# Lecture 2: Weak Interactions and BEC

Previous lecture:

- Ideal gas model gives a fair intuition for occurrence of BEC but is *unphysical* (infinite compressibility, shape of condensate....)

- Order parameter and its equation of motion
- Slow (low energy) scattering of atoms and interaction parameter
- Gross-Pitaevskii equation. Amplitude and phase of the order parameter
- Super current and superfluid velocity. Irrotational hydrodynamics.
- Solution of GPE in uniform and non-uniform case. Thomas-Fermi Approximation

# Order parameter

Density matrix

$$\rho^{(1)}(\mathbf{x},\mathbf{y}) = \langle \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{y})\rangle = \sum_\alpha n_\alpha \psi_\alpha^*(\mathbf{x})\psi_\alpha(\mathbf{y})$$

Field operator $$\hat{\Psi}(\mathbf{x}) = \sum_\alpha \psi_\alpha(\mathbf{x})\hat{a}_\alpha$$

Creation and annihilation operators $$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha,\beta} \quad [\hat{a}_\alpha, \hat{a}_\beta] = 0$$

$$\langle \hat{a}_\alpha^\dagger \hat{a}_\alpha \rangle = n_\alpha$$

# Order parameter

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# Neglecting quantum fluctuations

$$\alpha = 0 \qquad\qquad N_0 = \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \gg 1$$

Neglect commutator $[\hat{a}_0, \hat{a}_0^\dagger] = 1$

$$\hat{a}_0^\dagger = \hat{a}_0 = \sqrt{N_0}$$

- a (large) number

Classical field and quantum fluctuations:

$$\hat{\Psi}(\mathbf{x}) = \Psi_0(\mathbf{x}) + \delta\hat{\Psi}(\mathbf{x}) = \sqrt{N_0}\psi_0(\mathbf{x}) + \sum_{\alpha\neq 0} \psi_\alpha(\mathbf{x})\hat{a}_\alpha$$

# Equation of motion

$$\hat{H} = \int \mathrm{d}\mathbf{x} \left[ \frac{\hbar^2}{2m} \nabla \hat{\Psi}^\dagger \cdot \nabla \hat{\Psi} + U(\mathbf{x}, t) |\hat{\Psi}(\mathbf{x})|^2 \right] + \\ + \frac{1}{2} \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} |\hat{\Psi}(\mathbf{x})|^2 V(\mathbf{x} - \mathbf{y}) |\hat{\Psi}(\mathbf{y})|^2$$

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{x}, t) = [\hat{\Psi}, \hat{H}] =$$

$$= \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x}, t) + \int \mathrm{d}\mathbf{y} V(\mathbf{x} - \mathbf{y}) |\hat{\Psi}(\mathbf{y})|^2 \right] \hat{\Psi}(\mathbf{x}, t)$$

# Interaction term

Replacing the field operator by its 'classical' part $\hat{\Psi}(\mathbf{x},t) \longrightarrow \Psi_0(\mathbf{x},t)$

and assuming slow changes of $\Psi_0(\mathbf{x},t)$ we put

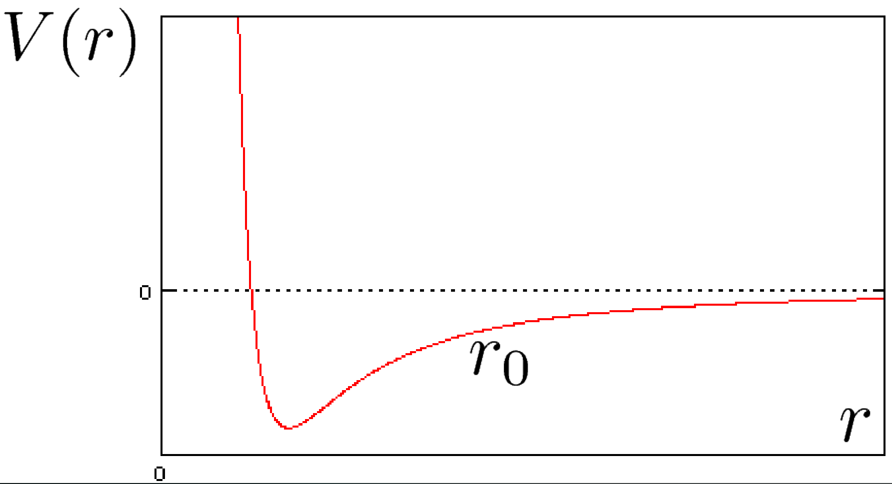
$$\hat{\Psi}(\mathbf{x},t)\int \mathrm{d}\mathbf{y} V(\mathbf{x}-\mathbf{y})|\hat{\Psi}(\mathbf{y})|^2 \longrightarrow g|\Psi_0(\mathbf{x},t)|^2\Psi_0(\mathbf{x},t)$$

