Problem Sheet 1 – Non Linearity, Chaos and Complexity Solutions

Sheet 1 Question 1

(i) Particle motion in **B** field

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\wedge\mathbf{B} \qquad \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

Normalise $v^* = \dfrac{v}{v_0}$, $\quad t = t^*T \qquad r = r^*L \qquad B = B^*B_0$

sub in

$$m\frac{d\mathbf{v}^*\cancel{v_0}}{dt^*T} = q\cancel{v_0}\mathbf{v}^*\wedge\mathbf{B}^*B_0$$

$$\frac{d\mathbf{v}^*}{dt^*} = T.\frac{qB_0}{m}\mathbf{v}^*\wedge\mathbf{B}^* \qquad \text{which is normalised if } T = \left(\frac{qB_0}{m}\right)^{-1} = \frac{1}{\Omega}$$

also

$$\frac{d\mathbf{r}^*}{dt^*}\frac{L}{T} = \mathbf{v}^*v_0 \quad \text{ie:} \qquad v_0 = \frac{L}{T}$$

$$\text{so} \qquad L = v_0T = \frac{v_0}{\Omega}$$

solving the equations yields circular motion about **B** with frequency Ω, radius L.

Frequency is independent of velocity (particle energy), whereas gryroradius (L) depends on velocity.

(ii) Wave equation (ID here)

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2}$$

Normalise: $$\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^{*2}}\frac{\psi_0}{T^2} = \frac{\partial^2\psi^*\psi_0}{\partial x^{*2}L^2}$$

which is normalised (dimensionless) if

$$\frac{\partial^2\psi^*}{\partial t^{*2}} = \frac{\partial^2\psi^*}{\partial x^2} \qquad \frac{L}{T} = c\,.$$

Therefore, c is characteristic velocity of all structures regardless of length scale and is independent of amplitude ψ. Solutions are of the form $\psi = f\left(x+ct\right)+g\left(x-ct\right)$.

(iii) Conservation of quantity Q with number density n

$$\frac{\partial\left(nQ\right)}{\partial t} = \nabla.\left(n\,Q\,\mathbf{v}\right),$$

where Q is carried by "particles" of density n.

Normalise

$$\frac{\partial\left(n^*Q^*\right)}{\partial t^*}\frac{1}{T}\,\frac{1}{L^3}Q_0 = \nabla.\left(n^*Q^*\mathbf{v}^*\right)\frac{1}{L}\,\frac{1}{L^3}\,\frac{L}{T}Q_0 \qquad \text{ie} \qquad \mathbf{v}^* = \frac{\mathbf{v}}{v_0} = \frac{\mathbf{v}}{\left(\frac{L}{T}\right)}$$

then: $$\frac{\partial}{\partial t^*}\left(n^*Q^*\right) = \nabla^*.\left(n^*Q^*\mathbf{v}^*\right).$$

There is no characteristic scale if $v_0 = \dfrac{L}{T}$ equation just specifies that structures on all length and timescales are conserved.

<u>Sheet 1 Question 2</u>

$$F = F_0 + F_1 M + F_2 M^2 + F_3 M^3 + F_4 M^4$$

can always be written as

$$F = F_0' + F_2'\left(M - M_0\right)^2 + F_3'\left(M - M_0\right)^3 + F_4'\left(M - M_0\right)^4$$

since both are general polynormals up to degree 4 then $M \to M - M_0$ is the required transformation.

(i) For <u>symmetry</u> $F_3 = 0$.

We then have (dropping 's)

$$F\left(M\right) = F_0 + \alpha\left(T - T_c\right)M^2 + \beta M^4$$

extrema

$$\frac{\partial F}{\partial M} = 2\alpha\left(T - T_c\right)M + 4\beta M^3 = 2M\left(\alpha\left(T - T_c\right) + 2\beta M\right)^2$$

ie: at $M = 0$ or $M^2 = \alpha\frac{\left(T_c - T\right)}{2\beta}$.

But M is real so:

$$M = \pm\sqrt{\frac{\alpha\left(T_c - T\right)}{2\beta}}$$ is an extreme for $T < T_c$

look for <u>minima</u>

$$\frac{\partial^2 F}{\partial M^2} = 2\alpha\left(T - T_c\right) + 12\beta M^2.$$

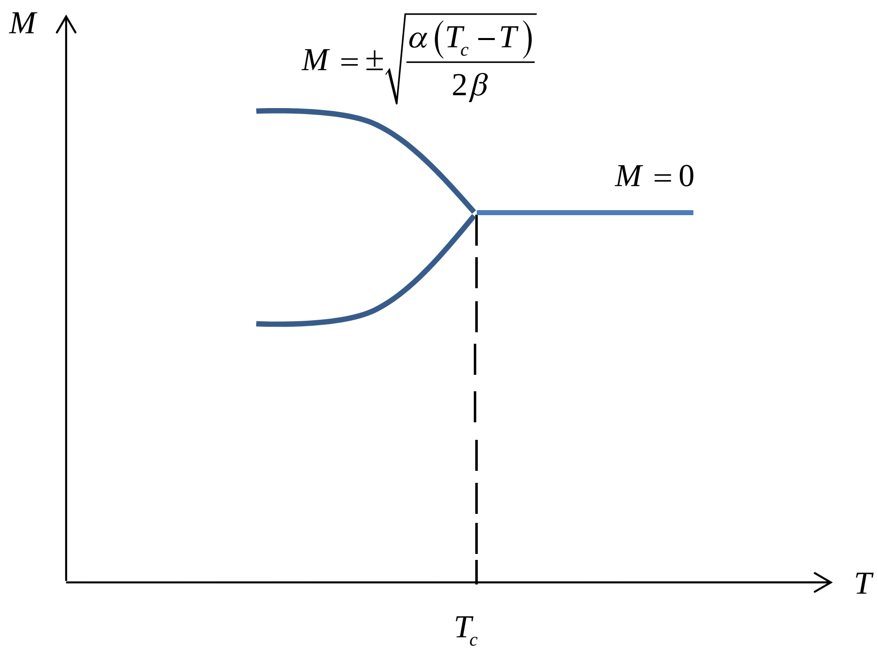
$M = 0$: min for $T > T_c$ max for $T < T_c$.

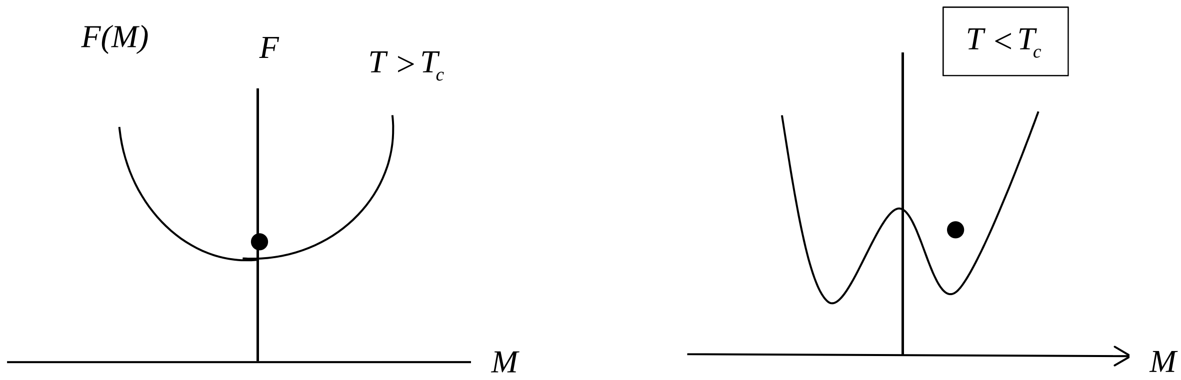
$$M = \pm\sqrt{\frac{\alpha\left(T_c - T\right)}{2\beta}}$$

$$\frac{\partial^2 F}{\partial M^2} = 2\alpha\left(T - T_c\right) + \cancel{12\beta}^{6}\,\frac{\alpha\left(T_c - T\right)}{\cancel{2\beta}} = -4\alpha\left(T - T_c\right)$$

min for $T < T_c$ max for $T > T_c$

pitchfork bifurcation at $T = T_c$





As we go from $T > T_c$ to $T < T_c$ system "falls" into one of the potential walls – which one is determined by fluctuations at $T = T_c$.

