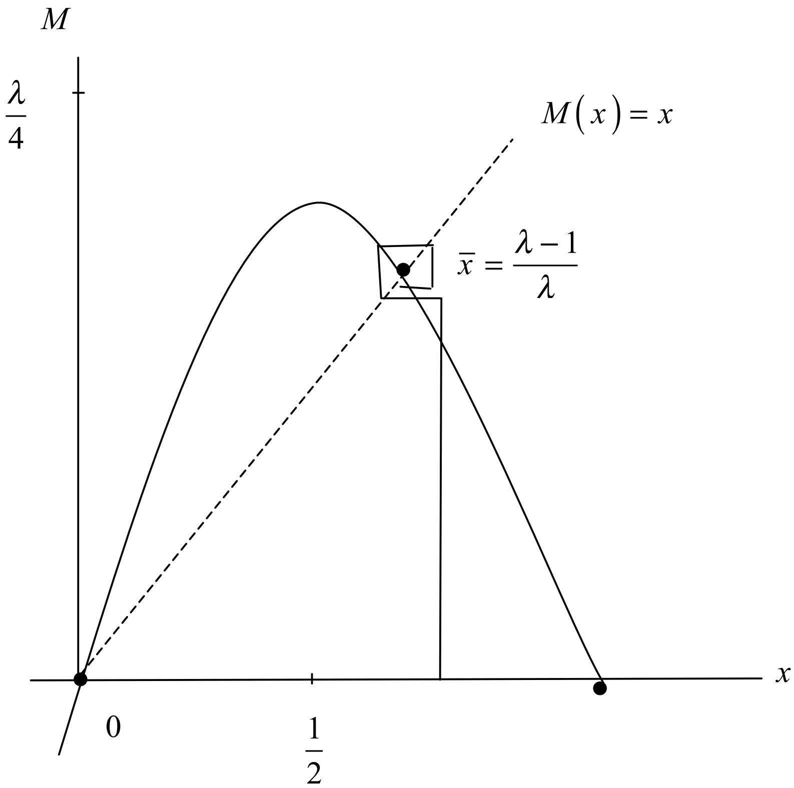
So, have shown there is global chaos at $\lambda = 4$

Problem – what happens as we go from $\lambda < 3$ attractor to $\lambda = 4$ - chaos?

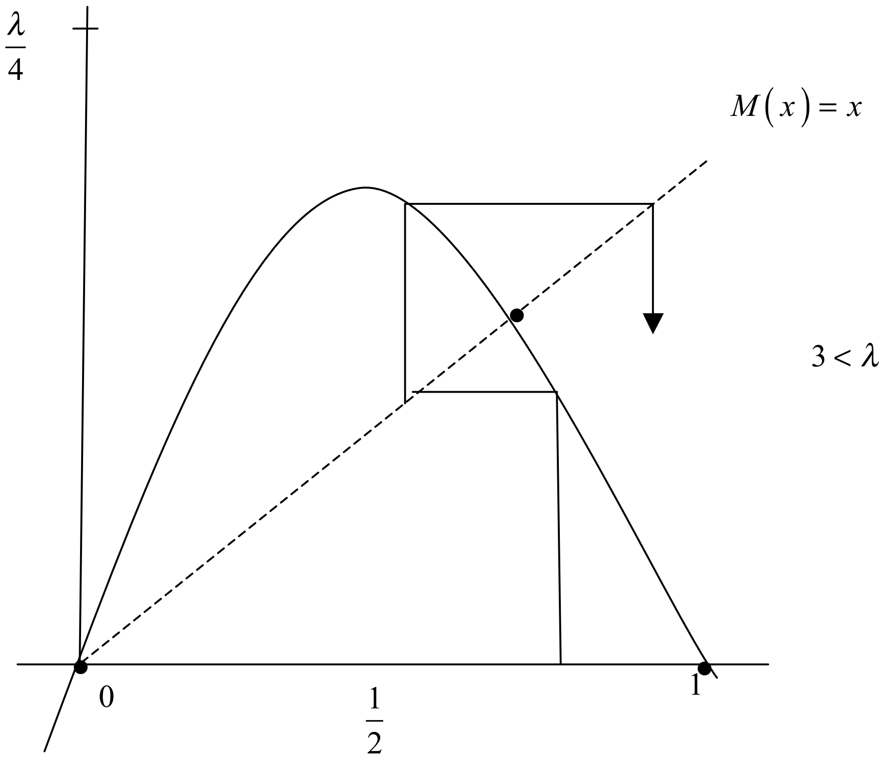
We will need to find the p^{th} iterated map M^p (or its essential properties).