Questions:

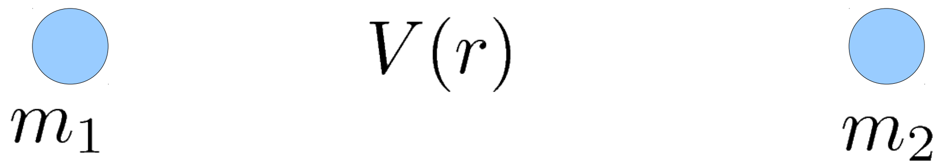
1. What is 'slow changes'?

2. What is the value of coupling?

$$g \sim \int \mathrm{d}\mathbf{x} V(\mathbf{x})$$



# Scattering theory



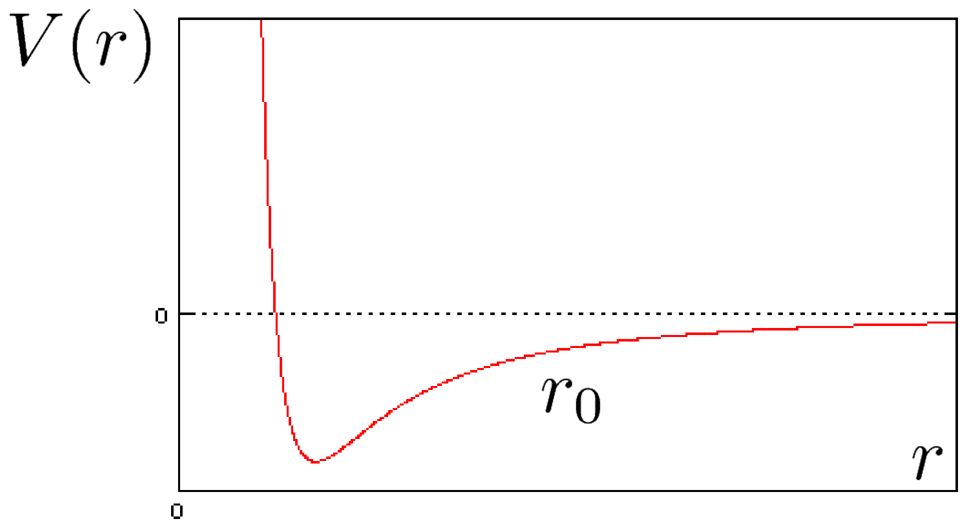

Schrödinger equation for effective particle with reduced mass

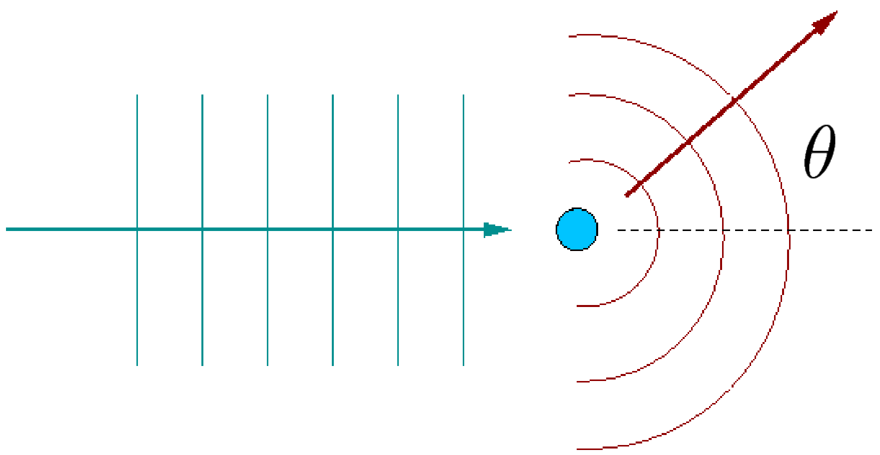
$$m^* = m_1 m_2/(m_1 + m_2)$$

and relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

$$\left(-\frac{\hbar^2}{2m^*}\nabla^2 + V(r)\right)\psi(\mathbf{r}) = \frac{\hbar^2 k^2}{2m^*}\psi(\mathbf{r})$$

