Limit Cycles

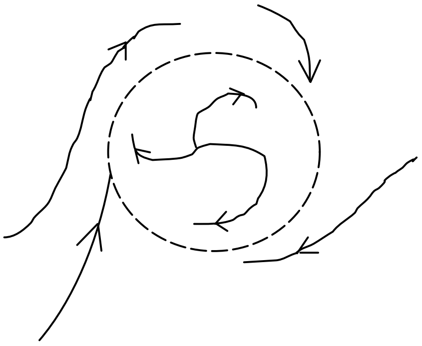
Have found that orbits cannot cross, can be attracted to (fixed points), etc. One other possibility is limit cycle.

ODE is 'well behaved' ie: all derivatives exist and are continuous –

Therefore, all orbits smoothly follow neighbours in phase space.

One other possibility only:

<u>limit cycle</u> →



orbits approach closed curve as $t \to \infty$

NB – complete description of all details is non trivial – here give the basics.

<u>Limit cycle – an example</u>

Consider
$$F = x + y - x\left(x^2 + y^2\right)$$
$$G = -(x - y) - y\left(x^2 + y^2\right)$$

Fixed point $F = G = 0$ is $\bar{x} = 0,\ \bar{y} = 0$

Stability analysis $x = \bar{x} + \delta x \qquad y = \bar{y} + \delta y$

$$\begin{matrix} F = \delta x + \delta y \\ G = -\delta x + \delta y \end{matrix} = \begin{matrix} a\delta x + b\delta y \\ c\delta x + d\delta y \end{matrix} \qquad p = a + d = 2 \qquad q = ad - bc = 2 \qquad \begin{matrix} p^2 < 4q \\ p > 0 \end{matrix}$$

- unstable spiral

<u>In addition</u> – to look elsewhere in phase plane, rewrite in polars

$$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2$$

use following identities

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \qquad x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dy}$$

then
$$\frac{dx}{dt} = x + y - x\left(x^2 + y^2\right)$$
$$\frac{dy}{dt} = -\left(x - y\right) - y\left(x^2 + y^2\right)$$

gives
$$r\frac{dr}{dt} = x^2 + xy - x^2\left(x^2 + y^2\right) - xy + y^2 - y^2\left(x^2 + y^2\right)$$
$$= r^2 - r^4$$

$$r^2\frac{d\theta}{dt} = -x^2 + yx - xy(x^2 + y^2) - xy - y^2 + xy(x^2 + y^2)$$
$$= -r^2$$

ie: $\dfrac{dr}{dt} = r\left(1 - r^2\right) \quad \dfrac{d\theta}{dt} = -1$

Integrate directly –

$$\theta = \theta_0 - t \qquad \left[r^2 = \frac{Ae^{2t}}{1 + Ae^{2t}} \right]$$

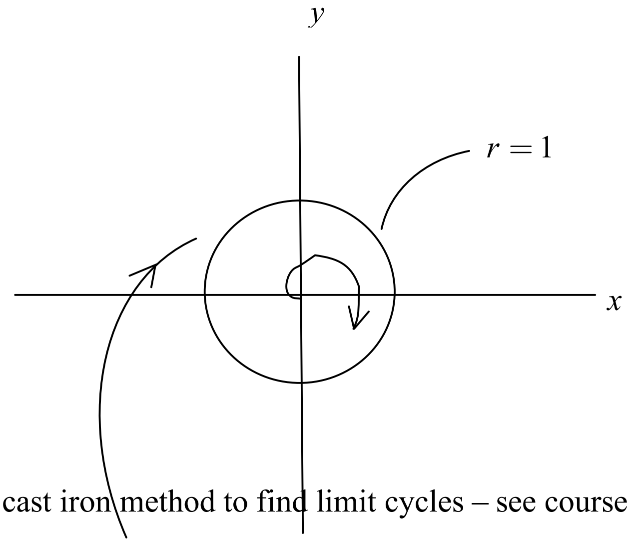
don't need to integrate *r* equation to see the limit cycle.

$\dfrac{dr}{dt} = 0 \qquad r = 1 \qquad$ for any $\theta \qquad$ (as well as $\overline{r} = 0$ the fixed point)

trajectory sits on circle $r = 1$.

