1st order non-linear differential equations

- Understand the "mechanics" by looking at simple example for least number of variables.
- One dimensional – ie: only one q, system has potential $V(q)$.
- 1st order differential equation.
- Non-linear.
- Then look at more complicated systems.

Two possibilities – deterministic, ie: given $q(t = t_0)$ at some t_0 we have an equation that uniquely determines $q(t)$ for all subsequent t. Only need to look at one trajectory.

- Stochastic – equation contains terms that are only known statistically (ie, due to random noise). Need to look at an ensemble of trajectories.

Deterministic

Consider example (but this method will solve any 1D ODE)

$$\frac{dq}{dt} = \alpha q - \beta q^3 \qquad \alpha, \beta \text{ constant} \quad \beta > 0$$

This is a simple model for:-

1) population dynamics

$q(t)$	=	population density
α	=	initial growth rate (eg: birth rate)
β	=	saturation term – hinders growth when population density is high.

- this is a simple model – could replace βq^3 by something more realistic

2) material in strong E field

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

for B uniform $\mathbf{J} = -\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

usually in EM assume linear response of media.

$\mathbf{J} = \sigma \mathbf{E}$ $\quad \sigma$ constant : Ohm's Law.

Nonlinear response $\quad \sigma = \sigma_0 + \sigma_1 E^2 \Rightarrow \frac{\partial E}{\partial t} = \alpha E - \beta E^3$

3) single mode laser $\quad q \propto I$ intensity of coherent radiation

$\alpha \equiv$ energy input (incoherent radiation)

$\beta \equiv$ energy losses

Use example to introduce **linear stability analysis**

$$\frac{dq}{dt} = \alpha q - \beta q^3 \qquad \text{consts: } \beta > 0 \text{, all } \alpha$$

Step 1:

seek time in dependent solutions "fixed points" $\overline{q}$

NB: not necessarily equilibria

here $\quad \dfrac{d\overline{q}}{dt} = 0 \quad$ ie $\quad \alpha\overline{q} - \beta\overline{q}^3 = 0 \quad$ or $\quad \overline{q}\left(\alpha - \beta\overline{q}^2\right) = 0$

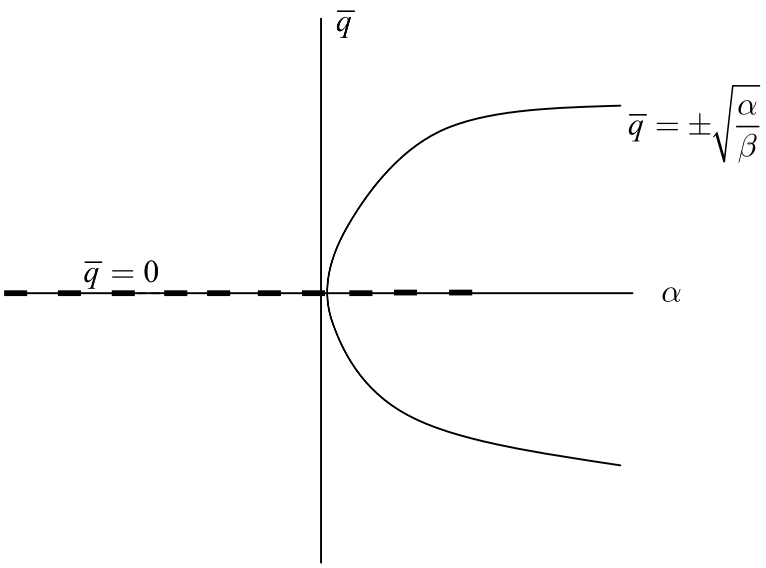
then $\quad \overline{q} = 0 \qquad \overline{q} = \pm\sqrt{\dfrac{\alpha}{\beta}} \qquad \alpha > 0,\ \beta > 0$

$\qquad \overline{q} = \pm i\sqrt{\dfrac{|\alpha|}{\beta}} \qquad \alpha < 0,\ \beta > 0$

but insist $\overline{q}$ real in our model.