# Asymptotic solution




Beyond the range $r_0$ of the interatomic potential the solution of the Schrödinger equation simplifies

$$\psi(\mathbf{r}) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

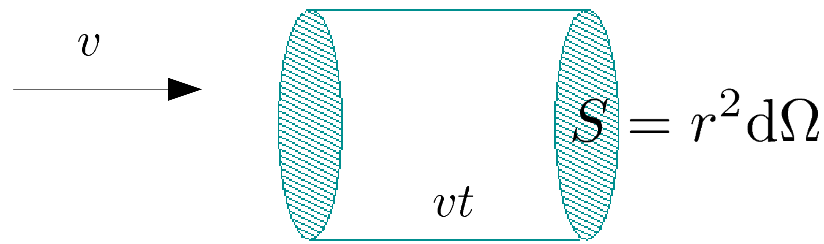
scattering amplitude

Incoming current density

$$j = \frac{\hbar}{m}\mathrm{Im}\psi^*\nabla\psi = \frac{\hbar k}{m} = v$$

# Scattering Amplitude

Cross section

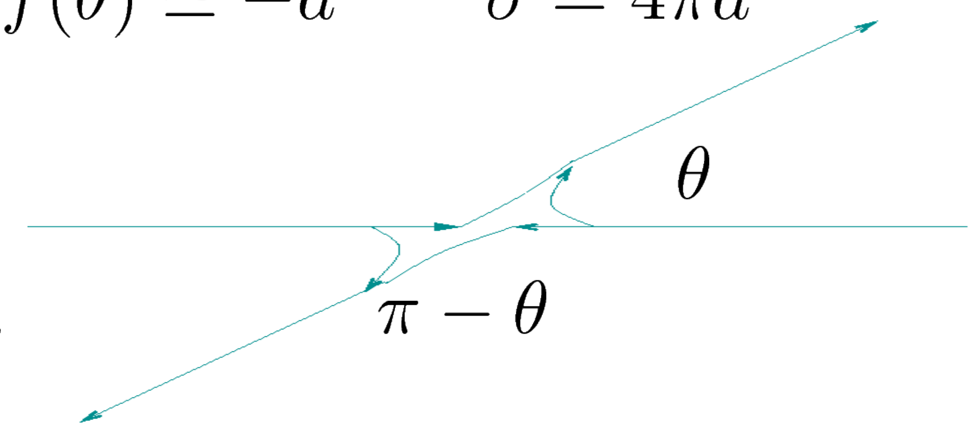



$$\mathrm{d}\sigma = \frac{|\psi(\mathbf{r})|^2 \mathrm{d}V}{vt} = |f(\theta)|^2 \mathrm{d}\Omega = |f(\theta)|^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

s-scattering, independent of angle $\quad f(\theta) \simeq -a \qquad \sigma = 4\pi a^2$

identical particles (bosons/fermions)

$$\mathrm{d}\sigma = |f(\theta) \pm f(\pi - \theta)|^2 \mathrm{d}\Omega$$



# Partial waves

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{\chi_{kl}(r)}{kr}$$

$P_l(x)$ - Legendre polynomials

Radial equation

$$-\frac{\mathrm{d}^2\chi_{kl}}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}\chi_{kl} + \frac{2m^*}{\hbar^2}V(r)\chi_{kl} = k^2\chi_{kl}$$

Large distances $r \gg r_0$

$$\chi_{kl} = A_l \sin\left(kr - \frac{\pi l}{2} + \delta_l\right)$$

# Comparison

$$f(\theta)\frac{e^{ikr}}{r} = \psi(\mathbf{r}) - e^{ikr\cos\theta} =$$

$$= \sum_{l=0}^{\infty} P_l(\cos\theta)\frac{A_l i^{-l} e^{-i\delta_l}}{2ikr}\left(e^{ikr+2i\delta_l} - e^{-ikr+i\pi l}\right)$$

$$-\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{2l+1}{2ikr}\left(e^{ikr} - e^{-ikr+i\pi l}\right)$$

Only outgoing wave $\quad A_l = (2l+1)i^l e^{i\delta_l}$

# Scattering amplitude

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)(e^{2i\delta_l}-1)$$

$$\sigma = 2\pi\int_0^{\pi}|f(\theta)|^2\sin\theta\mathrm{d}\theta = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

bosons: $l = 0, 2, 4, \ldots$ fermions: $l = 1, 3, 5, \ldots$

phase shifts $\delta_l = \delta_l(k)$

must be calculated from the solution of the Schrödinger equations.