<u>Iterates of the logistic map</u>

Sketch (use graphics)



slope < 1
stable
$1 < \lambda < 3$



$3 < \lambda < 4$

$\bar{x} = \frac{\lambda - 1}{\lambda}$ goes stable → unstable

attractor → repellor

as λ goes through 3.

Look at $M^2(x)$

$$\begin{aligned} M^2(x) &= M(M(x)) \\ &= \lambda M(1-M) \\ &= \lambda^2 x(1-x)(1-\lambda x(1-x)) \\ &\;\;\vdots \\ &= \lambda^2\left[x-(1+\lambda)x^2+2\lambda x^3-\lambda x^4\right] \end{aligned}$$

fixed points of $M^2(x)$ are $M^2(\overline{x})=\overline{x}$

or $\quad M^2(\overline{x})-\overline{x}=0$

or $\quad \lambda^2\left[\overline{x}-(1+\lambda)\overline{x}^2+2\lambda\overline{x}^3-\lambda\overline{x}^4\right]-\overline{x}=0$

so roots are $\quad \overline{x}=0$ and 3 others.

- 3 real or 1 real, 2 imaginary
- all depends on λ.

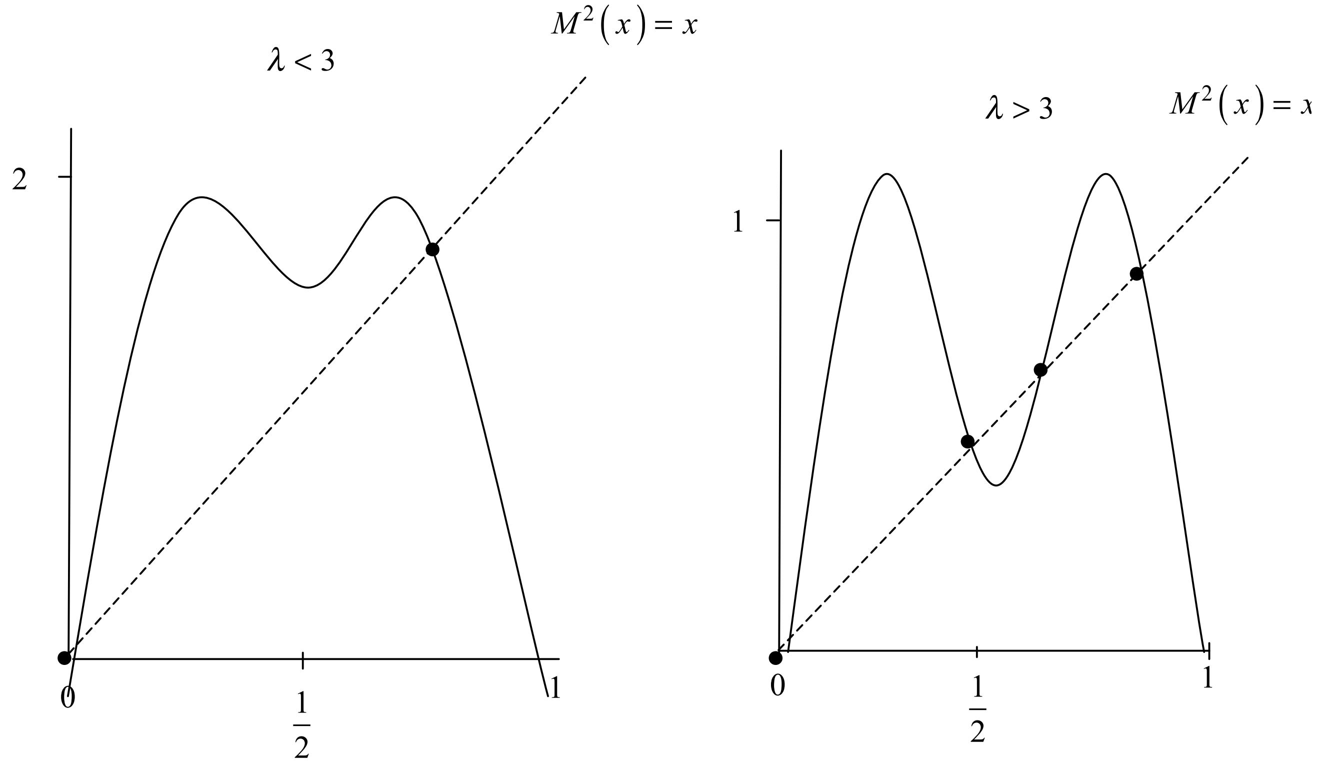
Sketch $M^2(x)$ - it is a quadratic