Then $\quad \overline{q} = 0, \quad \overline{q} = \pm\sqrt{\dfrac{\alpha}{\beta}} \qquad$ roots both real $\alpha > 0$

(all $\alpha, \beta > 0$), $\quad (\alpha > 0,\ \beta > 0)$



bifurcation at
$\overline{q} = 0$
$\alpha = 0$

Step 2

Examine stability of fixed points. Look in vicinity of $\overline{q}$, ie: $q(t) = \overline{q} + \delta q(t)$

$\overline{q} = \text{const}$ $\quad \delta q$ small

sub in original equation

$$\frac{d}{dt}(\delta q) = \alpha(\overline{q} + \delta q) - \beta(\overline{q} + \delta q)^3$$

$$= \alpha(\overline{q} + \delta q) - \beta(\overline{q} + \delta q)(\overline{q}^2 + 2\overline{q}\delta q + \delta q^2)$$
$$= \alpha(\overline{q} + \delta q) - \beta(\overline{q}^3 + 3\overline{q}^2\delta q + 3\overline{q}\delta q^2 + \delta q^3)$$

can write

=0

$$\frac{d}{dt}(\delta q) = (\alpha\overline{q} - \beta\overline{q}^3) + \delta q(\alpha - 3\beta\overline{q}^2) + 0(\delta q^2)$$

Linearize, ie: assume terms $0(\delta q^2) \ll 0(\delta q)$ behaviour is roughly linear if δq is small enough, then $\frac{d}{dt}(\delta q) = \delta q(\alpha - 3\beta\overline{q}^2) = \eta\delta q$

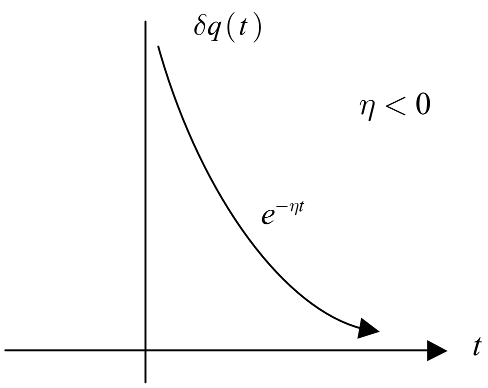
- this linear equation much easier to solve than original nonlinear equation for $q(t)$.

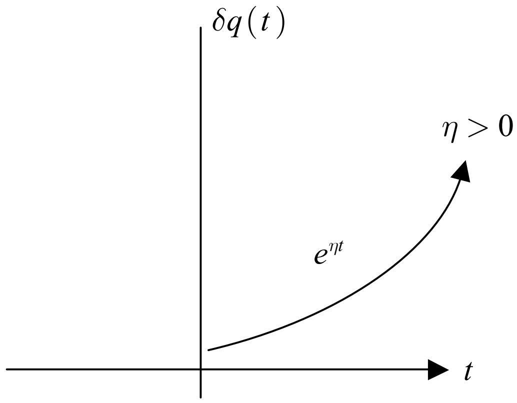
$\frac{d}{dt}(\delta q) = \eta\delta q$ has solution $\delta q(t) = \delta q_0 e^{\eta t}$

where δq_0 - integration constant

ie: $\delta q(t=0) = \delta q_0$

Now two possibilities:





$\delta q(t) \to 0$ as $t \to \infty$ $\qquad$ $\delta q(t) \to \infty$ as $t \to \infty$

hence $q(t) \to \overline{q}$ stable $\qquad$ $q(t) \to \infty$ unstable

So, if any small perturbation is introduced when system is at $\overline{q}$
stable fixed point – stays there $(\to \overline{q})$
unstable fixed point - moves away.

Real system (with fluctuations) will not be found at unstable fixed points – but tends to move towards stable fixed points once in vicinity.