The situation becomes simpler if the energy of scattered particles is small....

# Slow collisions

For $kr_0 \ll 1$ there exists a parametrically large region

$$r_0 \ll r \ll 1/k$$

where right hand side $\sim k^2$ is not important and $V(r) \sim 0$ :

Analysis of the solutions for each $l$ gives

$$f_l = \frac{1}{2ik}(e^{2i\delta_l} - 1) \sim k^{2l}$$

And one neglects all $l > 0$

# S-wave amplitude

$$\frac{\mathrm{d}^2\chi_{k0}}{\mathrm{d}r^2} = 0 \qquad\qquad \chi_{k0} = c_0(1 - r/a)$$

On the other hand one can already use the asymptotic form of wavefunction

$$\chi_{k0} = e^{i\delta_0}\sin\left(kr + \delta_0\right) = e^{i\delta_0}(\sin\delta_0 + k\cos\delta_0 r)$$

$$c_0 = e^{i\delta_0}\sin\delta_0 \qquad\qquad \tan\delta_0 \simeq \delta_0 = -ak \simeq f_0 k$$

# Born approximation

To calculate $a$ one has to solve Schrödinger equation for $r < r_0$

For small interaction potential $V(\mathbf{r})$ perturbation theory gives

$$f_0 = -a = -\frac{m^*}{2\pi\hbar^2} \int \mathrm{d}\mathbf{r} V(\mathbf{r})$$

Details of the potential are not important for small energy scattering as long as they yield the same value of the scattering length.

Let us define an effective potential $V_{\mathbf{eff}} = \frac{4\pi\hbar^2 a}{m}\delta(\mathbf{r})$

giving the same value of scattering length as $V(\mathbf{r})$ non-perturbatively

# Dilute atomic gas

Below BEC transition

$$k \sim k_T = 1/\Lambda_T \ll n^{1/3}$$

And collisions are always slow: $kr_0 \ll 1$

Interatomic interactions can be safely characterised by the corresponding scattering length $a$

Weak interactions (diluteness) condition is the condition on *gas paramete*r

$$n|a|^3 \ll 1$$

# Gross – Pitaevskii Equation

Replacing the field operator by its 'classical' part $\hat{\Psi}(\mathbf{x},t) \longrightarrow \Psi_0(\mathbf{x},t)$

and assuming slow changes of $\Psi_0(\mathbf{x},t)$ over the lenghts $\sim a$
we obtain GPE:

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x},t) + g|\Psi_0(\mathbf{x})|^2\right]\Psi_0(\mathbf{x},t)$$

with coupling constant $g = \int \mathrm{d}\mathbf{y} V_{\text{eff}}(\mathbf{y}) = \frac{4\pi\hbar^2 a}{m}$

$V(\mathbf{x}),\ V_{\text{eff}}(\mathbf{x}) = g\delta(\mathbf{x})$ have the same scattering length

# Condensate density and phase

The order parameter $\Psi_0(\mathbf{x}, t)$ has meaning of *macroscopic wavefunction*

$$\int \mathrm{d}\mathbf{x} |\Psi_0(\mathbf{x})|^2 = N_0$$

Condensate density $$n_0(\mathbf{x}) = |\Psi_0(\mathbf{x})|^2$$

Moreover the macroscopic wavefunction has a ***PHASE***

$$\Psi_0(\mathbf{x}, t) = \sqrt{n_0(\mathbf{x}, \mathbf{t})} e^{iS(\mathbf{x}, t)}$$

# Phase and current

Multiplying Gross Pitaevskii Equation

$$i\hbar\frac{\partial}{\partial t}\Psi_0(\mathbf{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x},t) + g|\Psi_0(\mathbf{y})|^2\right]\Psi_0(\mathbf{x},t)$$

by its complex conjugate and integrating by parts one gets *continuity equation*

$$\frac{\partial}{\partial t}n_0(\mathbf{x},t) + \nabla\cdot\mathbf{j}(\mathbf{x},t) = 0$$

for density and *super*current

$$\mathbf{j}(\mathbf{x},t) = \frac{\hbar}{2im}\left(\Psi_0^*\nabla\Psi_0 - \Psi_0\nabla\Psi_0^*\right) = n_0\frac{\hbar}{m}\nabla S$$

