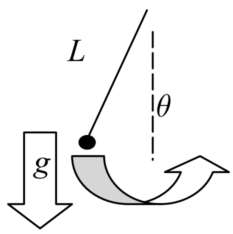
2nd order non-linear DE – Pendulum

Generalize what was done for ID, 1st order nonlinear DE – non- trivial differences.

Start by considering a familiar example: pendulum



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

with linear equation

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

for $\sin\theta \simeq \theta$, ie: small perturbations about rest position $\theta = 0$

- oscillates $\quad \theta = Ae^{i\omega t} \qquad \omega^2 = \frac{g}{L}$.

How to deal with 2nd order DE? Write as two coupled 1st order DEs:

$$y = \frac{d\theta}{dt}, \qquad \frac{dy}{dt} = -\omega^2 \sin\theta$$

Now we can find fixed points, linearize as before.

<u>Fixed points</u> $\quad \overline{y} = 0, \quad \sin\overline{\theta} = 0$ or $\overline{\theta} = n\pi \qquad n = -3, -2, -1, 0, 1, 2....$

<u>Stability/Classification</u>

Linearize, we write

$$y = \overline{y} + \delta y(t) = \delta y$$
$$\theta = \overline{\theta} + \delta\theta(t) = n\pi + \delta\theta$$

with $\delta y, \delta\theta$ small. Then linearized equations are:

$$\frac{d\delta\theta}{dt} = \delta y \qquad \frac{d\delta y}{dt} = -\omega^2(-1)^n \delta\theta$$

since $\sin(n\pi + \delta\theta) = \sin(n\pi)\cos\delta\theta + \cos n\pi \sin\delta\theta$

$$\cos(n\pi) = (-1)^n$$
$$\sin\delta\theta \approx \delta\theta$$

don't know behaviour of $y(\theta)$ yet.

Now, can write:

$$\frac{d^2\delta\theta}{dt^2} = -\omega^2(-1)^n\,\delta\theta$$

- the linearized 2nd order ODE. There are 2 possibilities:

<u>n even</u>

$$\frac{d^2\delta\theta}{dt^2} = -\omega^2\delta\theta \qquad \delta\theta = Ae^{i\omega t} + Be^{-i\omega t}$$

oscillatory solution

<u>n odd</u> $\qquad \dfrac{d^2\delta\theta}{dt^2} = +\omega^2\delta\theta \qquad \delta\theta = Ae^{\omega t} + Be^{-\omega t}$

- behaviour depends on the integration consts A, B

δy? $\qquad \delta y = \dfrac{d(\delta\theta)}{dt}$

<u>n even</u> $\qquad dy = i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t} = \omega Ae^{i(\omega t+\pi/2)} + \omega Be^{-i(\omega t+\pi/2)}$

$$i = e^{i\pi/2}, -i = e^{-i\pi/2}$$

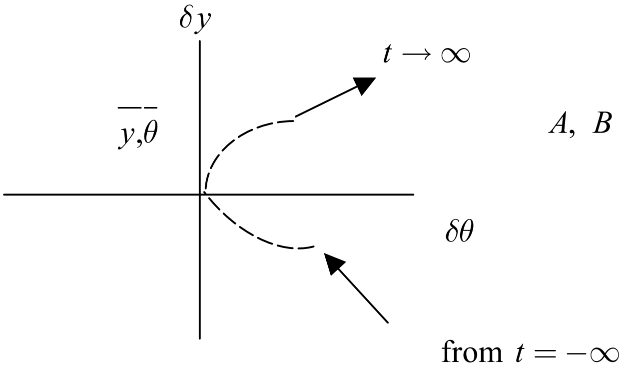
ie: out of phase $\pi/2$ with $\delta\theta$
oscillatory – about <u>centre</u> fixed point.