NB: "in vicinity" means when $q = \overline{q} + \delta q, \quad \delta q > \delta q^2$ (ie: linearization). Procedure is not valid for all $q(t)$, ok if near stable fixed point $(\delta q \to 0)$, fails as system moves away from unstable fixed point $(\delta q \to \infty)$.

<u>BUT</u> identifies nature of fixed points.

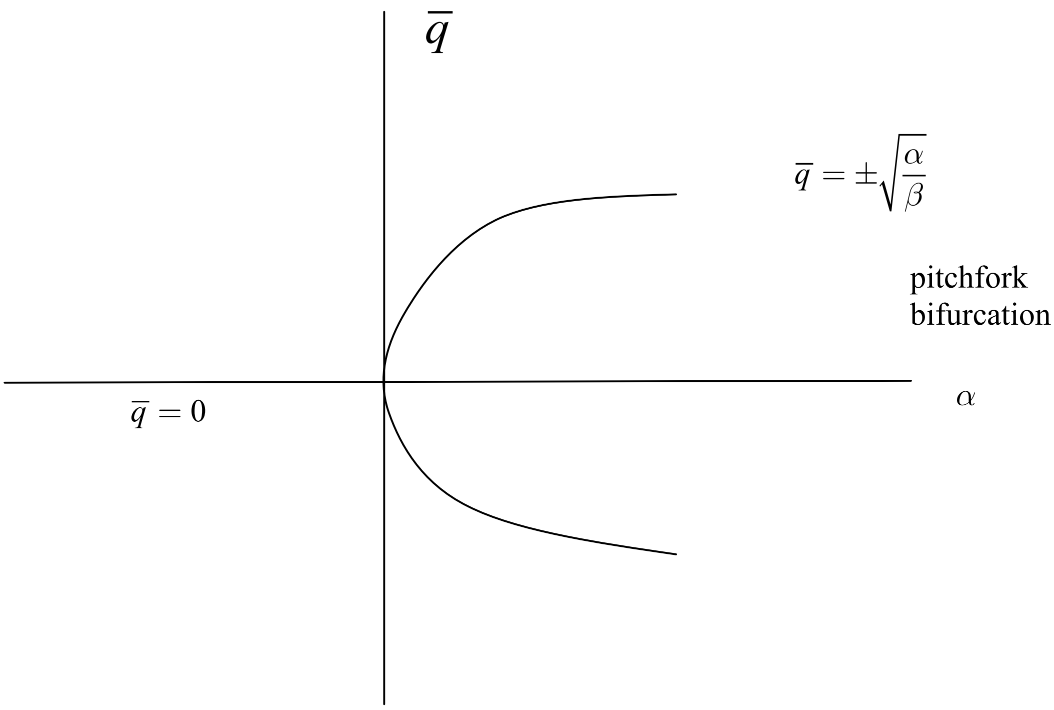
So, for our system $(\beta > 0)$

$$\eta = \alpha - 3\beta\overline{q}^2$$

then for $\overline{q} = 0$: $\quad \eta > 0 \quad \alpha > 0$ unstable; $\quad \eta < 0 \quad \alpha < 0$ stable

for $\overline{q} = \pm\sqrt{\dfrac{\alpha}{\beta}}$, $\alpha > 0$: $\quad \eta = \alpha - 3\alpha = -2\alpha$ stable

Stable fixed points:



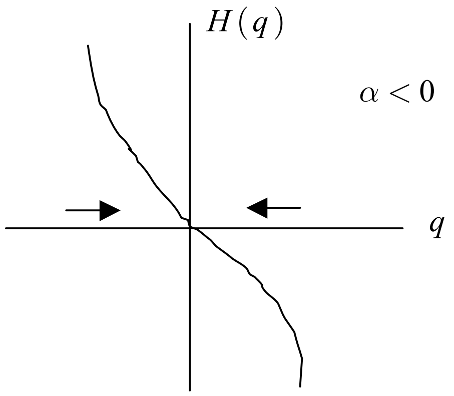
- Exactly the form obtained for $M(T)$ for 2nd order phase transition