# Hydrodynamic form of GPE

Substituting 'polar' representation of order parameter

$$\Psi_0(\mathbf{x},t) = \sqrt{n_0(\mathbf{x},\mathbf{t})}e^{iS(\mathbf{x},t)}$$

into GPE and separating real and imaginary parts

$$\frac{\partial}{\partial t}n(\mathbf{x},t) + \nabla\cdot\mathbf{j}(\mathbf{x},t) = 0$$

$$\hbar\frac{\partial}{\partial t}S + \left(\frac{m\mathbf{v}_s^2}{2} + U + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0$$

quantum pressure

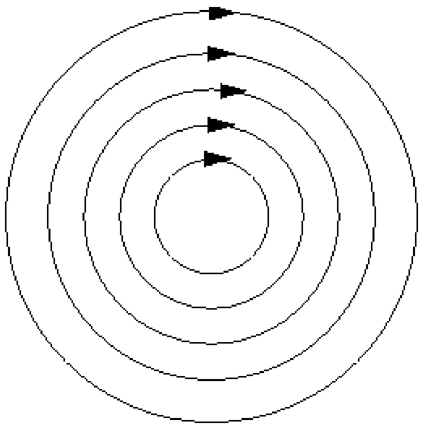
# Superfluid velocity

$$\mathbf{v}_s(\mathbf{x},t) = \frac{\hbar}{m}\nabla S(\mathbf{x},t)$$

is irrotational, i.e. $\nabla \times \mathbf{v}_s = \frac{\hbar}{m}\nabla \times \nabla S = 0$

for example consider uniform rotation $\mathbf{v}(\mathbf{r}) = \mathbf{\Omega} \times \mathbf{r}$

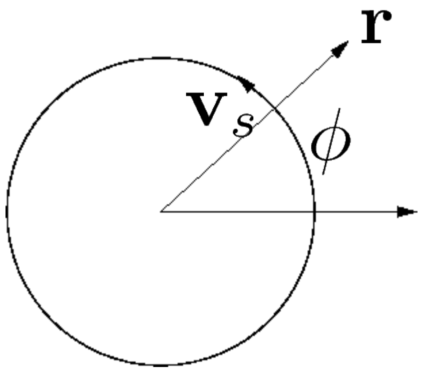
cannot be described by velocity field $\mathbf{v}_s(\mathbf{x},t)$



- rotation can only enter in form of singular points of phase where $n_0(\mathbf{x}) = 0$

# Vortex

Cylindrical coordinates




$$\Psi_s(r,\phi) = |\Psi_s(r)|e^{is\phi}$$

$$L_z = \hbar s N$$

$$\mathbf{v}_s = \frac{\hbar}{m}\frac{1}{r}\frac{\partial S}{\partial \phi}\hat{\phi} = \frac{\hbar}{m}\frac{s}{r}\hat{\phi}$$

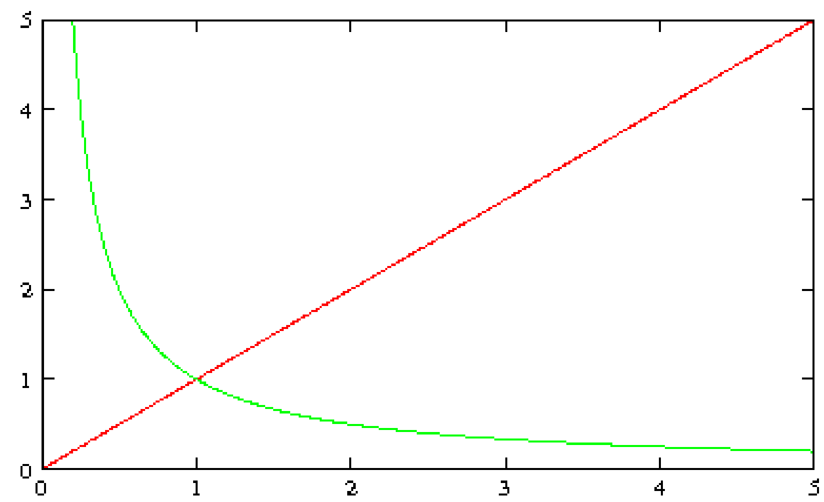
$$\oint \mathbf{v}_s \cdot \mathrm{d}\mathbf{l} = 2\pi s\frac{\hbar}{m}$$

