Lecture 4: Superfluidity

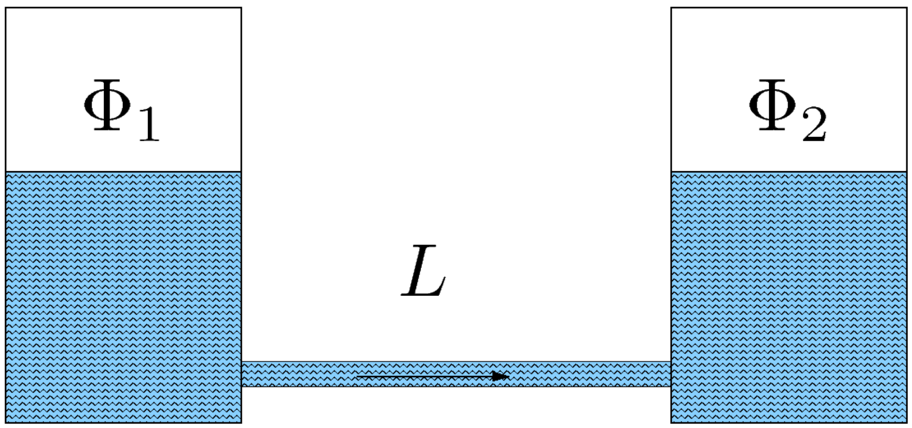
Previous lecture: Elementary excitations above condensate are phonons in the low energy limit.

This lecture

- Rotation of superfluid helium. Hess-Fairbank effect and persistent currents
- Landau criterion of superfluid flow
- Superfluid and normal components of a liquid
- Two fluid hydrodynamics. First and second sound
- Vortices in superfluids

Superfluidity

In 1938 Allen and Misener (Cambridge), Kapitza (Moscow) observed liquid ^{4}He flowing without dissipation in thin capillars below certain temperature



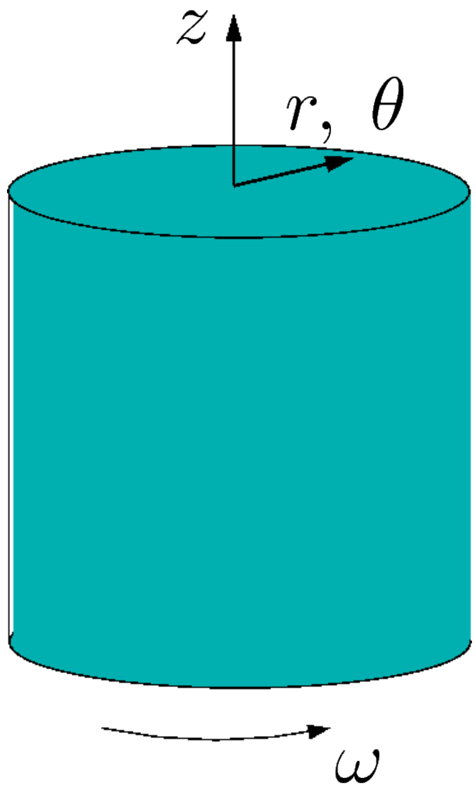
$$T_\lambda = 2.17\mathrm{K}$$

$$v_s = \frac{\hbar}{m}\frac{\Phi_2 - \Phi_1}{L}$$

Nowadays we understand by supefluidity a complex of phenomena. Here are two main manifestations of superfluidity

Rotation of superfluids

1. Hess-Fairbank effect.



Consider container with helium *above* superfluifd temperature T_λ rotating *slowly* with angular velocity ω

Helium behaves as a normal liquid and rotates with the container

$$\langle L_z \rangle = I_{\rm cl}\omega \quad I_{\rm cl} = \sum_i m r_i^2$$

Cool the liquid below T_λ while rotating the container.