- it is symmetric about $x=\frac{1}{2}$ (because M is)

asymptotes $\quad M^2(x)\to-\infty \quad x\to\pm\infty$

intercepts at $x=0,1$

<u>Sketch</u> $M^2(x)$



Fixed points:

$\bar{x} = 0$ $+1$ $\bar{x} = 0$ $+3$

Now if $\bar{x}$ is a fixed point of M

must be fixed point of M^2 | call this x^*

(converse is not time)

$\lambda < 3$ $x^* = \frac{\lambda - 1}{\lambda}$ is attractor $\left|\frac{d\left(M^2\right)}{dx}\right|_{x^*} < 1$

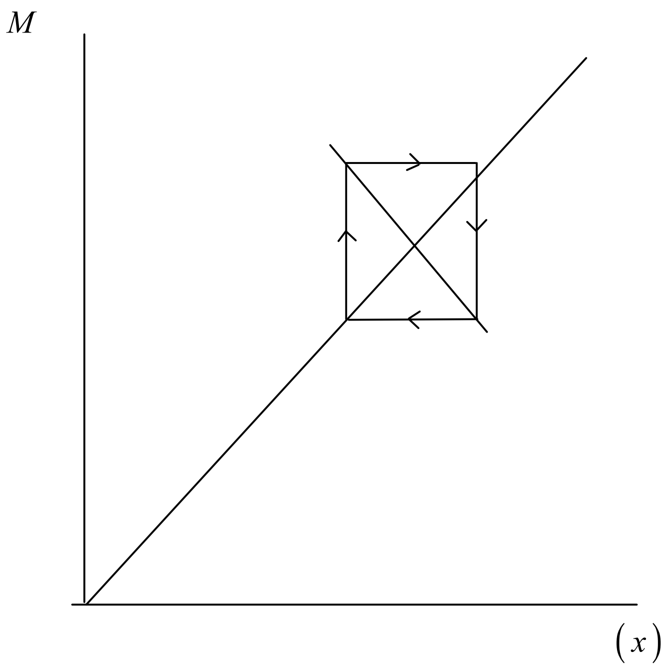
$\lambda > 3$ x^* is repellor $\left|\frac{d\left(M^2\right)}{dx}\right| > 1$ and $\left|\frac{dM}{dx}\right| > 1$

- but two new fixed points appear
  they may (may not) be attractive.

$3 < \lambda < 4 \qquad x^* = \dfrac{\lambda - 1}{\lambda}$ is repulsive fixed point of M, M^2....

but two new fixed points of M^2 appear at $\lambda > 3$ <u>not</u> fixed points of M .

If these are attractive this is period 2 cycle of M .



