High dimensional systems

Properties

- many interacting elements
- ‘microscopic’ rules for element-element interaction
- order and control parameters
- emergence (a class of which is scaling)
- systems may have dynamical steady states but in general may not be in equilibrium

- complex systems
  - → emergent behaviour is not just given by dynamics of one individual
  - → a theory of out-of-equilibrium statistical mechanics (understand/predict in a statistical sense)
    - behaviour of the ensemble

Examples- flocking birds have a microscopic rule: competition between order ('wanting to be like your neighbour') and disorder ('doing your own thing/random perturbation') and order/control parameters for the macroscopic behaviour that emerges (flocking)

<u>High dimensional systems- we will do the following</u>

1. Introduction – self-organisation (when a many d.o.f. system looks like a few d.o.f. system).

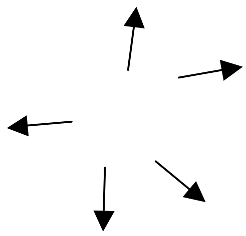
2. Order - disorder transitions (met one already- ferromagnet) → <u>order/control parameters</u>.

3. Emergence and self-similarity (fractals)

4. Dimensional analysis- Buckingham Pi theorem
   - how to identify order and control parameters.

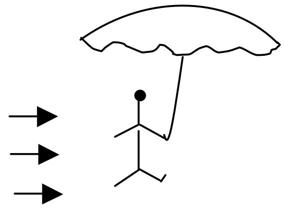
Self organisation ("enslaving")

n tourists



disordered

Tour guide



ordered

Motion of tourists 'enslaved' to tour guide.

Tour guide = order parameter.

- just follow the guide
- clearly desirable to treat system in this way –

  just solve 1 equation – tour guide
  instead of N – tourists.

How can this occur?

Consider tourist ODE:

$$\frac{dq_i}{dt} = -\alpha_i q_i + Ae^{-\lambda t} \qquad i = 1 \ldots n, \quad \alpha, \lambda > 0$$

q_i is ith tourist position

driving term ≡ tour guide

each tourist solution obtained by integrating directly – use integrating factor

ie: $$\frac{dy}{dt} + f(t)y = g(t) \qquad I.F. \quad \exp\left[\int f(t)\,dt\right]$$

multiply by I.F. gives a perfect integral.

So, here – dropping i subscript for the moment

$$\frac{dq}{dt} + \alpha q = Ae^{-\lambda t} \qquad \text{1.F} \quad e^{\int \alpha\, dt} = e^{\alpha t}.$$

So we have

$$\frac{dq}{dt}e^{\alpha t} + \alpha q e^{\alpha t} = Ae^{(\alpha-\lambda)t}$$

integrate:

$$qe^{\alpha t} = \frac{A}{\alpha-\lambda}e^{(\alpha-\lambda)t} + q_0 \qquad q_0 = \text{constant of integration}$$

or

$$q = \frac{A}{\alpha-\lambda}e^{-\lambda t} + q_0 e^{-\alpha t}$$

so, for each tourist:

$$q_i = \frac{A}{\alpha_i-\lambda}e^{-\lambda t} + q_{0i}\, e^{-\alpha_i t}$$

Now, if tourist timescale $\frac{1}{\alpha_i} \ll \frac{1}{\lambda}$ tour guide timescale, then for large t, $\alpha_i t \gg \lambda t$ (and $\alpha_i, \lambda > 0$)

$$q_i \simeq \frac{A}{\alpha_i}e^{-\lambda t}$$ - tourists just follow the tour guide.

Note that the competition between the terms:

$e^{-\lambda t}$	and	$e^{-\alpha_i t}$
order		disorder

is competition between order and disorder – and there is a parameter $\frac{\alpha_i}{\lambda} = \pi$

that tells you whether ordered $\pi \gg 1$
disordered $\pi \ll 1$

NB: could have instead sought stationary solution

$$\frac{dq_i}{dt} = 0 = -\alpha_i q_i + Ae^{-\lambda t}$$

$$\rightarrow q_i = \frac{A}{\alpha_i}e^{-\lambda t}$$ - same result!

Hence, the 'enslaved' solution is the time independent solution (steady state) – tourist solution quickly relaxes to follow the tour guide.

General statement of same thing

Any set of equations

$$\frac{dq_i}{dt} = H_i\left(q_j\left(t\right)\right) + S_i\left(t\right) \qquad \begin{array}{l} i-1...N \\ j=1...N \end{array}$$

ie: H_i are a function of the q_i and the S_i are the external driving terms.

IF all q_i interact on timescales << characteristic timescale of S_i solution is in "dynamic equilibrium"

$$H_i\left(q_j\left(t\right)\right) = -S_i\left(t\right)$$

Now just need to do algebra, not integrate ODEs.

Self organisation

No tour guide – just tourists, but they interact with each other.

Write equations in terms of tourists *c* of *m* q_c

$$\frac{dq_i}{dt} = H_i\left(q_j\right) = \alpha_i\left(q_i\right) + G\left(q_c\right)$$

where $q_c = \sum_N q_j$ say

Again, if q_c has slower timescale than all q_i

- q_i enslaved to q_c