<u>Buckingham II theorem- see also handout</u>

Physics from dimensional analysis.

Example – pendulum.

Say we don't know the equation of motion (pre Galileo 1602), we observe – guess/intuition, etc.

Period $\tau = f\left(m, L, g, \theta_0 ...\right)$

mass (m), length (L), grav. accel'n (g), angle of release (θ_0)

Now the dimensions must balance so

$$\tau \sim m^\alpha l^\beta g^\gamma \theta_0^\delta$$

ie: $\left[\mathrm{T}\right] \equiv \left[\mathrm{M}\right]^\alpha \left[\mathrm{L}\right]^\beta \left[\mathrm{LT}^{-2}\right]^\gamma \left[rad\right]^\delta$

$[rad]$: dimensionless

$\left[\mathrm{M}\right], \left[\mathrm{L}\right], \left[\mathrm{T}\right]$ physical dimensions, eg: kg, m, s

Equate powers

T:	$1 = -2\gamma$	$\gamma = -\frac{1}{2}$
M:	$0 = \alpha$	$\alpha = 0$
L:	$0 = \beta + \gamma$	$\beta = +\frac{1}{2}$

rads are dimensionless $\delta = 0$

So $\tau \sim \sqrt{\frac{l}{g}}$ independent of m and θ_0

Galileo's famous observation

Take this idea further –

"Buckingham Π Theorem" 1914

Procedure

1) Guess the relevant quantities

2) System with N variables, R independent dimensions, has a solution which is function of N-R groups, π_{N-R} of variables that dimensionless.

3) Consider the simplest π_k (Occam's razor).

Do this for the (full nonlinear) pendulum

Variable	dimension	what it is
θ_0	-	angle of release
m	$[\mathrm{M}]$	mass of bob
τ	$[\mathrm{T}]$	period
g	$[\mathrm{L}][\mathrm{T}]^{-2}$	grav. acceln
l	$[\mathrm{L}]$	length of pendulum

$N = 5$ variables
$R = 3$ dimensions $([\mathrm{M}],[\mathrm{L}],[\mathrm{T}])$

Therefore, $N - R = 2$ possible groups of variables $\pi_{1,2}$

Now θ_0 - dimensionless so let $\pi_1 = \theta_0$

None of the other variables has dimension M so π_2 must be independent of m.

So $\pi_2 = h(\tau, g, L)$ which is dimensionless;

e.g.: $$\pi_2 = \left(\frac{\tau^2 g}{L}\right)^\alpha \equiv \left(\frac{[\mathrm{T}]^2[\mathrm{L}][\mathrm{T}]^{-2}}{[\mathrm{L}]}\right)^\alpha$$

dimensionless

simplest choice $\alpha = 1$

$$\pi_2 = \frac{\tau^2 g}{L}$$

Then the pendulum's motion is given by the function

$$f(\pi_1,\pi_2) = f\left(\theta_0, \frac{\tau^2 g}{L}\right) = C$$

What is *f*?

Guess that $\pi_2 = f_0(\pi_1)$ or $\tau = f_1(\theta_0)\sqrt{\frac{l}{g}}$

NB: $f_1(\theta_0)$ is <u>universal</u>, ie the same for any l, g.

What is $f_1(\theta_0)$?

Recall for the nonlinear undamped pendulum there is a constant of the motion

$$E = \frac{y^2}{2} - \omega^2 \cos\theta$$

where $y = \frac{d\theta}{dt}, \omega = \sqrt{\frac{g}{l}}$.

Now θ_0 is the angle of release of the pendulum, ie: $y = 0$ at $\theta = \theta_0$

$$E = -\omega^2 \cos\theta_0$$

then

$$y^2 = \left(\frac{d\theta}{dt}\right)^2 = 2E + 2\omega^2\cos\theta = 2\omega^2\left(\cos\theta - \cos\theta_0\right)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\left(\cos\theta - \cos\theta_0\right)^{1/2}$$

$$dt = \frac{d\theta}{\left(\cos\theta - \cos\theta_0\right)^{1/2}}\sqrt{\frac{l}{2g}}$$

and the τ is time for pendulum to go $\theta = 0 - \theta_0 \times 4$

so $$\tau = \int_0^\tau dt = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\left(\cos\theta - \cos\theta_0\right)^{1/2}}$$

ie: $$f_1(\theta_0) = 2\sqrt{2}\int_0^{\theta_0}\frac{d\theta}{\left(\cos\theta - \cos\theta_0\right)^{1/2}}$$

so Π theorem → there is some $\tau = f_1(\theta_0)\sqrt{\frac{l}{g}}$

$f_1(\theta_0)$ - universal

energy equation → $f_1(\theta_0)$.