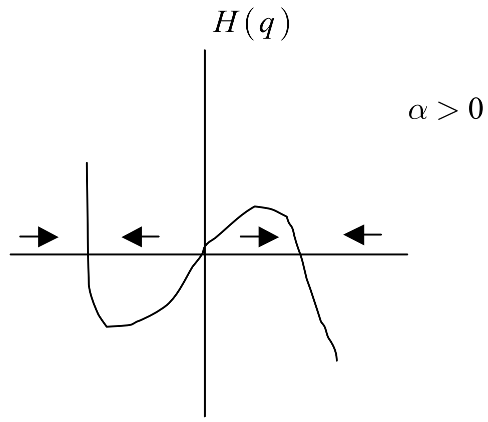
<u>Phase plane analysis</u> (for ID system)

Linearization is only locally correct (ie: close enough to $\bar{q}$)
seek <u>global</u> information. Phase plane analysis (Poincaré).

Write $\frac{dq}{dt} = H(q) \equiv \alpha q - \beta q^3$

Now sketch $H(q)$





Recall $\frac{dq}{dt} = 0$ ie: $H(q) = 0$ at $\bar{q} = 0$

$\bar{q} = \pm\sqrt{\frac{\alpha}{\beta}}$ $\alpha > 0$

and asymptotes are $q \to \infty, H \to -q^3 \to -\infty$ etc

Now for any q – read off $\frac{dq}{dt}$ $-ve \Rightarrow q$ decreases
ie: $H(q)$ $+ve \Rightarrow q$ increases
with time

mark with arrows.

Then for $\alpha < 0$ arrows always $\to q = 0$
- long time behaviour of $q(t)$ is $\to q = 0$ independent of initial $q(t=0), q = 0$ is attractor.

For $\alpha > 0$ - two possibilities.

If initially $q(t=0) > 0$, q will move towards $\bar{q} = +\sqrt{\frac{\alpha}{\beta}}$.

If initially $q(t=0)<0$, q will move towards $\overline{q}=-\sqrt{\frac{\alpha}{\beta}}$

- attractors.

This is the global property, ie: true for any q; consistent with linearization result.

Similarly, $\alpha>0$ has $q=0$ as a repellor – linear theory $\rightarrow q(t)$ will grow away from $\overline{q}=0$ exponentially with t.

Phase plane analysis $\Rightarrow q(t)$ will subsequently be attracted to $\overline{q}=\pm\sqrt{\frac{\alpha}{\beta}}$

{which sign depends on fluctuations in starting $q(t=0)$, ie: $q(t=0)=0+\varepsilon$, ε vanishingly small} – on sign of ε - 'sensitivity to initial conditions' – a trivial example. NOT chaos: this arises later in ID maps (difference equations). For differential equations we will see that 3^+D phase space is needed for chaos.

So have all behaviour of $q(t)$ without solving D.E. – at most needed to solve algebraic equations. All this (linearization, phase plane analysis) extends to higher dimensions. We will do 2D next.

SUMMARY: Technique for not solving nonlinear DE –

Any $\frac{dq}{dt}=H(q)$

1. Look for fixed points $H(\overline{q})=0$.
2. Linearize – put $q=\overline{q}+dq$ in $\frac{dq}{dt}=H(q)$, neglect terms $0(\partial q^2)$ - gives $\frac{d\delta q}{dt}=\eta\delta q$ and $\eta=\eta(\overline{q})$.
3. Classify the $\overline{q}$ according to sign of $\eta(\overline{q})$.
4. Sketch phase plane, ie: $\frac{dq}{dt}=H(q)$ *vz* q, obviously if $H(q)>0$ q increases with t;

   $H(q)<0$ q decreases

   $\rightarrow$ this yields full global dynamics – no actual integration of $q=\int H(q)\,dt$