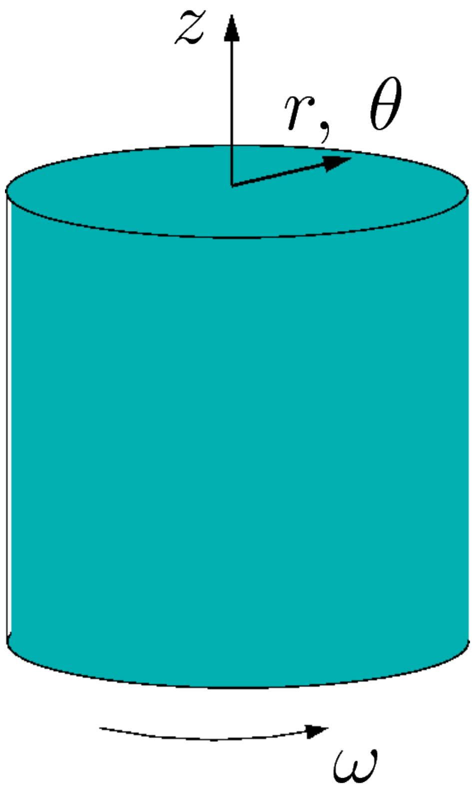
$$\langle L_z \rangle = f_{\rm n}(T) I_{\rm cl}\omega$$

Normal fraction $f_{\rm n}(T) < 1 \qquad f_{\rm n}(0) = 0$

- thermodynamic equilibrium

Rotation of superfluids

2. <u>Persistent current.</u>



Start with *fast* rotation of the container above T_λ.

Liquid follows (is in equilibrium with) the container.

Cool below T_λ and **stop** the container

The liquid **continues** to rotate with angular momentum

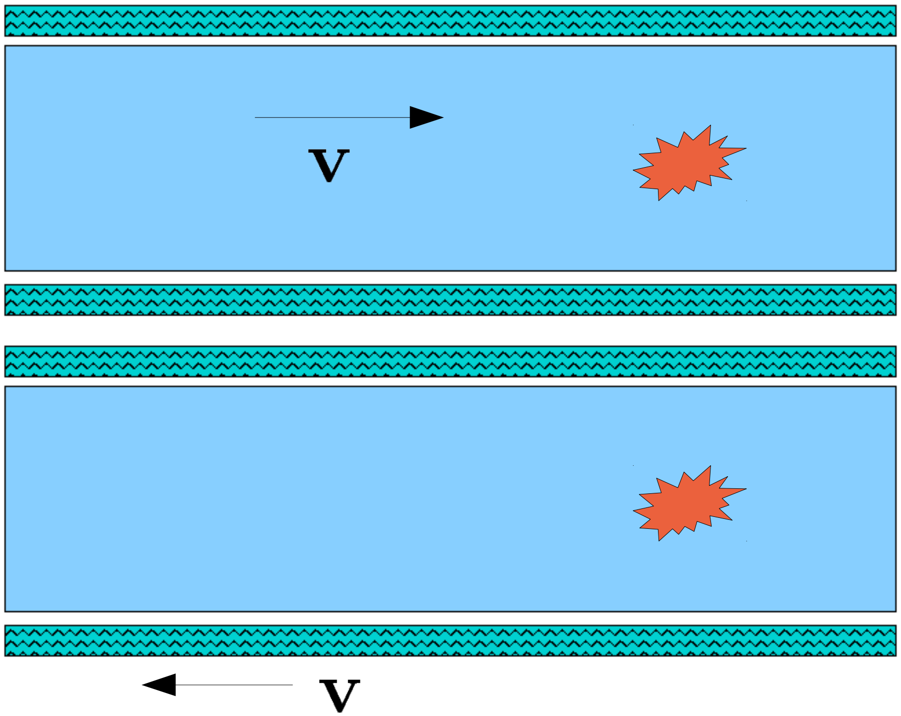
$$\langle L_z \rangle = f_s(T) I_{cl} \omega$$

can be changed reversibly

Superfluid fraction $f_s(T) = 1 - f_n(T)$

- long living metastable state.