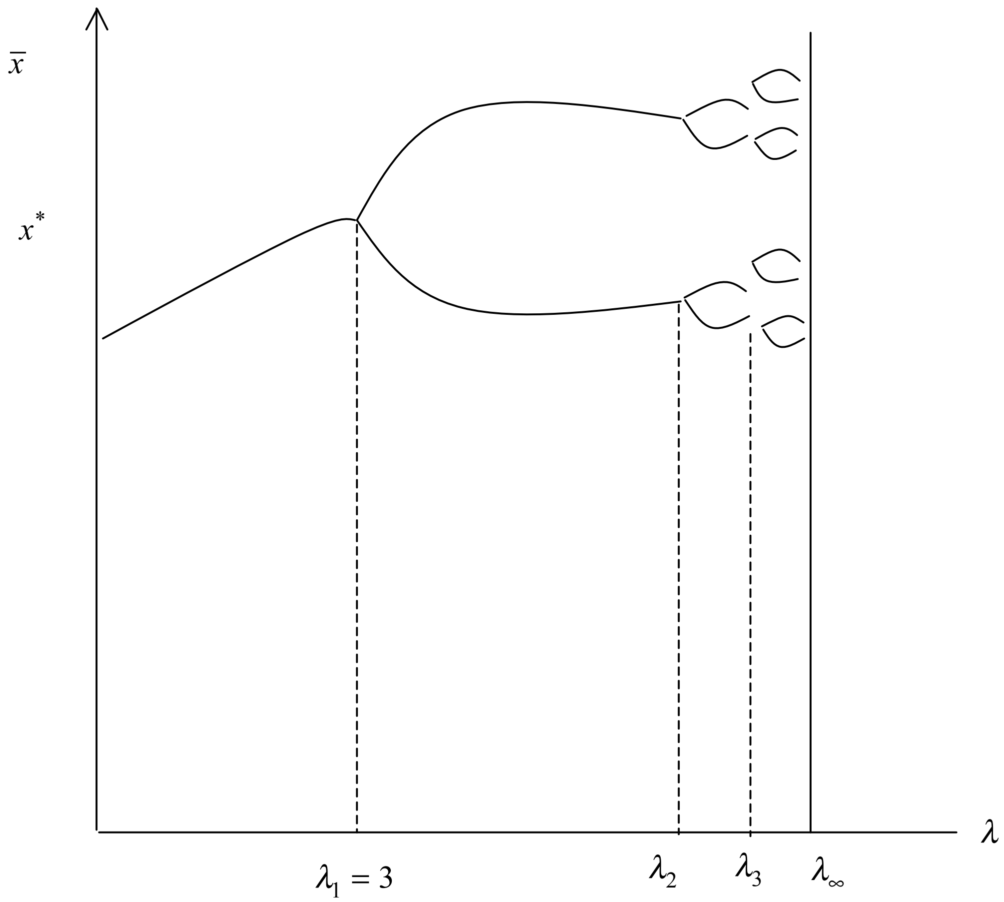
Period 2 in M is fixed point of M^2 .

Local gradients to $\bar{x}$, ie: λ must be "just right".

See handout for the full sequence….