(ii) Asymmetric, now $F_3 = \gamma \neq 0$

$$\frac{\partial F}{\partial M} = 2\alpha\left(T - T_c\right)M + 3\gamma M^2 + 4\beta M^3$$

extrema now $\dfrac{\partial F}{\partial M} = 0 = M\left\{2\alpha\left(T - T_c\right) + 3\gamma M + 4\beta M^2\right\}$

$$M = 0, \quad M = \frac{-3\gamma \pm \sqrt{\left(9\gamma^2 - 4.2\alpha\left(T - T_c\right).4\beta\right)}}{2.4\beta}$$

Two real values of M when

$$9\gamma^2 > 32\alpha\beta\left(T - T_c\right)$$

write M as $$M = \frac{-3\gamma \pm 3\sqrt{\gamma^2 - \gamma_c^2}}{8\beta}.$$

Consider

$$\frac{\partial^2 F}{\partial M^2} = 2\alpha\left(T - T_c\right) + 6\gamma M + 12\beta M^2$$

$M = 0$ is min for $T > T_c$.

For $M \neq 0$ extrema given by $2\alpha\left(T - T_c\right) + 3\gamma M + 4\beta M^2 = 0$ which gives $\dfrac{\partial^2 F}{\partial M^2} = 3\gamma M + 8\beta M^2$,

or
$$\frac{\partial^2 F}{\partial M^2} = M\left(\pm 3\sqrt{\gamma^2 - \gamma_c^2}\right)$$

Then in addition to $M = 0$ solution

$\gamma^2 > \gamma_c^2$ 2 real $M \neq 0$ roots, one max, one min

$\gamma^2 = \gamma_c^2$ $M = \dfrac{-3\gamma}{8\beta}$ $\gamma_c^2 = \dfrac{32\alpha\beta}{9}\left(T - T_c\right)$
$\Rightarrow T > T_c$.

$\gamma^2 < \gamma_c^2$ - M imaginary no max/min.

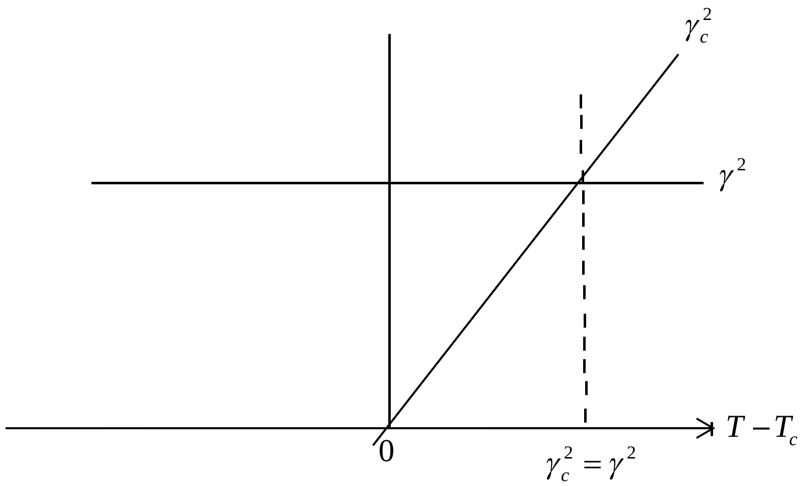
Also at $\gamma_c^2 = 0$ $T = T_c$ $M = \dfrac{-3\gamma \pm 3\gamma}{8\beta}$ ie: $M = 0$ (a)

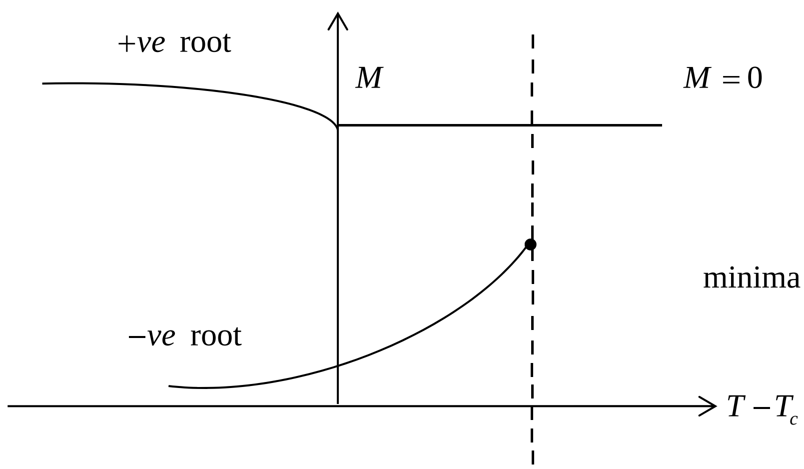
$M = \dfrac{-6\gamma}{8\beta}$ (b)

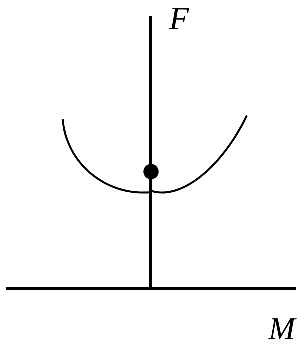
(b) is $(-)$ ve root hence $\dfrac{\partial^2 F}{\partial M^2} > 0$ is a min

(a) is inflexion. Finally, for $\gamma_c^2 < 0$ 2 real roots, both min and $M = 0$ is max

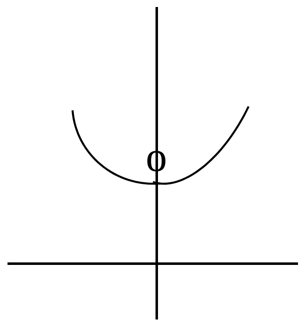
graphically

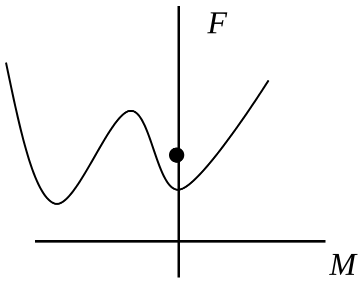




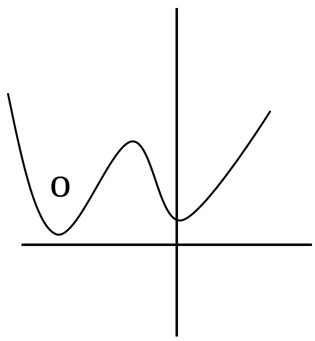


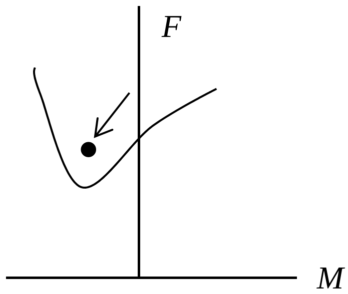
$T > T_c$
$\gamma_c^2 > \gamma^2$



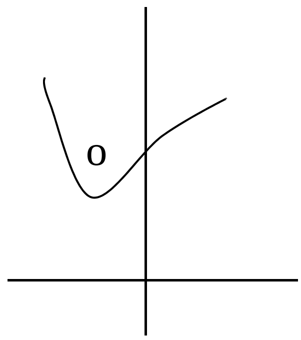


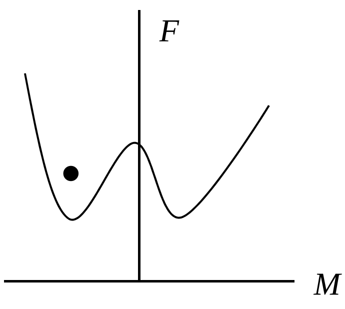
$T > T_c$
$\gamma_c^2 < \gamma^2$



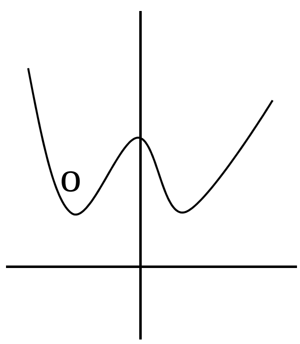


$T = T_c$
$\gamma_c = 0$





$T < T_c$



● – going from $T > T_c$ } hysteresis
o – going from $T < T_c$ }

Now fluctuations are unimportant.