<u>n odd</u> $\qquad \delta y = \omega Ae^{\omega t} - \omega Be^{-\omega t}$

$$\delta\theta = Ae^{\omega t} + Be^{-\omega t}$$

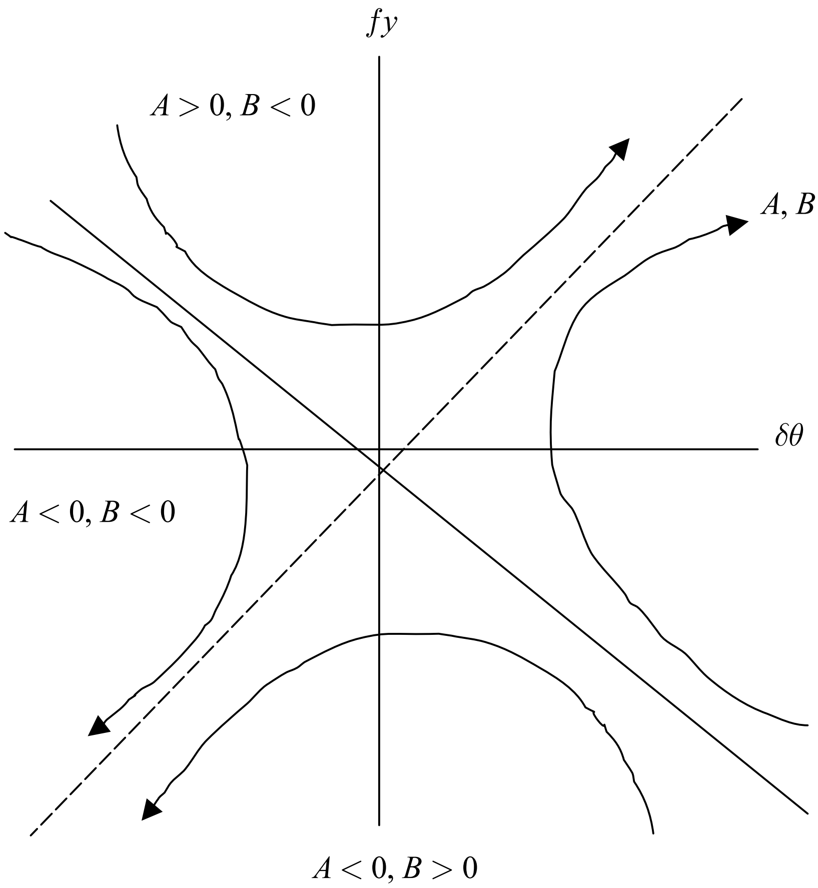
this is either attracted or repulsed from fixed point.

$t \to \infty$
"A" term dominates
$t \to -\infty$
"B" term dominates



$A,\ B > 0$

Consider all possible combinations of signs of A, B – four possibilities:



<u>Saddle point:</u>
Separatrix has lines given by

$$t \to \infty, \frac{\delta y}{\delta\theta} = \frac{\omega A e^{\omega t}}{A e^{\omega t}} = \omega$$

$$t \to -\infty, \frac{\delta y}{\delta\theta} = \frac{-\omega B e^{-\omega t}}{B e^{-\omega t}} = -\omega$$

to make phase plane diagram (global solution) we need some global properties.
Will now do a sketch in y, θ phase plane.

<u>Global properties for the pendulum:</u>

<u>Constant of the motion:</u>

Equation of motion $\qquad \dfrac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$

can integrate once $\qquad \dfrac{d\theta}{dt}\dfrac{d^2\theta}{dt^2} + \omega^2 \sin\theta \dfrac{d\theta}{dt} = 0$

$\times \dfrac{d\theta}{dt}$

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega^2 \cos\theta = E \ \underline{\text{or}}\ \frac{y^2}{2} - \omega^2 \cos\theta = E\,.$$

Also symmetry property:

$$\theta \to -\theta$$
$$y \to -y$$

Different IC give different E. Say we start the pendulum at rest – at θ_0 $\quad E = -\omega^2 \cos\theta_0$
- this is the maximum $|\theta|$ of the motion. θ is symmetric

$$+\theta_0 = -\theta_0$$

*NB: $\theta \to -\theta$ gives the same equation of motion, θ symmetry property.

