2^{nd} Order Nonlinear DE – Intro to General Phase Plane Analysis

We will generalize what was done in lectures for the single nonlinear pendulum (for a *complete* description of fixed point topology see for example course textbooks Ott, Drazin)

We have 2 kinds of information: **global** and **local**.

Global information

1) Express 2^{nd} order DE as two coupled 1^{st} order DE (which we can't in general integrate)

$$\frac{dx}{dt} = F(x,y) \qquad \frac{dy}{dt} = G(x,y)$$

Some notation: autonomous — F, G not explicitly t dependent
nonautonomous — F, G are explicitly t dependent

c.f. pendulum $\quad x \equiv \theta \quad \frac{d\theta}{dt} \equiv y \quad F \equiv y\,, \quad G \equiv -\omega^2 \sin x$

pendulum – dynamics given by trajectory in phase space i.e. y,θ space

2) Can then write $\quad \frac{dy}{dx} = \frac{G}{F}$

hence there is a uniquely defined tangent to the $y(x)$ "trajectory" everywhere *except* at $F = 0$ or $G = 0$ - the fixed points $\bar{x}, \bar{y}$.
So generally – trajectories can't cross except at the fixed points.

For the pendulum, other global information came from the symmetries of $y(\theta)$. So generally look at symmetries in F, G to understand topology.

N.B.: This framework encompasses Hamiltonian systems where one has $\underline{p}(\underline{q})$ trajectories in $\underline{p},\underline{q}$ phase space and for each component:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} \qquad H \text{ - Hamiltonian here}$$

$\underline{p},\underline{q}$ are canonical coordinates and $H = (\underline{p},\underline{q}) = \sum_j \frac{{p_j}^2}{2m} + V(q_j)$

(H constant)
- the pendulum is of this class.

Local information:

We can – i) find fixed points $F=0$, $G=0$:- $\bar{x},\bar{y}$
ii) linearize about fixed points to classify them.

- To consider the general case – let's consider that a fixed point $\bar{x},\bar{y}$ exists – classify all possible behaviour.

Classification of fixed point behaviour

Let $F=0$, $G=0$ have solution $\bar{x},\bar{y}$

consider vicinity of this fixed point $x=\bar{x}+\delta x$
$y=\bar{y}+\delta y$

we can Taylor expand F, G

$$F(x,y)=F(\bar{x},\bar{y})+\frac{\partial F(\bar{x},\bar{y})}{\partial x}\delta x+\frac{\partial F(\bar{x},\bar{y})}{\partial y}\delta y+O(\delta x^2,\delta y^2,\delta x\delta y)$$
$$G(x,y)=G(\bar{x},\bar{y})+\frac{\partial G(\bar{x},\bar{y})}{\partial x}\delta x+\frac{\partial G(\bar{x},\bar{y})}{\partial y}\delta y+O(\delta x^2,\delta y^2,\delta x\delta y)$$

but $F(\bar{x},\bar{y})$, $G(\bar{x},\bar{y})=0$

- assumed that F,G are twice differentiable.

So now need to solve equations of the form:

$$F=a\delta x+b\delta y$$
$$G=c\delta x+d\delta y$$

- *nonlinear* problem now reduced to *linear* algebra.

a,b,c,d are the differentials from Taylor expansion $a=\frac{\partial F(\bar{x},\bar{y})}{\partial x}$ etc.

So in the vicinity of $\bar{x},\bar{y}$

$$\frac{d\delta x}{dt}=a\delta x+b\delta y$$
$$\frac{d\delta y}{dt}=c\delta x+d\delta y$$

or

$$\frac{d\underline{\delta x}}{dx}=\underline{\underline{J}}\bullet\underline{\delta x}$$

where $\underline{\delta x}=\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$ $\underline{\underline{J}}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Use the method of normal modes – i.e. assume solutions of the form $\underline{\delta x} = e^{st}\underline{u}$, where

$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a constant vector (**eigenvector**) and s is **eigenvalue** (one eigenvalue per eigenvector).

then $\underline{\underline{J}} \cdot \underline{u} = s\,\underline{u}$. To rearrange, write the RHS as a dot product, i.e.:

$s\underline{u} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \underline{\underline{s}} \bullet \underline{u}$ so that $\underline{\underline{J}} \bullet \underline{u} = \underline{\underline{s}} \bullet \underline{u}$

or $\left(\underline{\underline{J}} - \underline{\underline{s}}\right).\underline{u} = 0$ so $\underline{\underline{J}} - \underline{\underline{s}} = \begin{pmatrix} a-s & b \\ c & d-s \end{pmatrix}$