iii) Van der Vaal

Expand for $bm << 1$
using

$$\ln(1-bm) \simeq -\left[bM + \left(\frac{bM}{2}\right)^2 + \left(\frac{bM}{3}\right)^3 .....\right]$$

Substitute into F

$$F = \frac{T}{b}\left[-\cancel{bM} - \left(\frac{bM}{2}\right)^2 - \left(\frac{bM}{3}\right)^3 + (bM)^2 + \left(\frac{bM}{2}\right)^3 + \left(\frac{bM}{3}\right)^4\right] + \cancel{MT} - \frac{aM^2}{2}$$

$$= M^2\left(\frac{bT}{2} - \frac{a}{2}\right) + M^3\frac{b^2}{6}T + b^3\frac{TM^4}{12}$$

then $\quad \alpha(T - T_c) \equiv \dfrac{bT - a}{2} = \dfrac{b}{2}\left(T - \dfrac{a}{b}\right)$

$$T_c = \frac{a}{b}.$$

Sheet 1 Question 3

(i) $$\frac{dq}{dt} = \sin q$$

fixed points $\sin \bar{q} = 0$ $\bar{q} = n\pi$ n integer

linearize about fixed points

$$q(t) = \bar{q} + \delta q$$

$$\frac{d\delta q}{dt} = \sin(\bar{q} + \delta q) = \sin \bar{q} \cancel{\cos \delta q} + \cos \bar{q} \sin \delta q = 0$$

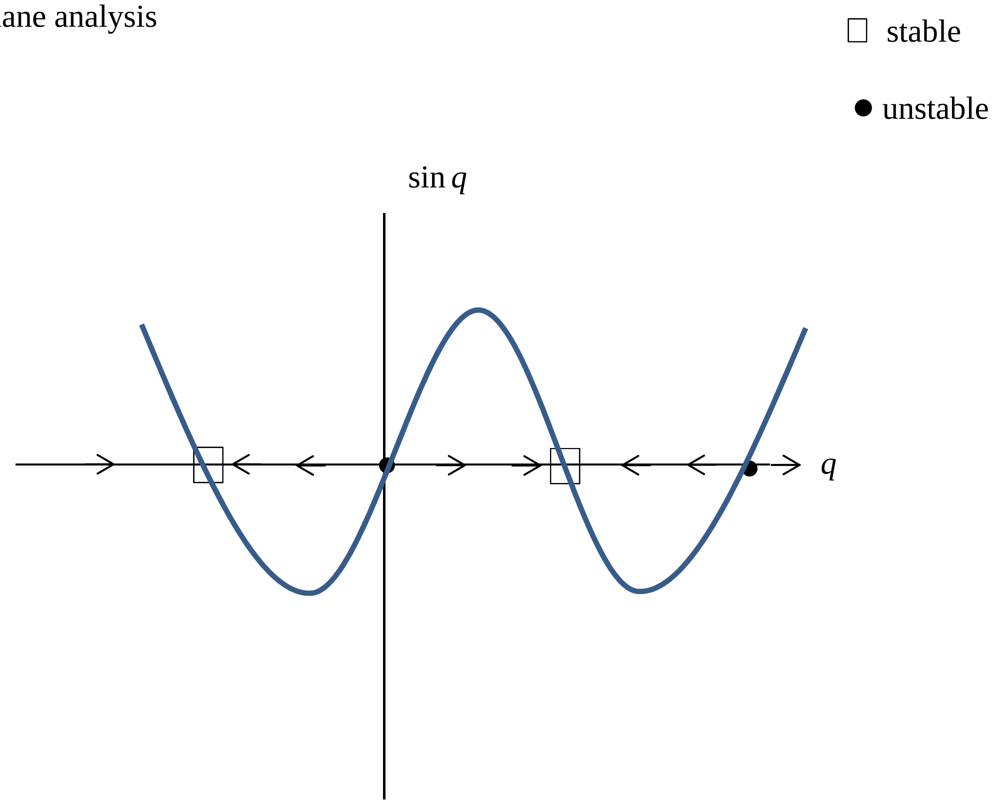
$\sin \delta q \simeq \delta q, \cos \delta q \simeq 0$ as δq is small

then $$\frac{d\delta q}{dt} = (-1)^N \delta q$$

solution is of form $\delta q = \delta q_0 e^{st}$

$s + ve$	for n even	–	unstable
$s - ve$	for n odd	–	stable

Phase plane analysis



$\rightarrow$ flow arrows $+ve\, q$ for $\frac{dq}{dt} - ve$ q increases with time

$-ve$ for $\frac{dq}{dt} - ve$ q decreases with time

ii) $\frac{dq}{dt} = \alpha q - \beta q^2$

fixed points $\alpha \bar{q} - \beta \bar{q}^2 = 0$

$$\bar{q}\left(\alpha - \beta \bar{q}\right) = 0$$

ie: $\bar{q} = 0$ or $\bar{q} = \frac{\alpha}{\beta}$.

Stability $q\left(t\right) = \bar{q} + \delta q\left(t\right)$

Sub in $\frac{d}{dt}\left(\delta q\right) = \alpha\left(\bar{q} + \delta q\right) - \beta\left(\bar{q} + \delta q\right)^2$

$$= \alpha \bar{q} - \beta \bar{q}^2 + \delta q\left(\alpha - 2\beta \bar{q}\right) + 0\left(\delta q^2\right)$$

neglect as small

but $\alpha \bar{q} - \beta \bar{q}^2 = 0$

So, $\frac{d\left(\delta q\right)}{dt} = \delta q\left(\alpha - 2\beta \bar{q}\right)$,

then, assuming that $\delta q = \delta q_0 e^{st}$

we will have $s + ve$ for $\alpha - 2\beta\, \bar{q} > 0$

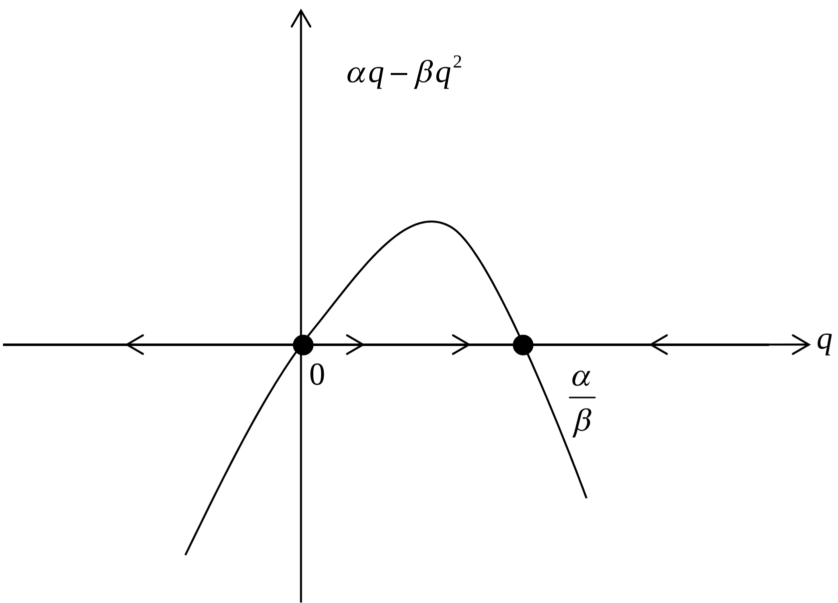
$s - ve$ for $\alpha - 2\beta\, \bar{q} < 0$.