Elementary excitations and viscosity



Dissipation = transformation of kinetic energy of the flow into excitations of the liquid

Energy of excitation in co-moving frame

$$E - E_0 = \varepsilon(\mathbf{p})$$

In the lab frame $$E' = E_0 + \varepsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v} + \frac{1}{2}Mv^2$$

Landau criterion

Energy of excitation $\varepsilon(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}$ must be positive

This is violated at some critical velocity $v = v_c$ $\mathbf{p} \cdot \mathbf{v} < 0$

and excitations appear spontaneously

Landau criterion of stable superflow $v < v_c$

$$v_c = \min_{\mathbf{p}} \frac{\varepsilon(\mathbf{p})}{p}$$

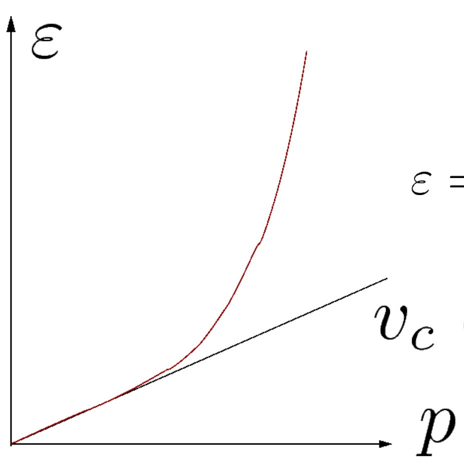
Below critical velocity the superfluid component and excitations do not interact

Phonons and superfluidity

Ideal gas has quadratic spectrum $\varepsilon = \dfrac{p^2}{2m}$ and therefore has *zero* critical velocity

If low energy excitations are phonons with sound velocity c then Landau criterion predicts superflow for $v < c$ and new phonons cannot be created from it

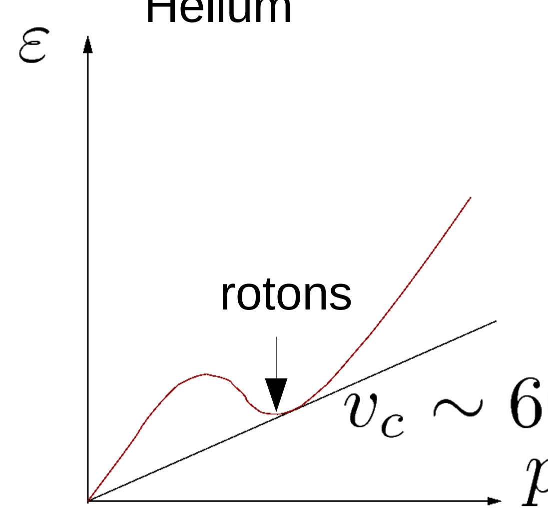
Bogoliubov spectrum



$$\varepsilon = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2gn\right)}$$