For nontrivial solutions $(u \neq 0)$ we have $\left|\underline{\underline{J}} - \underline{\underline{s}}\right| = 0$ i.e. $0 = \begin{vmatrix} a-s & b \\ c & d-s \end{vmatrix}$

working out the determinant:

$$0 = (a-s)(d-s) - bc$$

$$= s^2 - s(a-d) + ad - bc \qquad \text{giving} \quad s = \tfrac{1}{2}\left\{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right\}$$

which we will write as $s = \frac{1}{2}\left\{p \pm \sqrt{p^2 - 4q}\right\}$ for short.

There are 2 eigenvalues s_+, s_- so there are also 2 eigenvectors $\underline{u}_+, \underline{u}_-$

then the general solution for $\underline{\delta x} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$ is $\underline{\delta x} = C_1 e^{s_+ t}\underline{u}_+ + C_2 e^{s_- t}\underline{u}_-$ and C_1, C_2 are arbitrary constants.

In components this is:

$$\delta x = C_1 e^{s_+ t} u_+ + C_2 e^{s_- t} u_- \quad = \quad A e^{s_+ t} + B e^{s_- t}$$

$$\delta y = C_1 e^{s_+ t} v_+ + C_2 e^{s_- t} v_- \quad = \quad C e^{s_+ t} + D e^{s_- t}$$

and $\underline{u}_\pm$ both satisfy $\underline{\underline{J}} \cdot \underline{u} = s\underline{u}$ so since $\underline{u}_+ = \begin{pmatrix} u_+ \\ v_+ \end{pmatrix}$ etc, we have that

$$au_\pm + bv_\pm = s_\pm u_\pm$$

$$cu_\pm + dv_\pm = s_\pm v_\pm$$

if you need to find the $\underline{u}_+, \underline{u}_-$

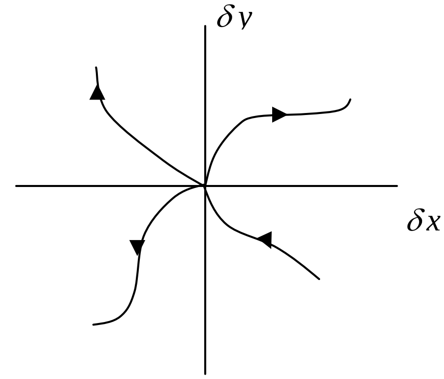
Now use this solution to classify all fixed points.

Classification

All depends on nature of the roots:

$s_\pm = \frac{1}{2}\left\{ p \pm \sqrt{p^2 - 4q} \right\}$

unstable



1. $p^2 > 4q, q > 0$

$s_\pm$ both real, distinct, same sign

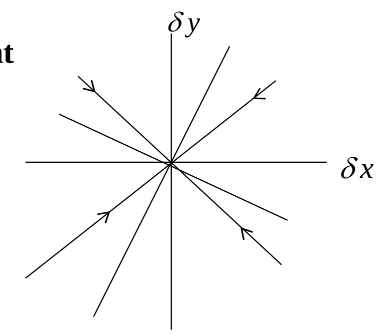
distinct so take $s_+ > s_-$ (say)

then for $t \to \infty, \underline{\delta x} \simeq C_1 e^{s_+ t} \underline{u}_+$ dominates, implying

for $p > 0$ $s_\pm > 0$ -explosive growth, $\underline{\delta x} \to \infty$ **unstable,**

for $p < 0$ $s_\pm < 0$ - decay $\underline{\delta x} \to 0$ **stable**

[special case $p^2 = 4q] \to$ **star point**



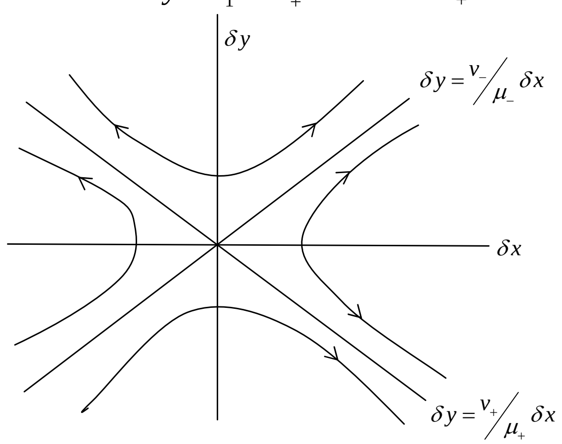
2. $p^2 > 4q \quad q < 0$

$s_\pm$ real, distinct –of opposite sign

consider $\underline{\delta x} = C_1 e^{s_+ t} \underline{u}_+ + C_2 e^{s_- t} \underline{u}_-$ and take $s_+ > 0 > s_-$