Take $\alpha, \beta > 0$

then $\bar{q} = 0$ is $s + ve$, ie: unstable (repellor)

$\bar{q} = \frac{\alpha}{\beta}$ is $s - ve$, ie: stable (attractor)

Phase plane – sketch $\frac{dq}{dt}$ vz q



Problem Sheet 2 – Non Linearity, Chaos and Complexity Solutions

Sheet 2 Question 1.

i) Undamped oscillator

$$\frac{d^2x}{dt^2} = -\omega^2 \sin x \, .$$

Can integrate this once $\times \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} \cdot \frac{dx}{dt} = -\omega^2 \sin x \frac{dx}{dt}$$

$$\rightarrow \quad \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \omega^2 \cos x = E = \text{constant.}$$

To obtain the dynamics – obtain fixed points, phase plane, etc.

first write as two coupled first order DE

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -\omega^2 \sin x$$

fixed points $\bar{y} = 0$, $\sin \bar{x} = 0$ or $\bar{x} = n\pi$.

Stability

Linearize $\begin{aligned} y &= \bar{y} + \delta y \qquad x = \bar{x} + \delta x \\ &= \delta y \end{aligned}$

then

$$\frac{d\delta x}{dt} = \delta y \qquad \frac{d\delta y}{dt} = -\omega^2 \sin\left(\bar{x} + \delta x\right)$$
$$= -\omega^2 \sin\left(n\pi + \delta x\right)$$

use

$$\sin\left(A + B\right) = \sin A \cos B + \cos A \sin B$$

$$\sin\left(n\pi + \delta x\right) = \underbrace{\cancel{\sin n\pi}}_{=0} \cos \delta x + \cos n\pi \sin \delta x$$

$\cos\left(n\pi\right) = \left(-1\right)^n$ and $\sin \delta x \simeq \delta x$ since δx small

so $$\frac{d\delta x}{dt} = \delta y \qquad \frac{d\delta y}{dt} = -\omega^2 \left(-1\right)^n \delta x \, .$$

Sufficiently simple to go direct to second order DE ie: $\quad \dfrac{d^2\delta x}{dt^2} = -\omega^2 \left(-1\right)^N \delta x$ for which we know solutions of form $\delta x = Ae^{\lambda t} + Be^{-\lambda t}$.

Then *n* even

$$\frac{d^2\delta x}{dt^2} = -\omega^2 \delta x \qquad \delta x = Ae^{i\omega t} + Be^{-i\omega t},$$

n odd

$$\frac{d^2\delta x}{dt^2} = +\omega^2 \delta x \qquad \delta x = Ae^{\omega t} + Be^{-\omega t}.$$

So, <u>*n* even</u> are <u>centre</u> fixed points

δx is oscillatory and $\delta y = \dfrac{d\delta x}{dt} = i\omega Ae^{i\omega t} - i\omega Be^{-\omega t}$

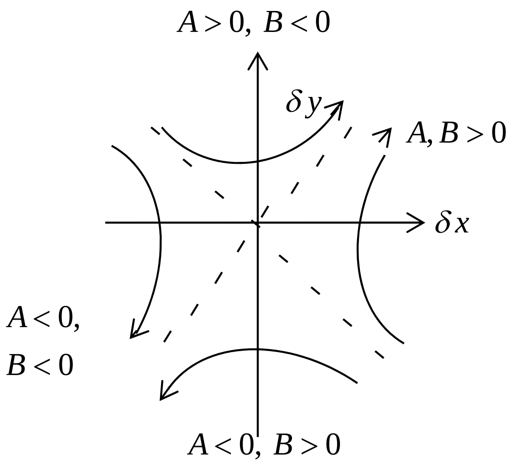
recall $\quad i = e^{\frac{i\pi}{2}}$ and $-i = e^{\frac{-i\pi}{2}}$ (complex numbers $x + iy = re^{i\theta}$)

So, $\quad \delta y = \omega A e^{i\left(wt+\frac{\pi}{2}\right)} + \omega B e^{-i\left(wt+\frac{\pi}{2}\right)}$

- out of phase $\dfrac{\pi}{2}$ with δx

n odd
$$\delta x = Ae^{\omega t} + Be^{-\omega t}$$
$$\delta y = \omega Ae^{\omega t} - \omega Be^{-\omega t}$$

Saddle point



Separatrix has lines given by

$$t \to \infty \qquad \frac{\delta y}{\delta x} = \frac{\omega Ae^{\omega t}}{Ae^{\omega t}} = \omega$$

$$t \to -\infty \qquad \frac{\delta y}{\delta x} = \frac{-\omega Be^{-\omega t}}{Be^{-\omega t}} = -\omega.$$

Topology: constant of the motion defines the phase plane orbits: and

$E = \dfrac{y^2}{2} - \omega^2 \cos x$ has symmetry in y and x

Phase plane: see lecture notes and handouts for sketch.

Separatrix has $x = \pm\pi \to \cos x = -1$ when $y = 0$, $E_c = \omega^2$ on the separatrix.

ii) <u>Damped oscillator</u>

$$\frac{d^2x}{dt^2}+\lambda\frac{dx}{dt}+\omega^2\sin x=0$$

Now we will have first order DE:

$$\frac{dx}{dt}=y$$

$$\frac{dy}{dt}=-\omega^2\sin x-\lambda y\,.$$

Fixed point $\bar{y}=0,\omega^2\sin\bar{x}=0$,

ie: as undamped case $\bar{y}=0,\quad \bar{x}=n\pi$.

Stability analysis

$$y=\delta y\quad x=\bar{x}+\delta x$$

So $\dfrac{d\delta x}{dt}=\delta y\quad \dfrac{d\delta y}{dt}=->\delta y-\omega^2(-1)^n\,\delta x$ (as before – same identities).

Now more complicated – solve using general formula as in lectures (given in detail here).

We write $\delta\mathbf{x}=\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$

then pair of equations are just

$$\frac{d\delta\mathbf{x}}{dt}=\mathbf{J}\cdot\delta\mathbf{x}\qquad \mathbf{J}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

where we use notation

$$\frac{d\delta x}{dt}=a\,\delta x+b\,\delta y\qquad\qquad \mathbf{J}=\begin{pmatrix}0 & 1\\ -\omega^2(-1)^n & -\lambda\end{pmatrix}$$

$$\frac{d\delta y}{dt}=c\,\delta x+d\,\delta y$$

We then have solutions of the form

$$\delta\mathbf{x}=C_1\,e^{S_+t}\mathbf{u}_+ + C_2\,e^{S_-t}\mathbf{u}_-$$

where the eigenvalues $s_\pm$ are solutions of $\begin{vmatrix} a-s & b \\ c & d-s \end{vmatrix} = 0$

ie: $0-(a-s)(d-s)-bc = s^2 - s(a+d) + ad - bc$

thus $s = \frac{1}{2}\left\{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right\}$

here, this is $s_\pm = \frac{1}{2}\left\{-\lambda \pm \sqrt{\lambda^2 - 4\left(\omega^2(-1)^n\right)}\right\}$.

Two cases:

n odd $\quad s_\pm = \frac{1}{2}\left\{-\lambda \pm \sqrt{\lambda^2 + 4\omega^2}\right\}$

n even $\quad s_\pm = \frac{1}{2}\left\{-\lambda \pm \sqrt{\lambda^2 - 4\omega^2}\right\}$

***n* odd**:

$s_\pm$ are real, distinct. $\quad s_\pm = \frac{1}{2}\left\{-\lambda \pm \lambda\sqrt{1 + \frac{4\omega^2}{\lambda^2}}\right\}$

for $\lambda + ve$ or $-ve$

$s_\pm$ are real and of opposite sign – saddle points (as before).

***n* even**:

$s_\pm$ may be complex $\quad s_\pm = \frac{1}{2}\left\{-\lambda \pm \lambda\sqrt{1 - \frac{4\omega^2}{\lambda^2}}\right\}$

complex if $4\omega^2 > \lambda^2$ otherwise real.

For $\lambda > 0$ – decay to stable fixed point
$\lambda < 0$ – growth – unstable fixed point

If $4\omega^2 > \lambda^2$ these are spiral.

Note that if $\lambda = 0$ we have $s_\pm = \pm\omega$ $\quad$ n odd – saddle and
$s_\pm = \pm i\omega$ $\quad$ n even- circle fixed points

So, essentially here, circle points $\rightarrow$ spiral fixed points for $4\omega^2 > \lambda^2$.

