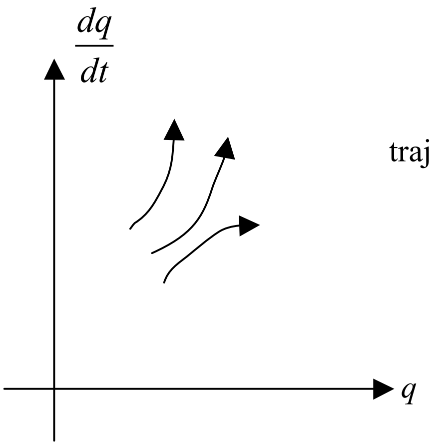
Maps and chaos

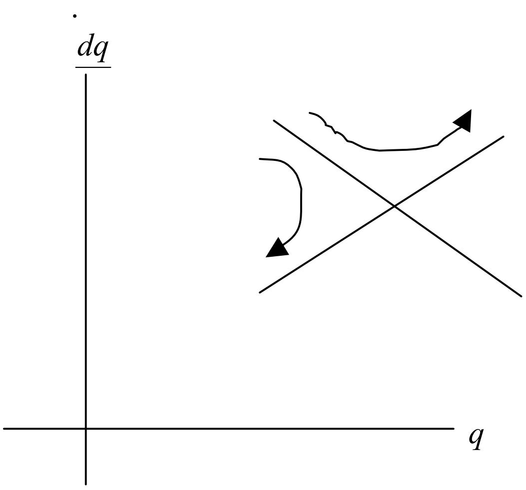
So far have discussed continuous 1st, 2nd order DE

- found trajectories in phase space are "well behaved"



trajectories cannot cross

only possibility for divergence of trajectories.



eg: at saddle

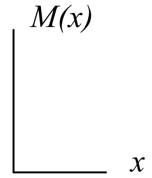
This does not permit stochastic (mixing up/shuffling) diffusion in phase space. This is only possible (in plane of paper) if we add another dimension(out of plane of paper) ie in 3 or more dimensions.

Now look at simplest systems that show route to chaos:

<u>Difference equations</u>

i) Show how they "work" – what are their properties; example of chaos.
ii) Show how they are fundamentally different to the analogous <u>differential</u> equations.
iii) Quantify the route to chaos – Lyapunov exponents.
iv) Introduce Universality – the bifurcation route to chaos, Feigenbaum numbers.

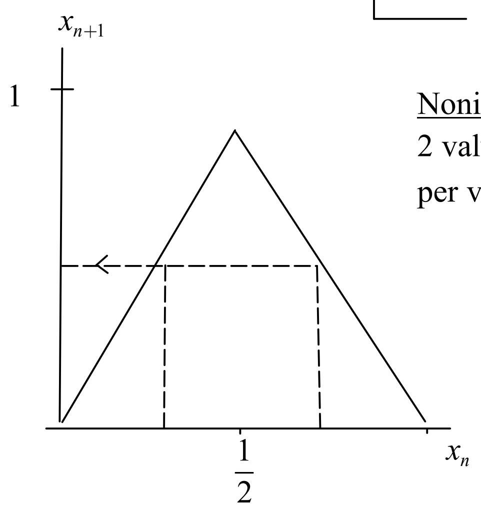
<u>1D Noninvertible maps</u> – the easiest place to start.



Piecewise linear 1D maps - the tent map

$$x_{n+1} = 1 - 2\left|x_n - \frac{1}{2}\right|$$
$$\equiv M(x)$$

- tells us how $x+1^{th}$ step depends on x^{th} step



<u>Noninvertible</u>
2 values of x_{n+1}
per value of

Write as:

$$x_{n+1} = \begin{cases} 2x_n & x_n \le \frac{1}{2} \\ 2(1-x_n) & x_n \ge \frac{1}{2} \end{cases}$$

What happens when we iterate the map?

If (i) $x_n < 0$ $\quad x_{n+1} < 0$ and $|x_{n+1}| > |x_n|$
ie: $x \to -\infty$

If (ii) $x_n > 1$ $\quad x_{n+1} < 0$, so $x_{n+2} < 0$ $\quad |x_{n+2}| > |x_{n+1}|$
as in (i) ie: $x \to -\infty$.

So, just consider the interval $x = [0,1]$ - iterates of x are bounded in this range.

We can still look for fixed points, linearise to examine stability as before → local behaviour.

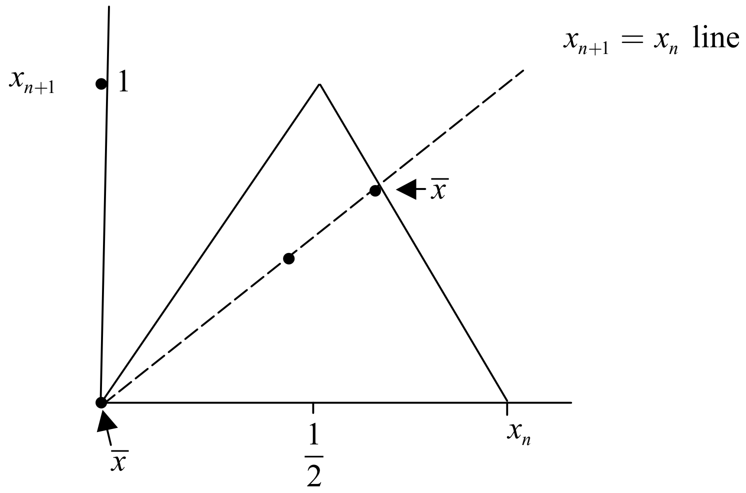
Interesting difference will be in global behaviour of maps.