For $\begin{array}{ll} r > 1 & r\left(1 - r^2\right) < 0 \\ r < 1 & r\left(1 - r^2\right) < 0 \end{array}$ by inspection.

Therefore, solution is attracted to $r = 1$ circle.



either attracted in from $r \to \infty$ or out from repulsive fixed point at $r = 0$ $\left(\overline{x} = 0,\ \overline{y} = 0\right)$

No single cast iron method to find limit cycles – see course texts for some advanced methods.

Example of limit cycle – Van der Pol oscillator

Van der Pol, 1926 – Electric circuit with valve (model of heatbeat)

Identical to Rayleigh, 1883 – Nonlinear Vibrations

1st experimentally shown limit cycle

$$\frac{d^2x}{dt^2} + \varepsilon\left(x^2 - 1\right)\frac{dx}{dt} + x = 0$$

cause of trouble

Write as $\quad \frac{dx}{dt} = y \qquad \frac{dy}{dt} = -x \quad - \varepsilon\left(x^2 - 1\right)y$

If $\varepsilon = 0$ → linear pendulum $\omega = 1$.

Symmetries – invariant for $\varepsilon \to -t$; $\quad \varepsilon \to -\varepsilon$

Therefore, solve for $\varepsilon > 0$
- reverse time for $\varepsilon < 0$

ie: $\varepsilon > 0$ growth is $\varepsilon < 0$ damping, etc.

Fixed points

$$\bar{x} = 0, \quad \bar{y} = 0$$

Stability

$$x = \bar{x} + \delta x$$
$$y = \bar{y} + \delta y$$

$$F = \frac{dx}{dt} = y$$

$$G = \frac{dy}{dt} = -x - \varepsilon\left(x^2 - 1\right)y$$

$$\longrightarrow$$

$$\frac{d\delta x}{dt} = \delta y$$

$$\frac{d\delta y}{dt} = -\delta x + \varepsilon \delta y$$

or work out

$$\frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial y} = 1 \qquad \frac{\partial G}{\partial x} = -1 - 2\varepsilon x \quad \frac{\partial G}{\partial y} = -\varepsilon\left(x^2 - 1\right)$$

Evaluate at $\bar{x}, \bar{y} = 0$ $\qquad \dfrac{\partial F}{\partial x} = 0 \quad \dfrac{\partial F}{\partial y} = 1 \quad \dfrac{\partial G}{\partial x} = -1 \quad \dfrac{\partial G}{\partial y} = \varepsilon$

then

$$p = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \varepsilon$$

$$q = \frac{\partial F}{\partial x} \cdot \frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} = 1$$

$\varepsilon > 0 \qquad p > 0 \qquad q > 0$ unstable and spiral if $p^2 < 4q$.

Guess there is more ….

Since damping term is $\varepsilon\left(x^2 - 1\right)$

this is +ve for large x	(damping)
changes sign as $x \to 1$	(growth)
is zero at $x = 1$!	(neither!)

Solve – multiple timescale analysis (Rowlands, appendix)
- method of averaging (Drazin, p 193) - handout for result

<u>Pendulum by formula</u>

We have

$$\frac{d\theta}{dt} = 0\,\delta\theta + 1.\delta y \equiv F \qquad\qquad J = \begin{pmatrix} 0 & 1 \\ -\omega^2\left(-1\right)^n & 0 \end{pmatrix}$$

$$\frac{dy}{dt} = -\omega^2\left(-1\right)^n \delta\theta + 0\delta y \equiv G$$

<u>or</u>

$$\frac{dy}{dt} = 0\,\delta y - \omega^2\left(-1\right)^n \delta\theta \qquad\qquad J = \begin{pmatrix} 0 & -\omega^2\left(-1\right)^n \\ 1 & 0 \end{pmatrix}$$

$$\frac{d\theta}{dt} = 1.\delta y + 0\ \delta\theta$$

- same thing since

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad \begin{aligned} F &= a\delta x + b\delta y \\ G &= c\delta x + d\delta y \end{aligned}$$

$$p = a + d = 0$$

$$q = ad - bc = \omega^2\left(-1\right)^n$$

So, for *n* even $q > 0$ centre, *n* odd $q < 0$ saddle (see handout)