Topology

Look for symmetries in original DE.

$$\frac{d^2x}{dt^2}+\lambda\frac{dx}{dt}+\omega^2\sin x=0$$

$$x\to -x \quad -\frac{d^2x}{dt^2}+(-1)\lambda\frac{dx}{dt}+\omega^2\sin x(-1)=0$$

Same equation $\to$ $x>-x$ is this symmetry by reflection? Check what happens to y (below).

$$t\to -t \qquad (-1)^2\frac{d^2x}{dt^2}+(-1)\lambda\frac{dx}{dt}+\omega^2\sin x=0$$

$t\to -t$ is $\lambda\to -\lambda$,

ie: damping and increasing $t\equiv$ growth and decreasing t

Sufficient to sketch one of these and note that

$y=\dfrac{dx}{dt}$ so $x\to -x$ gives $y\to -y$ rotational symmetry.

See course handout for sketch

<u>Sheet 2 Question 2</u>

Lotka-Volterra

In our original notation

$$\frac{dx}{dt} = (\lambda - \alpha y)x$$

$$\frac{dy}{dt} = -(\eta - \beta x)y$$

<u>Fixed points</u>

$$(\lambda - \alpha\bar{y})\bar{x} = 0 \qquad \bar{x} = 0 \text{ or } \bar{y} = \frac{\lambda}{\alpha}$$

$$-(\eta - \beta\bar{x})\bar{y} = 0 \qquad \bar{y} = 0, \text{ or } \bar{x} = \frac{\eta}{\beta}$$

ie: $\bar{x} = 0, \bar{y} = 0 \qquad \bar{x} = \dfrac{\eta}{\beta},\ \bar{y} = \dfrac{\lambda}{\alpha}$.

<u>Stability</u> – linearise

$$x = \bar{x} + \delta x \qquad y = \bar{y} + \delta y$$

$$\frac{d\delta x}{dt} = \lambda(\bar{x} + \delta x) - \alpha(\bar{y} + \delta y)(\bar{x} + \delta x)$$

$$= \underbrace{\cancel{\lambda\bar{x} - \alpha\bar{y}\bar{x}}}_{=0} + (\lambda - \alpha\bar{y})\delta x - \alpha\bar{x}\delta y - \alpha\,\cancel{\delta x\delta y}$$

$$\frac{d\delta x}{dt} = (\lambda - \alpha\bar{y})\delta x - \alpha\bar{x}\delta y$$

$$\frac{d\delta y}{dt} = -\eta(\bar{y} + \delta y) + \beta(\bar{x} + \delta x)(\bar{y} + \delta y)$$

$$= \underbrace{\cancel{-\eta\bar{y} + \beta\bar{x}\bar{y}}}_{=0} + \delta y(-\eta + \beta\bar{x}) + \delta x(\beta\bar{y}) + \beta\,\cancel{\delta x\delta y}$$

$$\frac{d\delta y}{dt} = (-\eta + \beta\bar{x})\delta y + \beta\bar{y}\delta x$$

again – can use formula but shown in full here: write in the form $\dfrac{d}{dt}\delta\mathbf{x} = \mathbf{J}\cdot\delta\mathbf{x}$

then in notation of notes $\quad \mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\lambda - \alpha\bar{y}) & -\alpha\bar{x} \\ \beta\bar{y} & (\beta\bar{x} - \eta) \end{bmatrix}$

with eigenvalues

$$s_\pm = \frac{1}{2}\left\{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right\}$$

Consider two fixed points

$$\bar{x} = 0, \bar{y} = 0 \qquad \mathbf{J} = \begin{bmatrix} \lambda & 0 \\ 0 & -\eta \end{bmatrix}$$

$$s_\pm = \frac{1}{2}\left\{(\lambda-\eta) \pm \sqrt{\underbrace{(\lambda-\eta)^2 + 4(\lambda\eta)}_{\lambda^2 - 2\lambda\eta + \eta^2 + 4\lambda\eta = (\lambda+\eta)^2}}\right\}$$

$$s_\pm = \frac{1}{2}\left\{(\lambda-\eta) \pm (\lambda+\eta)\right\}$$

ie: $s_+ = \lambda \quad s_- = -\eta$ <u>saddle point</u>.

Consider fixed point

$$\bar{x} = \frac{\eta}{\beta} \quad \bar{y} = \frac{\lambda}{\alpha}$$

$$\mathbf{J} = \begin{bmatrix} 0 & \dfrac{-\alpha\eta}{\beta} \\ \dfrac{\beta\lambda}{\alpha} & 0 \end{bmatrix}$$

$$S_\pm = \frac{1}{2}\left\{\pm\sqrt{0 - 4\left(\frac{\beta\lambda}{\alpha}\right)\left(+\frac{\alpha\eta}{\beta}\right)}\right\}$$
$$= \pm\sqrt{-\lambda\eta}$$

ie: wholly imaginary – <u>centre</u> fixed point.

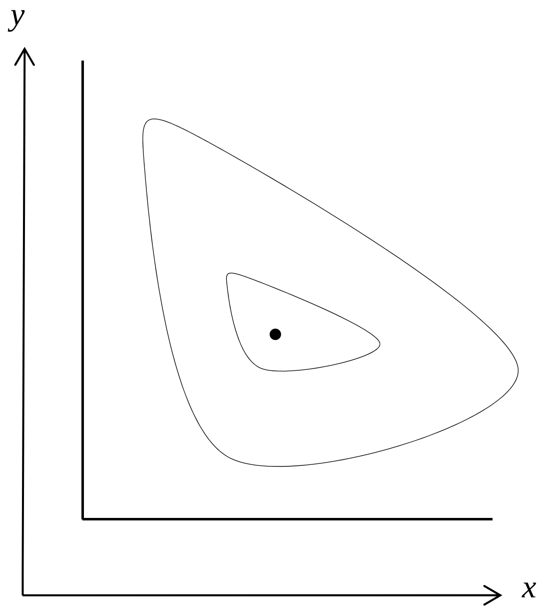
Topology: no t symmetry since

$$t \to -t \quad -\frac{dx}{dt} = (\lambda - \alpha y)x$$

$$-\frac{dy}{dt} = -(\eta - \beta x)y$$

Similarly, no symmetries in $x-y$ except change of sign in $\lambda, \eta, \beta, \alpha$ – unrealistic.

Phase plane:



$$C = (\eta \ln R - \beta R) - (\alpha F - \lambda \ln F)$$

$$\frac{dC}{dt} = \frac{\eta}{R}\frac{dR}{dt} - \beta\frac{dR}{dt} - \alpha F\frac{dF}{dt} + \lambda\frac{1}{F}\frac{dF}{dt}$$

$$= (\lambda - \alpha F)(\eta - \beta R) - (\lambda - \alpha F)(\eta - \beta R)$$

$$= 0.$$

Hence C is a constant and different values of C specify trajectories (closed) about the centre fixed point.