Π Theorem and Turbulence

Kolmogorov 1941 ideal incompressible Hydrodynamic turbulence

Variable	dimension	description
$E(k)$	$[\mathrm{L}]^3[\mathrm{T}]^{-2}$	energy/unit wave number
ε_0	$[\mathrm{L}]^2[\mathrm{T}]^{-3}$	rate of energy input
k	$[\mathrm{L}]^{-1}$	wave number

$N = 3$ $R = 2$ so one group

$$\pi_1 = \frac{E^3 h^5}{\varepsilon_0^2} = \frac{\left([\mathrm{L}]^3[\mathrm{T}]^{-2}\right)^3}{\left([\mathrm{L}]^2[\mathrm{T}]^{-3}\right)^2} \times \frac{1}{[\mathrm{L}]^5}$$

choose $\pi_1 = \text{const}$

$$\rightarrow \quad E = C\varepsilon_0^{2/3}\, k^{-5/3}$$

NB: this determines the universal $-5/3$ exponent of ideal Kolmogorov turbulence

– does not depend on details eg viscosity.
- does not determine *C*.

[Actually can get *C* from Navier-Stokes- Kolmogorov's "4/5 law"

Now add a variable – Magnetohydrodynamic (MHD) turbulence

Variable	Dimension	Description
$E(k)$	$[L]^3[T]^{-2}$	energy/unit wave n_0
ε_0	$[L]^2[T]^{-3}$	rate of energy $\frac{1}{p}$
k	$[L]^{-1}$	wave n_0
V	$[L][T]^{-1}$	characteristic speed

$$\pi_1 = \frac{E^3k^5}{\varepsilon_0^2} \qquad \pi_2 = \frac{V^2}{Ek} = \frac{[L]^2[T]^{-2}}{[L]^3[T]^{-2}[L]}$$

Now, let $\pi_1 \sim \pi_2^{\alpha}$ since we are interested in turbulence which is scaling

so $\quad E(k) \sim k^{-(5+\alpha)/(3+\alpha)}$

α is now not determined – depends on the detailed phenomenology

anomalous scaling:

may not be universal.

NB we can obtain the control parameter for turbulence in the same way:

Reynolds number for fluid turbulence

Variable	Dimension	Description
L_0	$[L]$	driving scale
η	$[L]$	dissipation scale
u	$[L][T]^{-1}$	bulk driving (flow) speed
υ	$[L]^2[T]^{-1}$	viscosity
$P = 4 \quad R = 2$	$M = 2$	

$$\pi_1 = \frac{UL_0}{\upsilon} = \frac{[L][T]^{-1}[L]}{[L]^2[T]^{-1}} = \text{Reynolds Number}$$

$$\pi_2 = \frac{L_0}{\eta}$$

how can we relate π_1, π_2?

Insist on steady state so energy transfer rate same on all scales (energy rate in = energy rate out).

Mass normalised so

energy rate in $\quad \varepsilon_{inj} = \frac{U^2}{T_0} = \frac{U^3}{L_0} \qquad U = \frac{L_0}{T_0}$

energy rate out – at viscous scale – obtain from Navier Stokes equation

$$\varepsilon_{diss} = \frac{\upsilon^3}{\eta^4}$$

then steady state $\Rightarrow \varepsilon_{inj} \sim \varepsilon_{diss}$ so

$$\frac{U^3}{L_0} \sim \frac{\upsilon^3}{\eta^4}$$

or $\quad \frac{U^3 L_0^3}{\upsilon^3} \sim \frac{L_0^4}{\eta^4} \qquad \frac{UL_0}{\upsilon} \sim \left(\frac{L_0}{\eta}\right)^{4/3}$

ie steady state implies that: $\boxed{\pi_1 = \pi_2^{4/3}}$

thus the number of excited modes or degrees of freedom (level of disorder) increases with the Reynolds number.