then as $t \to \infty$ $\underline{\delta x} \approx C_1 e^{s_+ t} \underline{u}_+$, as $t \to -\infty$ $\underline{\delta x} \approx C_2 e^{s_- t} \underline{u}_-$

so $t \to \infty,$ $\begin{matrix} \delta x = c_1 e^{s_+ t} u_+ \\ \delta y = c_1 e^{s_+ t} v_+ \end{matrix}$ $\delta y = \frac{v_+}{u_+} \delta x$ ditto $t \to -\infty, \delta y = \frac{v_-}{u_-} \delta x$



moves from one characteristic to the other as $t \to -\infty$ to $t \to +\infty$

Saddle point

(only time you need to evaluate u,v)

3. $p^2 < 4q \quad p \neq 0$

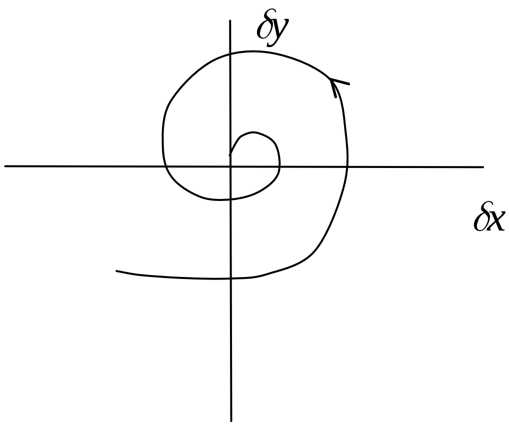
complex conjugate pair $s_{\pm} = \alpha \pm i\beta \qquad \alpha = p/2 \quad \beta = \left|\frac{p^2 - 4q}{2}\right|^{1/2}$

so that $\underline{\delta x} = C_1 e^{\alpha t} e^{+i\beta t} \underline{u}_+ + C_2 e^{\alpha t} e^{-i\beta t} \underline{u}_- = e^{\alpha t}[C_1 e^{+i\beta t} \underline{u}_+ + C_2 e^{-i\beta t} \underline{u}_-]$

if p, α +ve then we have explosive growth **spiral fixed point**

if - ve -then decay

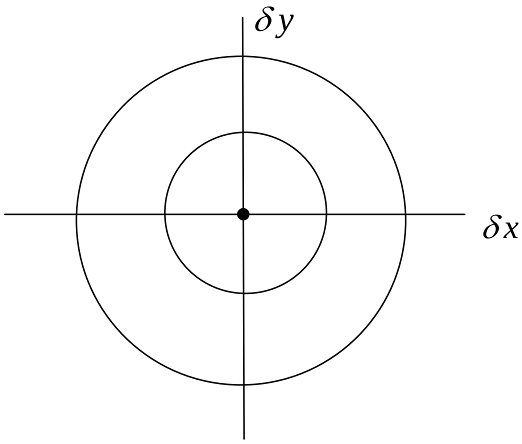
in addition - there is an oscillating part with 'frequency' β