Sheet 2 Question 3

Proof of existence of a limit cycle:

given $$\frac{dx}{dt} = x - y - x\left(x^2 + 2y^2\right), \frac{dy}{dt} = x + y - y\left(x^2 + y^2\right)$$

convert to plane polar coordinates r, θ use

$$x = r\cos\theta \quad y = r\sin\theta$$

and $$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \quad x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}$$

then $$r^2\frac{d\theta}{dt} = x\left[x + \cancel{y} - \cancel{y(x^2} + y^2)\right] - y\left[\cancel{x} - y - \cancel{x(x^2} + 2y^2)\right]$$

$$= x^2 + y^2 + xy^3 = r^2 + r^4\cos\theta\sin^3\theta$$

$$r\frac{dr}{dt} = x\left[x - \cancel{y} - x(x^2 + 2y^2)\right] + y\left[\cancel{x} + y - y(x^2 + y^2)\right]$$

$$= x^2 + y^2 - x^4 + 3y^2x^2 - y^4$$
$$= x^2 + y^2 - (x^2 + y^2)^2 - x^2y^2$$

$$= r^2 - r^4 - r^4\cos^2\theta\sin^2\theta .$$

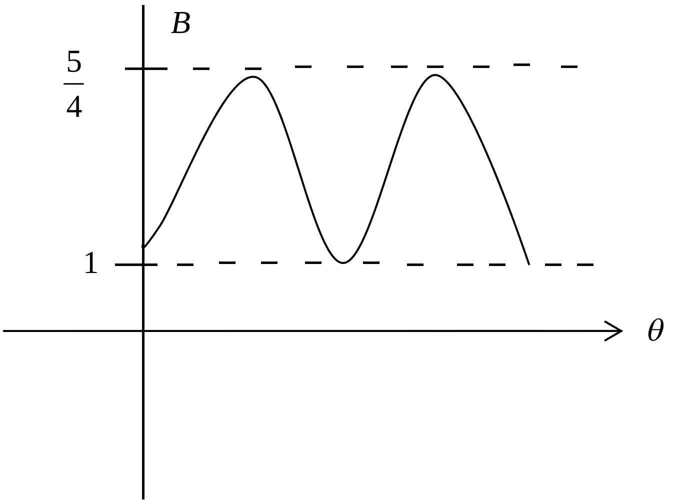
Identity:

$$\sin\left(A + B\right) = \sin A\cos B + \cos A\sin B$$
$$\sin 2A = 2\sin A\cos A$$

Giving
$$r^2\frac{d\theta}{dt} = r^2 + r^4\frac{1}{2}\sin^2\theta\sin 2\theta$$
$$r\frac{dr}{dt} = r^2 - r^4\left(1 + \frac{1}{4}\sin^2 2\theta\right)$$

now $$r\frac{dr}{dt} = r^2 - r^4\underbrace{\left(1 + \frac{1}{4}\sin^2 2\theta\right)}_{B} = r^2\left(1 - r^2 B\right)$$

Bracket B is bounded $\left[1, \frac{5}{4}\right]$



hence

$r \to \infty \quad \frac{dr}{dt} < 0$

$r \to 0 \quad \frac{dr}{dt} > 0$

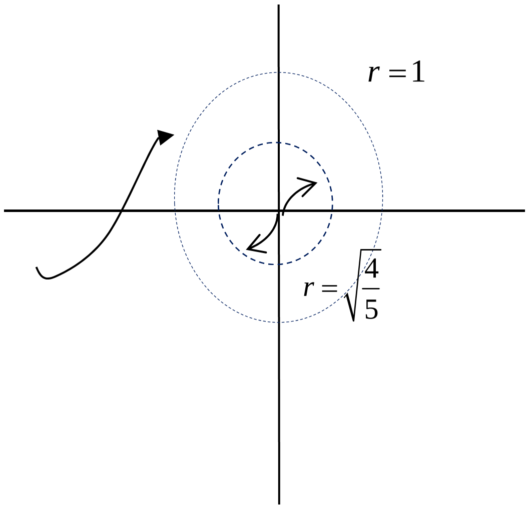
for any θ

Minimum value of $B = 1$ has $\frac{dr}{dt} = 0$ for $r = 1$

Maximum $\quad B = \frac{5}{4}$ has $\frac{dr}{dt} = 0$ for $r = \sqrt{\frac{4}{5}}$

If $\quad r > 1, \frac{dr}{dt} < 0$

If $\quad r < \sqrt{\frac{4}{5}}, \frac{dr}{dt} > 0$



orbits are attracted into the annulus for any θ

and $\quad \frac{d\theta}{dt} \neq 0$ in annulus

therefore, limit cycle.

Problem Sheet 3 – Non Linearity, Chaos and Complexity Solutions

Sheet 3 Question 1

Lyapunov exponent.

For a general map $x_{n+1} = f\left(x_n\right)$

This has iterates $x_1 \ldots x_n$ initial condition x_0 so $x_1 = f\left(x_0\right)$, $x_2 = f\left(x_1\right)$, etc.

For initially neighbouring points $\overline{x}_0 = x_0 + \varepsilon_0$, x_0 with $\varepsilon_0 << 1$.

After one iterate $\overline{x}_1 = f\left(\overline{x}_0\right) = f\left(x_0 + \varepsilon_0\right) = f\left(x_0\right) + \varepsilon_0 \dfrac{df}{dx}\left(x_0\right) \ldots$ by Taylor expansion.

Now, two points separated by ε_1 after one iterate, i.e.

$\overline{x}_1 = x_1 + \varepsilon_1 = f\left(x_0 + \varepsilon_0\right) = f\left(x_0\right) + \varepsilon_0 \dfrac{df}{dx}\left(x_0\right) + \ldots$ so $\varepsilon_1 = \varepsilon_0 \; f'\left(x_0\right)$ to first order in ε_0.

Generally, for j^{th} iterate we have $\overline{x}_j = x_j + \varepsilon_j$ thus $\varepsilon_j = \varepsilon_{j-1} \; f'\left(x_{j-1}\right)$ provided $\varepsilon_j << 1 \quad 0 < j < n$.

Then,

$$\begin{aligned} \overline{x}_n = x_n + \varepsilon_n &= x_n + \varepsilon_{n-1} \; f'\left(x_{n-1}\right) \\ &= x_n + \varepsilon_{n-2} \; f'\left(x_{n-2}\right) f'\left(x_{n-1}\right) \\ &= x_n + \varepsilon_0 f'\left(x_0\right) f'\left(x_1\right) \ldots . f'\left(x_{n-1}\right) \end{aligned}$$

or

$$\overline{x}_{n+1} = x_{n+1} \varepsilon_0 f'\left(x_0\right) \ldots f'\left(x_n\right)$$

$$\overline{x}_n = x_n + \varepsilon_0 \prod_{j=0}^{n-1} f'\left(x_j\right)$$

Now write $f'\left(x_j\right) = e^{\ln\left[f'\left(x_j\right)\right]}$

and neglecting signs of f' we can write $\overline{x}_n = x_n + \varepsilon_0 \exp\left[\sum_{j=0}^{n-1} \ln\left|f'\left(x_j\right)\right|\right]$

and hence Lyapurov exponent defined as: $\lambda = \lim\limits_{n\to\infty} \dfrac{1}{n} \sum_{j=0}^{n-1} \ln\left|f'\left(n_j\right)\right|$

which is a measure of exponential divergence $\overline{x}_n - x_n = \varepsilon_0 e^{n\lambda}$

If $\lambda < 0$ then $\overline{x}_n \to x_n$ for large *n*, converging – this is <u>attractor</u> (attractive fixed point).

If $\lambda > 0$ – exponential divergence for large *n*. repellor (repulsive fixed point).

<u>Sheet 3 Question 2</u>

The map $$x_{n+1} = \frac{x_n}{a} \qquad 0 < x < a$$

$$x_{n+1} = \frac{(1 - x_n)}{(1-a)} \qquad a < x < 1$$

where $0 < a < 1$.

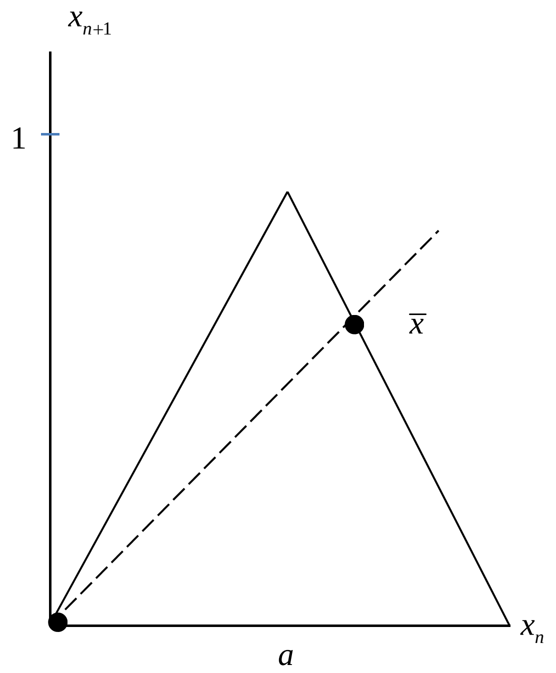
Consider <u>fixed points</u> $\bar{x} = 0$ and

$\bar{x}$ in the range $[a, 1]$

ie: $\bar{x} = \dfrac{1-\bar{x}}{(1-a)}$

$\bar{x} - a\bar{x} = 1 - \bar{x}$

or $(2-a)\bar{x} = 1$



thus fixed points $\bar{x} = 0 \quad \bar{x} = \dfrac{1}{(2-a)}$.