$$\nabla \times \mathbf{v}_s = 2\pi s\frac{\hbar}{m}\delta(\mathbf{r})\hat{z}$$

$$n(r) = |\Psi_s(r)|^2 \sim r^{2s}$$

# Vortices

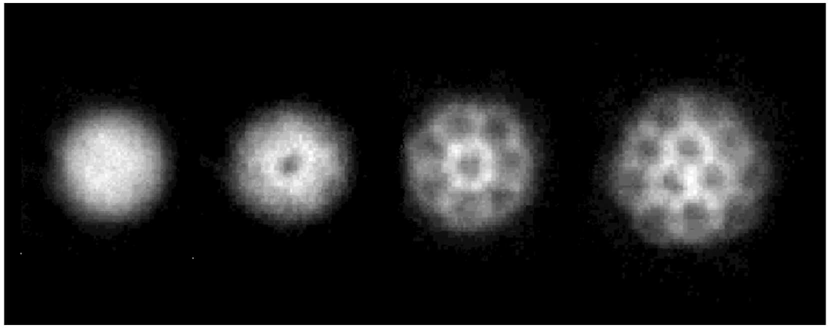
Superfluid velocity behaves differently from rigid rotation




Vortex configuration costs more energy

$$\Delta E = E_{s=1} - E_{s=0}$$

J. Dalibard, 2001



But can be favoured in the rotating frame

$$E_{\text{rot}} = E_0 - \Omega L_z$$

above critical rotation $\Omega_c$

# Time dependence and chemical potential

Density matrix at large distances

$$\langle \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{y})\rangle \qquad \rightarrow \qquad \langle \hat{\Psi}^\dagger(\mathbf{x})\rangle\langle \hat{\Psi}(\mathbf{y})\rangle = \Psi_0^\dagger \Psi_0$$

$$\Psi_0 = \langle \hat{\Psi} \rangle = \langle N|\hat{\Psi}|N+1\rangle$$

$$\Psi_0(t) = e^{-i\mu t/\hbar}\langle N|\hat{\Psi}|N+1\rangle$$

Chemical potential $\mu = E_{N+1} - E_N$

# Uniform case

Non Linear Schrödinger Equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_0(\mathbf{x}) - \mu\Psi_0(\mathbf{x}) + g|\Psi_0(\mathbf{x})|^2\Psi_0(\mathbf{x}) = 0$$

is solved with uniform solution $\Psi_0 = \sqrt{n}$

Mean field $\mu = \frac{\partial E_0}{\partial N} = gn$ $\quad P = -\partial E_0/\partial V = gn^2/2$

* $E_0 = \frac{1}{2}Ngn$ Compressibility and sound velocity $\frac{1}{mc^2} = \frac{\partial n}{\partial P} = \frac{1}{gn}$

# Condensate in a box

Close to the boundary

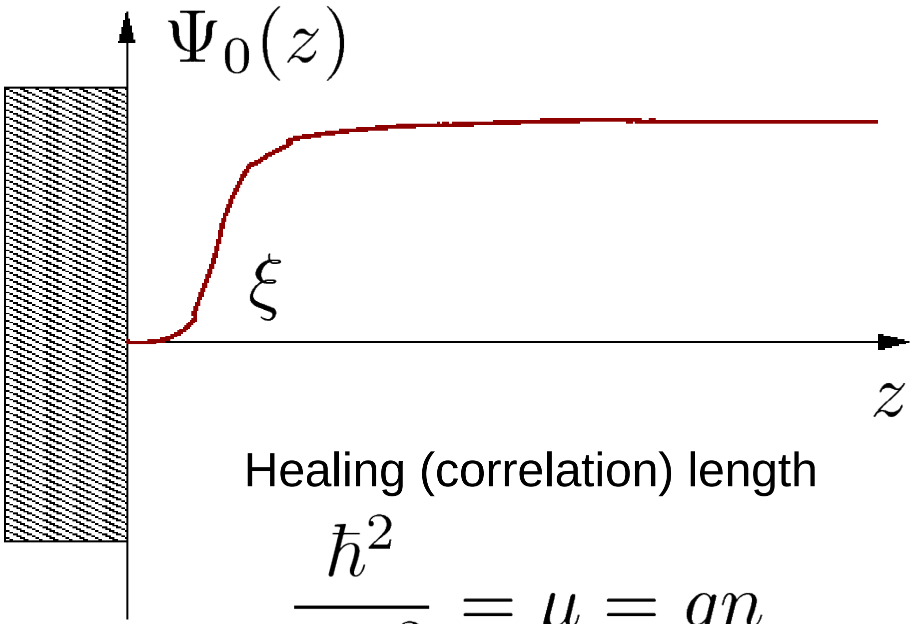
$$\Psi_0(\mathbf{x}) = \sqrt{n} f(z/\xi)$$