$$v_c \sim 1\mathrm{cm/s}$$

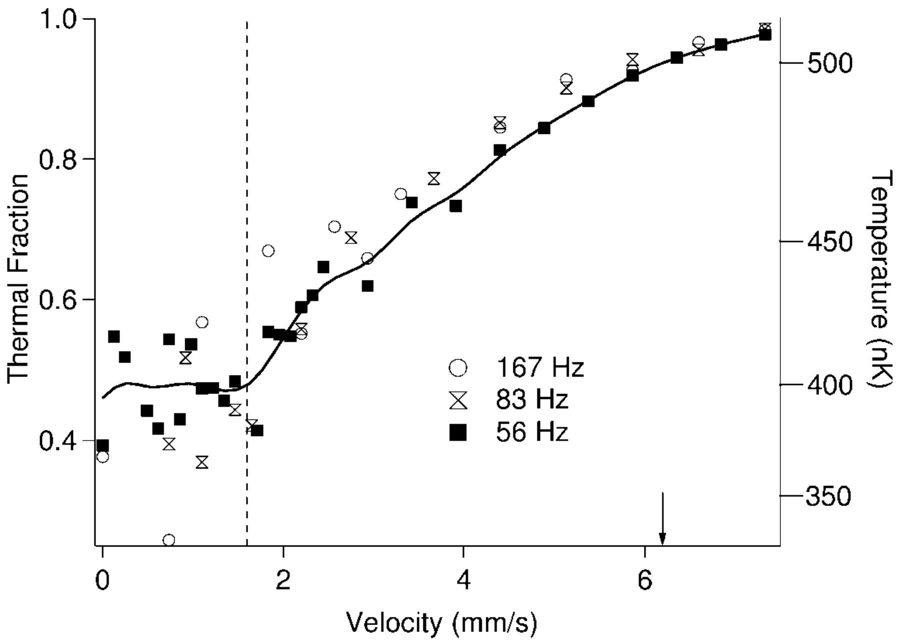
Helium

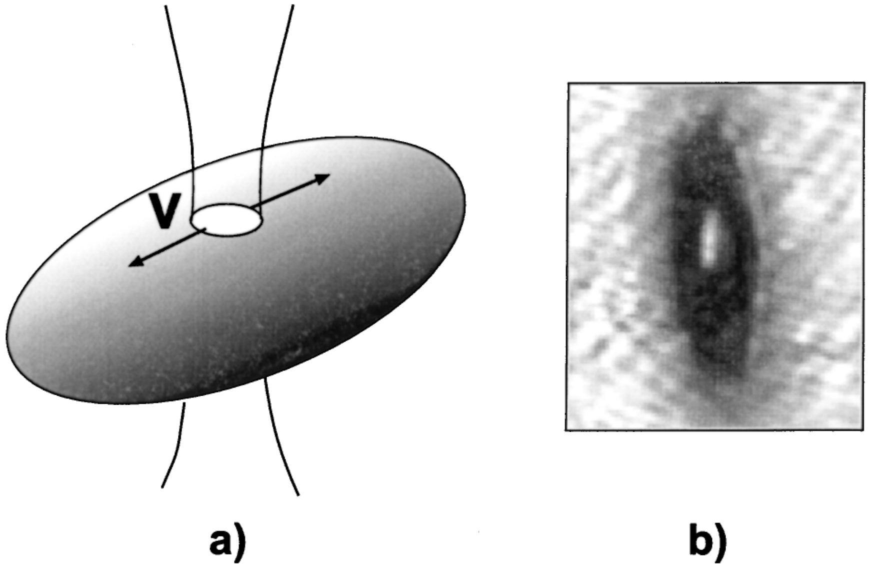


$$v_c \sim 60\mathrm{m/s}$$

Experiments

Measurements of critical velocity
in BEC – Ketterle group at MIT, 1999





Two fluid model

Below T_λ and for sufficiently low velocities the liquid has effectively two components:

Superfluid	density	ρ_s	velocity	$\mathbf{v}_\mathrm{s}$	$\nabla \times \mathbf{v}_\mathrm{s} = 0$
Normal	density	ρ_n	velocity	$\mathbf{v}_\mathrm{n}$	*(will be) in equilibrium with walls*

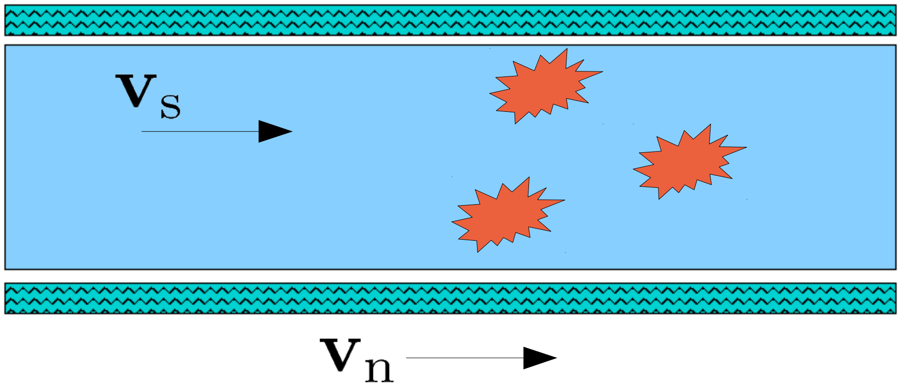
Total mass density $\rho = \rho_\mathrm{s} + \rho_\mathrm{n}$