<u>Stability</u>

Linearize $\quad x_n = \bar{x} + \delta x_n \qquad x_{n+1} = \bar{x} + \delta x_{n+1}$.

sub into $$x_{n+1} = \frac{(1-x_n)}{(1-a)}$$

$$\bar{x} + \delta x_{n+1} = \frac{(1 - \bar{x} - \delta x_n)}{1-a}$$

$$\cancel{\bar{x}} + \delta x_{n+1} = \cancel{\frac{(1-\bar{x})}{(1-a)}} - \frac{\delta x_n}{(1-a)}$$

ie: $\quad \delta x_{n+1} = \dfrac{-\delta x_n}{1-a} = \dfrac{\delta x_n}{(a-1)}$ hence unstable for <u>all</u> $0 < a < 1$: $\quad \delta x_{n+1} = \dfrac{1}{(a-1)^{n+1}} \delta x_0$

$\underline{\underline{M^2(x)}}$

Find "folding points" such that $M^2(x)=0$ or $M^2(x)=1$.

$\underline{\underline{M^2(x)=0}}$

Clearly, $M^2(x)=0$ for $M(x)=0$ or 1 ie: $M(x)=0$ $\quad x_R=0$ $\quad$ or $\quad x_R=a$

$M^2(x)=1$

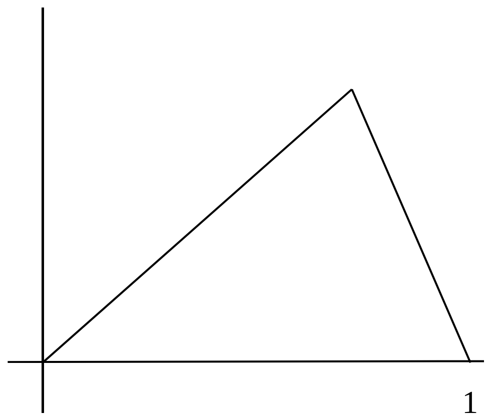
Since $M(a)=1$ we seek x_R such that $M(x_R)=a$.

Two possibilities

$$0<x<a \qquad M(x)=\frac{x}{a} \qquad a=\frac{x_R}{a} \qquad x_R=a^2$$

$$a<x<1 \qquad M(x)=\frac{1-x}{1-a} \qquad a=\frac{1-x_R}{1-a} \qquad x_R=1-a(1-a)$$

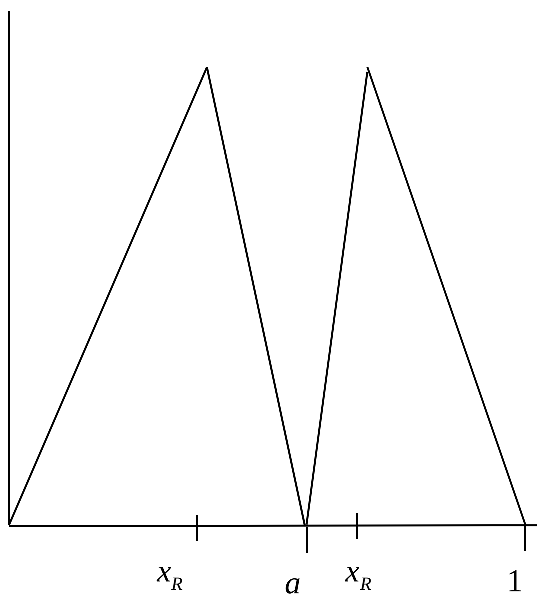
Sketch:



here $a>\frac{1}{2}$ thus

$a^2>\frac{a}{2}$ (try it!).

$1-a(1-a)<\frac{1-a}{2}$



Same topology as symmetric case (stretching and folding) just asymmetric.

Lyapunov exponent for $M(x)$

Fixed point is in the range $[a,1]$

so $\quad M(x)=\dfrac{(1-x)}{(1-a)}$

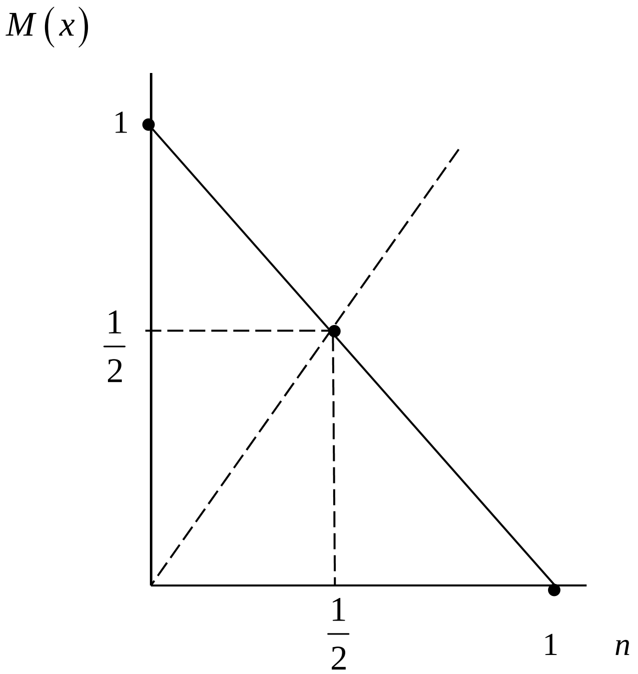
$$\frac{dM}{dx}=\frac{1}{1-a} \quad \text{and} \quad 0<a<1$$

so $\quad \dfrac{dM}{dx}>1 \quad$ hence $\quad \lambda=\ln\left[\dfrac{1}{1-a}\right]$

$\lambda>0$ exponential divergence

Special cases $\quad a=0$ and $a=1$

$\underline{a=0}$



Now $M(x)=1-x$

fixed point $\bar{x}=1-\bar{x}$

$$\bar{x}=\frac{1}{2}$$

gradient $\dfrac{dM}{dx}=-1$ everywhere.

Lyapunov exponent $\lambda=\ln|-1|=0$

$\lambda=0$ is marginally stable –

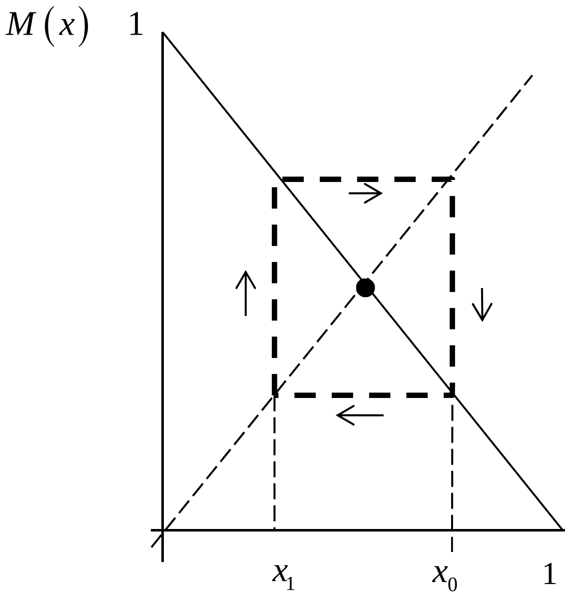
now $\quad M(\bar{x})=\bar{x}=\dfrac{1}{2}$

for any $\quad 0<x<1, \quad x\neq\dfrac{1}{2} \quad$ write $\quad \bar{x}_0=\bar{x}+\varepsilon$