Logistic map and Feigenbaum numbers

As λ increases there are successive period doublings



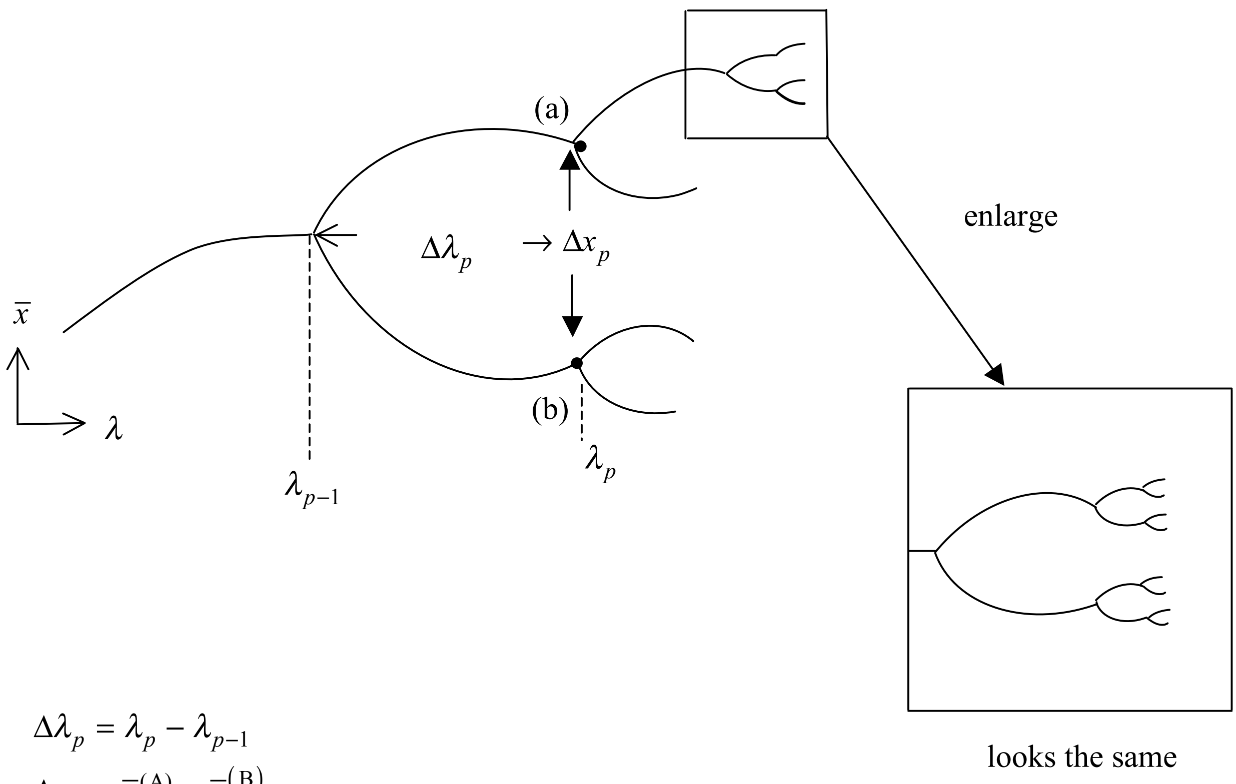
Pairs of bifurcated fixed points

Separated by Δx_p appear at λ_p

- looks "self similar".

Self – similar?

A piece of the sequence



$$\Delta\lambda_p = \lambda_p - \lambda_{p-1}$$
$$\Delta x_p = \bar{x}_p^{(A)} - \bar{x}_p^{(B)}$$

If self similar then

$$\frac{\Delta\lambda_p}{\Delta\lambda_{p+1}} = \text{const} \qquad \text{ratio independent of } p$$

$$\frac{\Delta x_p}{\Delta x_{p+1}} = \text{const} \qquad \text{indepenent of } p$$

Feigenbaum's result (numerical) – found by playing on calculator!

$$\frac{\Delta\lambda_p}{\Delta\lambda_{p+1}} \to \delta_F = 4.66920....$$

$$\frac{\Delta x_p}{\Delta x_{p+1}} \to \alpha_F = 2.59029.....$$

- these are "universal" numbers δ_F, α_F

- same for many such maps – originally thought for <u>all</u> maps. actually turn out to be for all <u>quadratic</u> maps. (Find why this is so.)

- this is a 'universality class' (show this next).
First note since δ_F, α_F are constants

show next

there is a termination to the sequence at λ_∞.

beyond this – global chaos.

(∞ bifurcations, ∞ close together)

+ some interesting islands of periodic 'superstable' behaviour.

What is λ_∞? (*just depends on δ_F and a const.*)

We have $\dfrac{\Delta\lambda_p}{\Delta\lambda_{p+1}} = \dfrac{\lambda_p - \lambda_{p-1}}{\lambda_{p+1} - \lambda_p} = \delta_F$

and $\lambda_\infty = \lim_{p\to\infty} \lambda_p$

now $\Delta\lambda_p = \dfrac{\Delta\lambda_{p-1}}{\delta_F} = \dfrac{\Delta\lambda_{p-2}}{\delta_F^2}$, etc.

So we write

$$\begin{aligned}\lambda_\infty &= \lambda_p + \Delta\lambda_{p+1} + \Delta\lambda_{p+2} + \ldots.\\ &= \lambda_p + \frac{\Delta\lambda_p}{\delta_F} + \frac{\Delta\lambda_p}{\delta_F^2} + \ldots.\end{aligned}$$

but $\delta_F > 1$ so roughly (lowest order in δ_F)

$$\lambda_\infty \simeq \lambda_p + \frac{\Delta\lambda_p}{\delta_F} = \lambda_p + \frac{\Delta_0}{\delta_F^p}$$

or $$\lambda_p = \lambda_\infty - \frac{C}{\delta_F^p}$$ C = some const, ie: Δ_0

thus λ_∞ depends on details of the maps, ie: C- Not universal.
Universal behaviour is the period doubling $\left(\lambda_F, \delta_F\right)$, sequence and the existence of some λ_∞.

<u>Other universality classes:</u>

There is a family of maps

$$f\left(x\right) = 1 - a\left|x\right|^q$$

- different δ_F, α_F one for each q.