4. $p^2 < 4q \quad p = 0$ (special case of (3)) so $q > 0$

now $s_{\pm} = \pm iq$

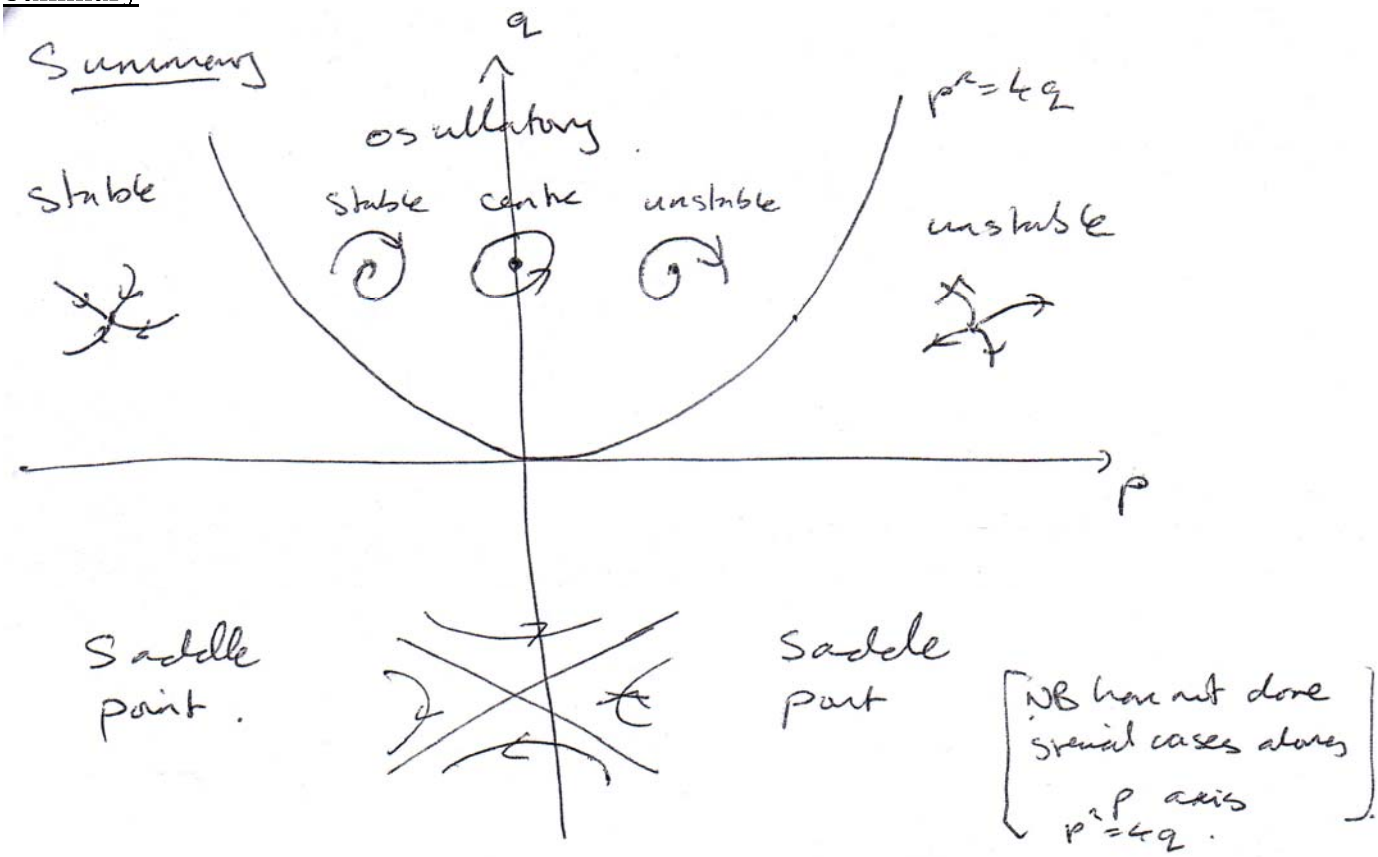
Substitute into case 3 above $\alpha = 0 \quad \beta = q$ then $\underline{\delta x} = C_1 e^{+iqt} \underline{u}_+ + C_2 e^{-iqt} \underline{u}_-$ oscillates with 'frequency' q.



No growth or decay – **centre fixed point**

c.f. simple pendulum

Summary

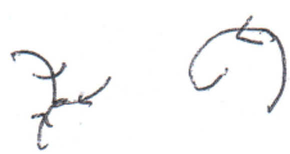


Recall $\quad p = a + d \qquad q = ad - bc$

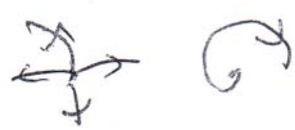
i.e. $\quad p = \dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} \qquad q = \dfrac{\partial F}{\partial x} \cdot \dfrac{\partial G}{\partial y} - \dfrac{\partial F}{\partial y} \cdot \dfrac{\partial G}{\partial x}$ $\qquad$ all evaluated at the fixed point $\bar{x}, \bar{y}$

Nomenclature

Attractors, Attractive fixed points:
stable nodes



Repellors, Repulsive fixed points:
unstable nodes



centre, **saddle**– not an attractor or repellor