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} f + f^3 - f = 0$$

$$f(0) = 0, \qquad f(\infty) = 1$$

$$f(z) = \tanh(z)$$

*




Healing (correlation) length

$$\frac{\hbar^2}{m\xi^2} = \mu = gn$$

$$\xi = \frac{\hbar}{\sqrt{mgn}}$$

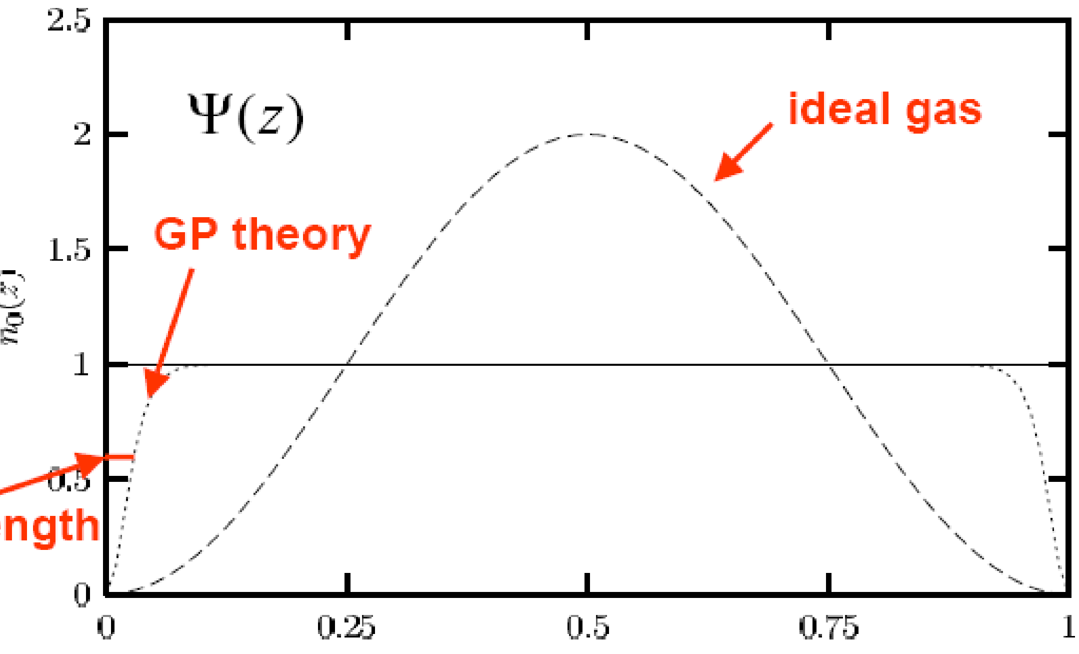
# Nonlinearity is important

$$\Psi_0 = \sqrt{n}\tanh(z/\xi)$$

Compare with ground state of free particle in a box: $\Psi_0(z) \sim \sin \pi z / L$

Typical values

$$\xi \sim 1\mu\mathrm{m} \ll L \sim 100\mu\mathrm{m}$$



# Harmonic trap

$$V(\mathbf{x}) = \frac{1}{2} m \omega_{\mathrm{ho}}^2 x^2$$

typical length $a_{\mathrm{ho}} = \sqrt{\frac{\hbar}{m\omega_{\mathrm{ho}}}}$

Non interacting particles $n(\mathbf{x}) = |\Psi(\mathbf{x})|^2 \sim \exp\left(-\frac{x^2}{a_{\mathrm{ho}}^2}\right)$

***Thomas-Fermi*** approximation: assume the condensate changes on much larger lengthscales

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_0(\mathbf{x}) - (\mu - U(\mathbf{x}))\Psi_0(\mathbf{x}) + g|\Psi_0(\mathbf{x})|^2\Psi_0(\mathbf{x}) = 0$$

