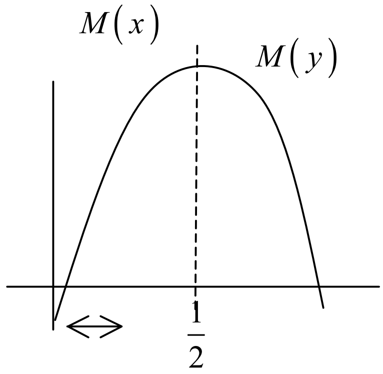
Logistic map - *do many iterations of the map*

Why is it universal? An example of RG



shift and rescale

- linear transformation
  symmetry about $y = 0$

First transform $M(x)$ to get simpler for

$$M(x) = x_{n+1} = \lambda x_n (1 - x_n)$$

transform with $y = \frac{\lambda}{a}\left(x - \frac{1}{2}\right)$

$y = 0$ is at the max.

so $\frac{ay}{\lambda} + \frac{1}{2} = x$

sub in to $M(x)$ - find a

$$\frac{ay_{n+1}}{\lambda} + \frac{1}{2} = \lambda\left[\frac{ay_n}{\lambda} + \frac{1}{2}\right]\left[1 - \frac{ay_n}{\lambda} - \frac{1}{2}\right]$$

$$\frac{ay_{n+1}}{\lambda} = \lambda\left[-\frac{a^2 y_n^2}{\lambda^2} + \frac{1}{4}\right] - \frac{1}{2}$$

$$y_{n+1} = \frac{\lambda^2}{a}\left[-\frac{a^2 y_n^2}{\lambda^2} + \frac{1}{4}\right] - \frac{1}{2}\cdot\frac{\lambda}{a}$$

$$= -ay_n^2 + \left[\frac{\lambda^2}{4a} - \frac{1}{2}\frac{\lambda}{a}\right]$$

$y_{n+1} = 1 - ay_n^2$ if $\frac{1}{a}\left[\frac{\lambda^2}{4} - \frac{\lambda}{2}\right] = 1$

or $a = \lambda\left[\frac{\lambda}{4} - \frac{1}{2}\right]$

i.e. $a = \frac{\lambda}{4}[\lambda - 2]$

* important (for later) we make a particular simplifying choice of $a(\lambda)$.

So now work with

$$y_{n+1} = 1 - ay_n^2 = M(y)$$

then
$$\begin{aligned} M^2(y) = y_{n+2} &= 1 - ay_{n+1}^2 \\ &= 1 - a\left(1 - ay_n^2\right)^2 \end{aligned}$$

$$\begin{aligned} y_{n+2} &= 1 - a\left[1 - 2ay_n^2 + a^2 y_n^4\right] \\ &= 1 - a + 2a^2 y_n^2 - a^3 y_n^4 \end{aligned}.$$

We just are interested in behaviour about the maximum in $M(x)$. This is $x = \frac{1}{2}$ ie: $y = 0$

so can neglect y^4 term.

Now $y_{n+2} \simeq (1-a) + 2a^2 y_n^2 = M^2(y)$.

Let's transform this back to the form for $M(y)$

write
$$\frac{y_{n+2}}{(1-a)} = 1 + \frac{2a^2 y_n^2}{(1-a)}$$

change variables
$$\tilde{y}_{m+1} = \frac{y_{n+2}}{(1-a)} \qquad \tilde{y}_m = \frac{y_n}{(1-a)}$$

* where one step in $m \equiv 2$ steps in n.....

Then sub in

$$\tilde{y}_{m+1} = 1 + \frac{2a^2}{(1-a)}(1-a)^2 \tilde{y}_m^2$$

and if we choose $\tilde{a} = -2a^2(1-a)$.

We have

$$\tilde{y}_{m+1} = 1 - \tilde{a}\tilde{y}_m^2$$ $\equiv M^2(y)$ and $M(\tilde{y})$, i.e. $M^2(y)$ is in same form as $M(\tilde{y})$.

NB: each step $n \to m$ is a <u>rescaling</u>.

We can do this as many times as we like – thus all the maps:

$$M^1(y),\ M^2(y),\ M^4(y),\ M^8(y),\ M^{2^P}(y)$$

really going from one bifurcation to the next – in both a and λ

Can be written in the form

$$\tilde{y}_{k+1} = 1 - a_p \tilde{y}_k^2 \equiv M^{2^P}(y)$$

where $a_{p+1} = 2a_p^2(a_p - 1)$ - a map for a_p!

So, every time we go to the next bifurcation
rescale <u>a</u> [recall $a = a(\lambda)$].

Now we know that sequence does not go on forever, it terminates at some λ_∞.

Since $a = a(\lambda)$ there is some "fixed point" of a_p corresponding to λ_∞.