Mass current $m\mathbf{j} = \rho_\mathrm{s}\mathbf{v}_\mathrm{s} + \rho_\mathrm{n}\mathbf{v}_\mathrm{n}$

Kin. energy $e_\mathrm{k} = \frac{1}{2}\rho_\mathrm{s}v_\mathrm{s}^2 + \frac{1}{2}\rho_\mathrm{n}v_\mathrm{n}^2$

Normal component and excitations

Consider non-interacting excitations at temperature T moving with velocity $\mathbf{v}_\mathrm{n}$



$$N_\mathbf{p} = \frac{1}{e^{\beta\epsilon'(\mathbf{p})} - 1}$$

$$\epsilon'(\mathbf{p}) = \epsilon(\mathbf{p}) + \mathbf{p}\cdot(\mathbf{v}_\mathrm{s} - \mathbf{v}_\mathrm{n})$$

Momentum per unit volume

$$m\mathbf{j} = \rho_\mathrm{s}\mathbf{v}_\mathrm{s} + \rho_\mathrm{n}\mathbf{v}_\mathrm{n} = \rho\mathbf{v}_\mathrm{s} + \frac{1}{V}\sum_\mathbf{p} \mathbf{p} N_\mathbf{p}$$

$$\rho_\mathrm{n}(\mathbf{v}_\mathrm{n} - \mathbf{v}_\mathrm{s}) = \frac{1}{V}\sum_\mathbf{p} \mathbf{p} N_\mathbf{p}$$

Normal component in Bogoliubov theory

For low temperatures only phonons are present $\epsilon(p) \simeq cp$

expanding for low velocities

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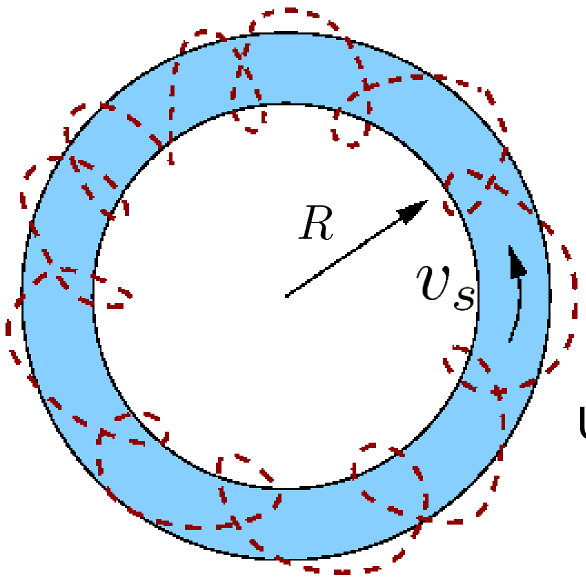
$$\rho_{\mathrm{n}} = -\frac{1}{3V}\sum_{\mathbf{p}} p^2 \frac{\mathrm{d}N_{\mathbf{p}}(\epsilon)}{\mathrm{d}\epsilon} = -\frac{2}{3c}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} pN(cp/T) \sim T^4$$

For high temperatures $T > \mu = gn$ excitations are particle-like $\epsilon(p) \simeq \frac{p^2}{2m}$

$$\rho_{\mathrm{n}} = m\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} N(p^2/2mT) \sim mn_T$$