# Local Density Approximation

$$\Psi_0(\mathbf{x}) = \sqrt{n_{\mathrm{TF}}(\mathbf{x})}$$

$$\mu - U(\mathbf{x}) = g n_{\mathrm{TF}}(\mathbf{x}) > 0 \qquad n(0) = \mu/g$$

Inverted parabola

$$n_{\mathrm{TF}}(\mathbf{x}) = n(0)\left(1 - \frac{x^2}{R_{\mathrm{TF}}^2}\right) \qquad \frac{m\omega_{\mathrm{ho}}^2 R_{\mathrm{TF}}^2}{2} = \mu$$

Thomas Fermi Radius $$R_{\mathrm{TF}} = \sqrt{\frac{2\mu}{m\omega_{\mathrm{ho}}^2}} \gg \xi(0) = \hbar/\sqrt{mgn(0)}$$

# Thomas – Fermi parameter

* $$N = \int d\mathbf{x}\, n_{\mathrm{TF}}(\mathbf{x})$$

$$\mu = \frac{\hbar\omega_{\mathrm{ho}}}{2}\left(\frac{15Na}{a_{\mathrm{ho}}}\right)^{2/5}$$

$$R_{\mathrm{TF}} = a_{\mathrm{ho}}\left(\frac{15Na}{a_{\mathrm{ho}}}\right)^{1/5}$$

If $N\dfrac{a}{a_{\mathrm{ho}}} \gg 1$

$\mu \gg \omega_{\mathrm{ho}}$ and $R_{\mathrm{TF}} \gg a_{\mathrm{ho}}$

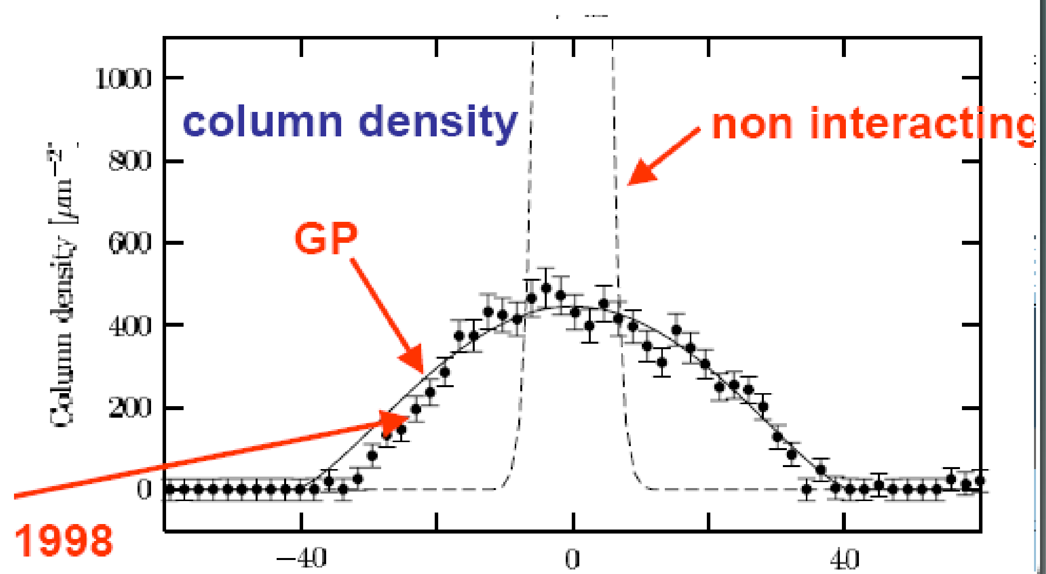
$$\frac{\xi(0)}{R_{\mathrm{TF}}} = \frac{a_{\mathrm{ho}}^2}{R_{\mathrm{TF}}^2} \ll 1$$

Local density approximation is valid

# Thomas-Fermi density profile

Quantum pressure $$-\frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n} \sim \frac{\hbar^2}{mR_{\mathrm{TF}}^2} \ll \frac{\hbar^2}{m\xi^2} = \mu$$

$$n_{\mathrm{TF}}(\mathbf{x}) = n(0)\left(1 - \frac{x^2}{R_{\mathrm{TF}}^2}\right)$$



# Conclusions of Lecture 2

- Order parameter has amplitude and phase and its dynamics is governed by Gross – Pitaevskii Equation
- Interactions enter through the scattering length of the potential
- Phase plays important role and leads to unusual irrotational hydrodynamics. In particular rigid rotation is forbidden
- The typical scale on which order parameter change its value is governed by healing length
- If the external trap is sufficiently slow function of coordinates one neglects quantum pressure term in GPE and uses Thomas-Fermi approximation. Usually it works fine for large number of particles