Fixed points are where x_n doesn't change as $n \to \infty$ so

$$\bar{x}_{n+1} = \bar{x}_n \text{ is fixed point}$$

$$= \bar{x} \qquad\qquad M(x) = x$$

Graphically:



Fixed point is in range $\frac{1}{2} \le x_n \le 1$ ie: $x_{n+1} = 2(1 - x_n)$

$$\bar{x} = 2(1 - \bar{x})$$

so fixed point is $x = 2 - 2\bar{x}$

$$\bar{x} = \frac{2}{3}$$

also in range $0 \le x_n \le \frac{1}{2}$ fixed point $\bar{x} = 0$ here $x_{n+1} = 2x_n$

Stability/classification. Can still write $\begin{aligned} x_n &= \bar{x} + \delta x_n \\ x_{n+1} &= \bar{x} + \delta x_{n+1} \end{aligned}$ ie: δx is small but now discontinuous.

sub in to

$$x_{n+1} = 2\left(1 - x_n\right)$$

$$\bar{x} + \delta x_{n+1} = 2\left(1 - \bar{x} - \delta x_n\right)$$

$\bar{x} = \frac{2}{3}$

$$\delta x_{n+1} = 2 - 3\bar{x} - 2\delta x_n$$

$$= -2\,\delta x_n$$

So $\delta x_{n+1} = -2\,\delta x_n$

also $\delta x_n = -2\delta x_{n-1}$

so $\delta x_{n+1} = -2.2\,\delta x_{n-1} = -2.2.2\,\delta x_{n-2}$

$= [-2]^{j+1}\,\delta x_{n-j} = [-2]^{n+1}\,\delta x_0$ where δx_0 is the initial condition

Hence, the fixed point is unstable – oscillates.

Similarly, $\bar{x} = 0, \delta x_{n+1} = 2\,\delta x_n$ - unstable $[x_{N+1} = 2x_N]$

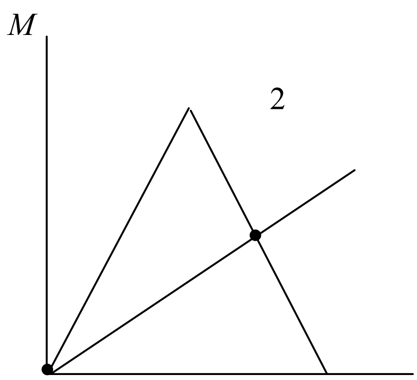
Consider <u>global</u> behaviour.

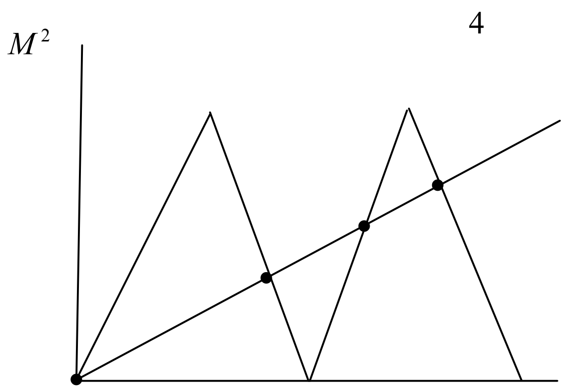
<u>Iterate many times</u>

Look at one interate: $x_{n+1} = M(x_n)$
(some notation here)
two iterates: $x_{n+2} = M^2(x_n)$
$= M(M(x_n))$
p iterates $x_{n+p} = M^p(x_n)$

where $$M(x) = \begin{cases} 2x & x \le \frac{1}{2} \\ 2(1-x) & x \ge \frac{1}{2} \end{cases}$$

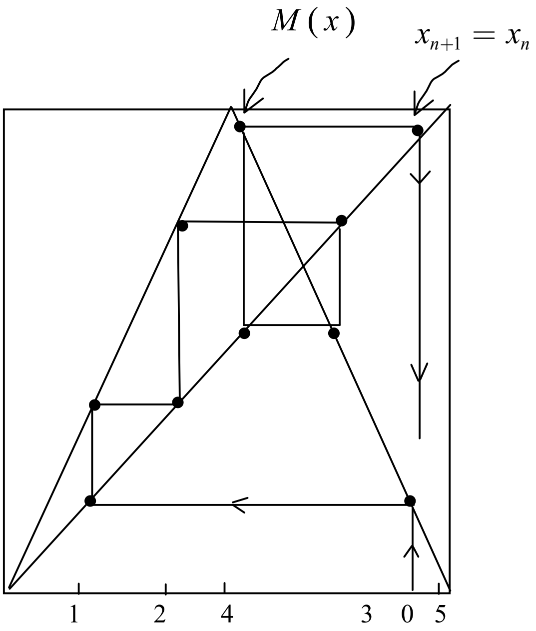
<u>Fixed points</u> – notice that

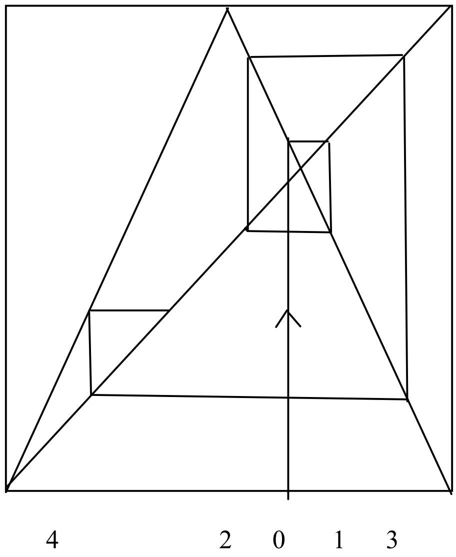




No of fixed points doubles each iterate.

Another way of looking at $M^p(x)$ graphically:





vertical lines are $M(x_n)$

horizontal $x_{n+1} = M(x_n)$

note that the iterates are 'shuffled'