This is just the fixed point of $a_{p+1} = 2a_p^2(a_p - 1)$.

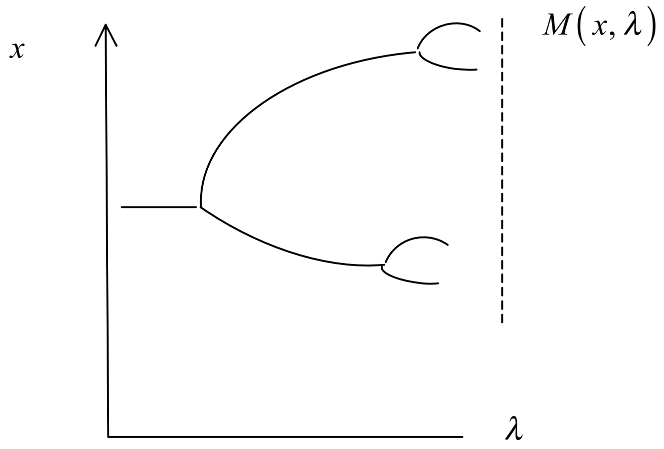
When $a_{p+1} = a_p = \bar{a}$ then $\bar{a} = 2\bar{a}^2(\bar{a} - 1)$

$$0 = 2\bar{a}^2 - 2\bar{a} - 1$$

ie: $$\bar{a} = a_c = \frac{1+\sqrt{3}}{2}$$

$$a_p > 0 \text{ for all } p$$

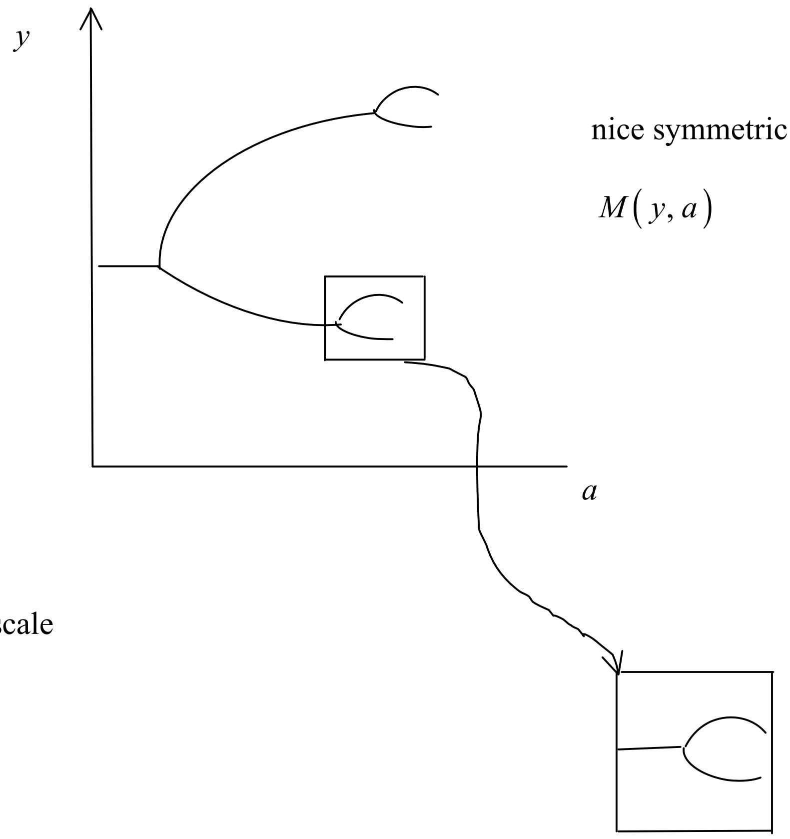
So RG (in this context) started with



transform – linear transformation $x \rightarrow y$ $\lambda \rightarrow a$

Note that – finally
- equivalent procedure is Taylor expansion close to fixed points of M^{2^P}

$\rightarrow$ universality $\rightarrow$
applies to any map of this sequence (2^P)



nice symmetric map

$M(y, a)$

then rescale

$M(\tilde{y}) = 1 - \tilde{a}\tilde{y}^2$

Note that the same topology – pitchfork bifurcation – occurs every 2^P iterates – ie: once for every M^{2^P} as we vary $\tilde{a}$. (eg: $M \to M^2$ - one bifurcation).

Hence, the rescaling $M \to M^2 \to M^4,\ M^8,\ M^{2^P}$ generates the entire bifurcation sequence.

ie: $M(y) = 1 - a_p y^2$ with rescale $a_{p+1} = 2a_p^2(a_p - 1)$.

Then just look for fixed point in ↗ to find a_c, λ_c.

This procedure will work for any $M = 1 - a|x|^q$ - we expanded to $0(x^2)$