Topology of y, θ plane

We have $\dfrac{d\theta}{dt} = y \quad \dfrac{dy}{dt} = -\omega^2 \sin\theta$

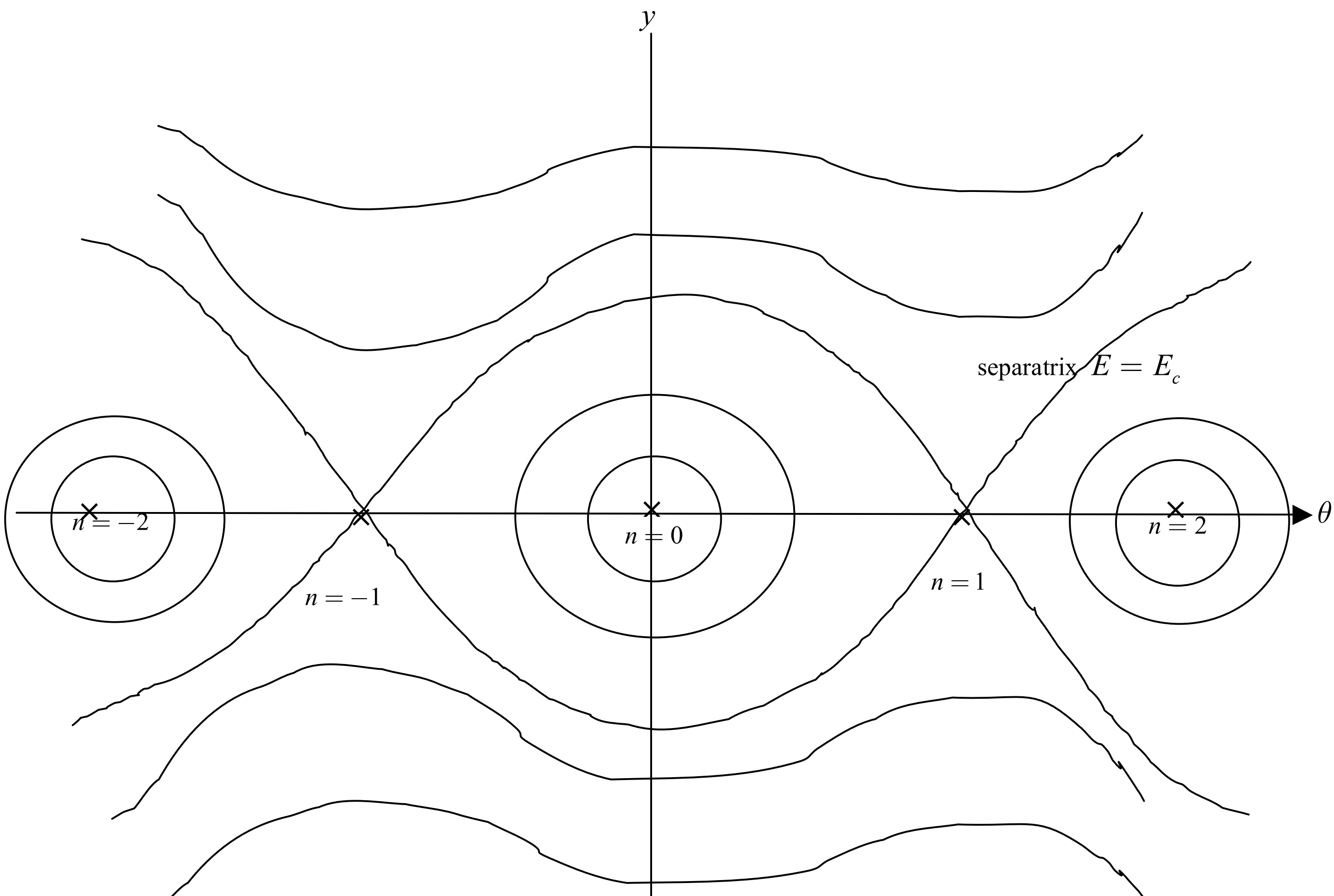
So $\dfrac{dy}{d\theta} = \dfrac{-\omega^2 \sin\theta}{y}$

So, unless $y = 0,\ -\omega^2 \sin\theta = 0$ (the fixed points)

$\dfrac{dy}{d\theta}$ is uniquely defined. This is tangent to $y(\theta)$ lines

therefore, $y(\theta)$ lines cannot cross- ***General property.***

Sketch of the phase plane



<u>Properties of the phase plane:</u>

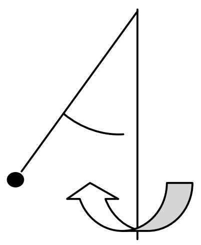
- lines don't cross except at fixed points $\quad \bar{y} = 0,\ \bar{\theta} = n\pi$
- $n = 0$, n even are centres (oscillatory solutions)
- n odd are saddle points

$$E = \frac{y^2}{2} - \omega^2 \cos\theta \qquad \text{constant} \qquad \text{- symmetric in } y,\ \theta$$

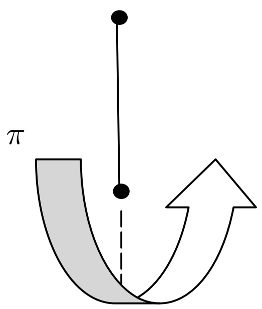
- y is bounded for a given θ_0 at $y = \dfrac{d\theta}{dt} = 0 \qquad E = E_0 = -\omega^2 \cos\theta_0$
- now the separatrix corrects the saddle points, e.g. passes through $\theta_0 = \bar{\theta} = \pi,\ y = \bar{y} = 0$

This is the line where $E = E_c = \omega^2$, critical value of E

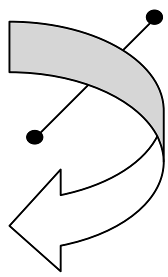
- critical value of E separates bonded from unbounded orbits.