$$M(x_0)=1-\bar{x}-\varepsilon=x_1$$

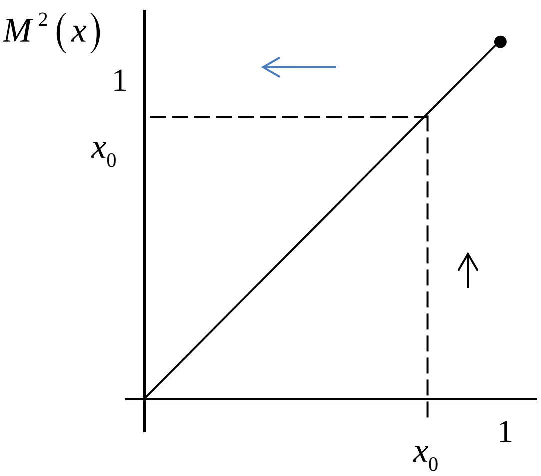
$$M^2(x_0)=M(x_1)=1-(1-\bar{x}-\varepsilon)=\bar{x}+\varepsilon=x_0$$

hence $M^2(x_0) = x_0$ these are period two orbits



graphically

or by simply calculating $M^2(x) = 1 - (1 - x) = x$



This is a <u>return</u> map $M^2(x) = x$

<u>a=1</u>

$M(x) = x$ again, a return m

Note that $\dfrac{dM}{dx} = \dfrac{d(x)}{dx} = 1$ so Lyapunov exponent $\gamma = \ln|1| = 0$ marginally stable

true for both orbits of $M(x, a = 1)$ and of $M^2(x, a = 0)$ [period 2 orbits of M]

Sheet 3 Question 3

We have

$$\frac{dg}{dt} = \lambda_g g - eR \qquad \frac{dR}{dt} = \lambda_b g - \alpha FR$$

and from Lotka-Volterra equations $\frac{dF}{dt} = \left(\eta - \beta R\right)F$

fast growing grass $\lambda_g \gg \lambda_B$

then we assume the grass is enslaved to the rabbits –

$$\frac{dg}{dt} = 0 \qquad \lambda_g g - eR = 0 \qquad g = \frac{eR}{\lambda_g}$$

giving $\frac{dR}{dt} = \frac{e\lambda_b}{\lambda_g} R - \alpha FR = \left(\lambda - \alpha F\right)R$

where $\lambda = \frac{e\lambda_b}{\lambda_g}$

which are the original Lotka-Volterra equations so dynamics of foxes and rabbits are the same and the grass is enslaved to rabbits.

Problem Sheet 4 – Non Linearity, Chaos and Complexity Solutions

Sheet 4 Question 1

(a) $B = 0$ case

$$F(M) = \alpha(T - T_c)M^2 + \beta M^4$$

minima $\quad M = 0, \; M = \pm\sqrt{\frac{\alpha(T_c - T)}{2\beta}}$

Thus, if we normalise M to some $\tilde{M}$ $\quad M^* = \frac{M}{\tilde{M}}$

$$M^* = \pm\sqrt{\frac{\alpha T_c}{2\beta\tilde{M}^2}\left(1 - \frac{T}{T_c}\right)}$$

Two dimensionless groups $\quad \pi_1 = \frac{\alpha T_c}{2\beta\tilde{M}^2} \quad \pi_2 = \frac{T}{T_c}$.

$B = B_0$ case

$F(M) = \alpha(T - T_c)M^2 + \gamma M^3 + \beta M^4$ extrema at $M = 0$ and

$$M = \frac{-3\gamma \pm \sqrt{9\gamma^2 - 32\alpha\beta(T - T_c)}}{8\beta}.$$

Normalise M to $\tilde{M}$ $\quad M^* = \frac{M}{\tilde{M}}$

$$M^* = \frac{-3\gamma}{8\beta\tilde{M}} \pm \left[\frac{9\gamma^2}{(8\beta\tilde{M})^2} - \frac{32\alpha\beta T_c}{(8\beta\tilde{M})^2}\left(1 - \frac{T}{T_c}\right)\right].$$

3 dimensionless groups

$$\pi_1 = \frac{3\gamma}{8\beta\tilde{M}} \qquad \pi_2 = \frac{32\alpha\beta T_c}{(8\beta\tilde{M})^2} \qquad \pi_3 = \frac{T}{T_c}.$$

(b) Microscopic model

Quantity	**dimension**	**what it is**
m	$\left[M^c\right]=\dfrac{[M]^{1/2}}{[L]^{1/2}[T]}$	Magnetization/spin
η	$[L]$	Spin separation
L_0	$[L]$	box size
Δt	$[T]$	time step
ε	$\left[M^c\right][T]^{-1}$	average charge in magnetization due to random fluctuations per spin
B_0	$\left[M^c\right]$ $\left(\text{since Tesla} = \dfrac{[M]^{1/2}}{[L]^{1/2}[T]}\right)$	externally applied field

In absence of B_0 $N=5$ $R=3$ 2 groups

With applied B_0 $N=6$ $R=3$ 3 groups

These are:

$$\pi_1=\frac{\varepsilon}{m}\Delta t \qquad \pi_2=\frac{L_0}{\eta} \qquad \pi_3=\frac{B_0}{m},$$

so in absence of applied B_0 we have π_1 and π_2 only. With applied B_0 we have π_3 as well.

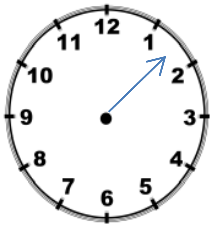
Then we can identify

$$\frac{\varepsilon}{m}\Delta t\equiv\frac{T}{T_c} \qquad \frac{\alpha T_c}{2\beta\tilde{M}^2}\equiv\frac{L_0}{\eta} \qquad \frac{3\gamma}{8\beta\tilde{M}}\equiv\frac{B_0}{m}.$$

Sheet 4 Question 2

Fireflies

Fly around at random, and each has a "clock" to tell it when to flash



cycle length τ_c

firefly flashes as t=12 say……

all start at random time τ_s

flash duration τ_d

Quantity	**diversion**	**what it is**
τ_c	$[T]$	cycle length
$\langle \tau_s \rangle$	$[T]$	average start time
τ_d	$[T]$	duration
R	$[L]$	interaction radius
N_f	–	No of flashes to reset
L_0	$[L]$	Size of box
Δt	$[T]$	timestep
v	$[L][T]^{-1}$	speed
N	–	number of fireflies

$N = 9$ $R = 2$

7 parameters
(most are trivial)

There are some 'trivial' and 'non-trivial' parameters here.

<u>Trivial</u>

$\pi_1 = \dfrac{R}{L_0}$	if	$\pi_1 > 1$	fireflies all see each other
$\pi_2 = \dfrac{v\Delta t}{L_0}$		$\pi_2 > 1$	fireflies cross box in one timestep
$\pi_3 = \dfrac{R}{v\Delta t}$		$\pi_3 < 1$	fireflies rush past each other
$\pi_4 = \dfrac{\tau_d}{\tau_c}$		$\pi_4 > 1$	fireflies 'always switched on'
$\pi_5 = \dfrac{\tau_s}{\tau_c}$			– only relevant if <u>no</u> synchronization – otherwise system 'forgets' initial phase
$\pi_6 = \dfrac{\tau_c}{\Delta t}$, $\pi_7 = \dfrac{\tau_d}{\Delta t}$		need $\pi_6, \pi_7 \gg 1$	to resolve the dynamics

Thus, to realise the 'interesting' dynamics on computer we need

$$\pi_1 \ll 1, \qquad \pi_2 \ll 1, \qquad \pi_3 \ll 1, \qquad \pi_4 \ll 1, \qquad \pi_{6,7} \gg 1\,.$$

In this case these are 'trivial'.

<u>Non-trivial parameters</u>

For synchronization a firefly must see N_f flashes within *R* – at least 'some of the time'.
Let number of flashes seen with *R* be α

$$\alpha = \frac{R^2 N}{L_0^2}\frac{\tau_d}{\tau_c}$$

number within *R*

fraction of these 'on'

want $\alpha \geq N_f$ for synchronization.

Thus, non-trivial parameters are $\pi_1 = \alpha, \pi_2 = N_f$ and for synchronization $\alpha \geq N_f$.