$E < E_c$
oscillates about stable (circle) fixed point $\theta < \pi$



$\theta = \pi$ unstable (saddle) fixed point



$E > E_c$
unbounded
motion

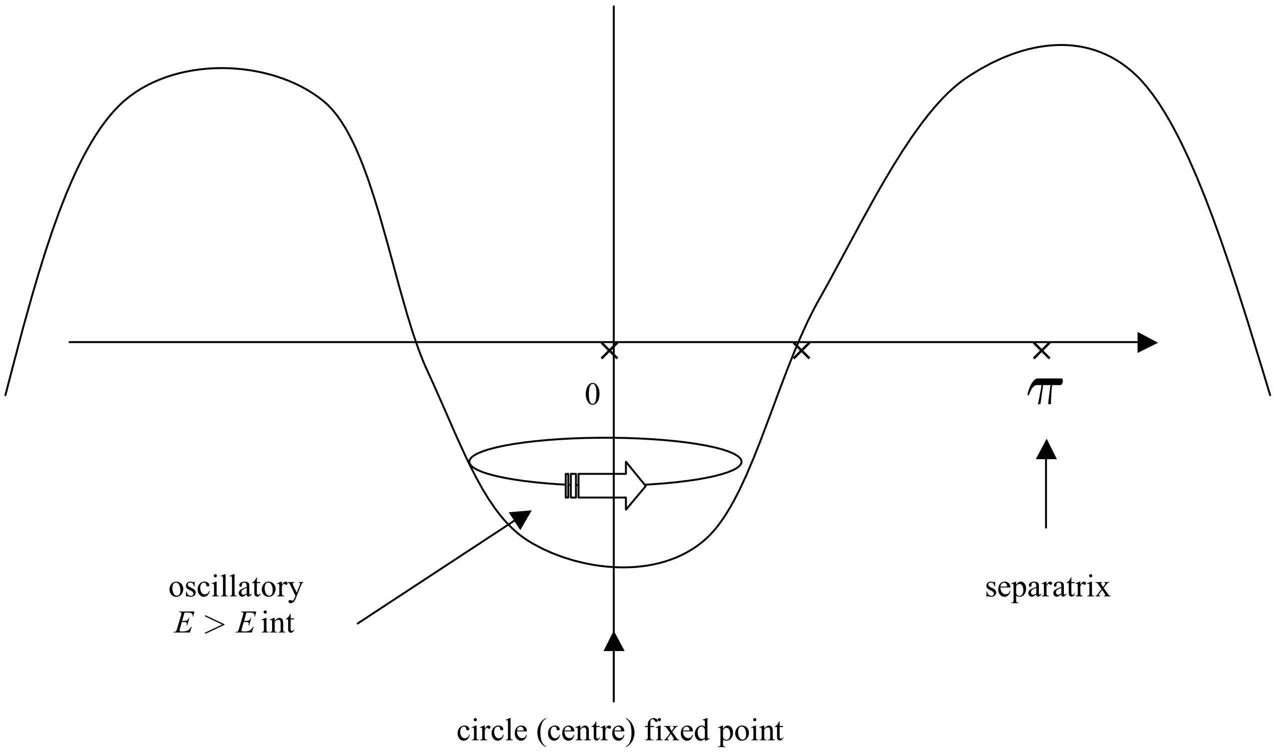
Global behaviour obtained without integrating equation

NB: integral of $\dfrac{1}{2}\left(\dfrac{d\theta}{dt}\right)^2 - \omega^2 \cos\theta = E$

Can be done analytically- Jacobian Elliptic functions – look up if interested.

Another way to think of E – in terms of potential function V

$$E = \frac{y^2}{2} + V(\theta) \qquad V = -\omega^2 \cos\theta$$



If $E > E_c$ - unbounded motion.

This is conservative system <u>but</u> now also know what happens for weak damping.

'weak' ≡ on timescale $> 1/\omega$
particle slowly loses energy.
See handout for sketch

All of the above generalizes to any 2nd order nonlinear DE- for general properties see the handout accompanying the lecture.