NB. In general condensate density $\neq$ superfluid density

Superfluid component - Winding number



Onsager – Feynman quantisation relation

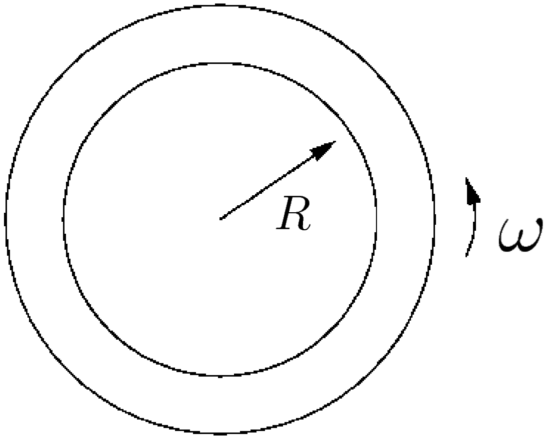
$$2\pi n = \oint \mathrm{d}\mathbf{l} \cdot \nabla\Phi = \frac{m}{\hbar} \oint \mathrm{d}\mathbf{l} \cdot \mathbf{v}_{\mathrm{s}}$$

Uniform flow $\quad \mathbf{v}_{\mathrm{s}} = \frac{\hbar n}{mR}\hat{\theta}$

Superfluid velocity takes only discrete values. The number n is called 'winding' number as it gives the number of times the phase 'winds' around the toroidal system

cf. Flux quantisation in superconductors

Rotation of superfluids



Consider annular geometry and let interactions with the rotating container be described by the potential

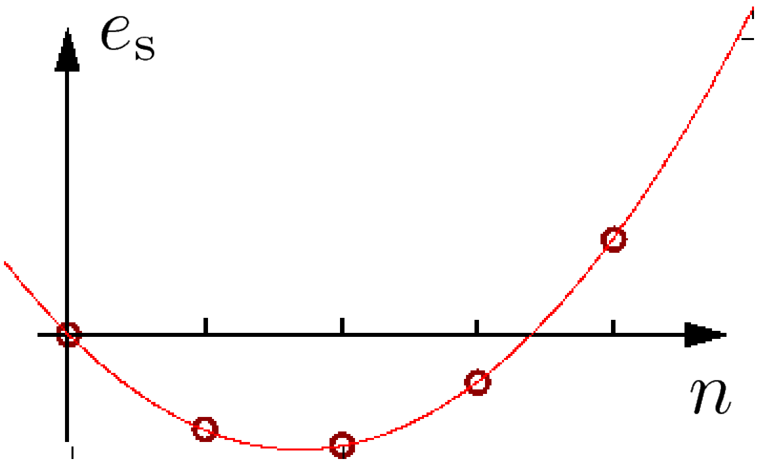
$$V(\mathbf{r}, t) = V(r, z, \theta - \omega t)$$

To find equilibrium (in the rotating frame) state we need to minimise

$$H_{\mathrm{eff}} = H_{\mathrm{kin}} + V(\mathbf{r}) - \omega L_z$$

Angular momentum $\mathrm{L} = \mathrm{r} \times \mathrm{p}$ $\qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$

Energy of superfluid component



$$e_s = \frac{1}{2}\rho_s v_s^2 - \rho_s v_s R\omega$$

$$= \rho_s(T)R^2\left(\frac{1}{2}n^2\omega_c^2 - n\omega_c\omega\right)$$

$$\omega_c = \frac{\hbar}{mR^2}$$

The system will choose $n = \mathrm{int}\left[\frac{\omega}{\omega_c} + \frac{1}{2}\right]$ to minimise its energy

If $\omega < \omega_c/2$ then $n = 0$

Hess-Fairbank effect

Above T_λ only normal component exists $\rho = \rho_{\rm n}$

Angular momentum is classical: $L = M v_{\rm n} R = m N R^2 \omega$

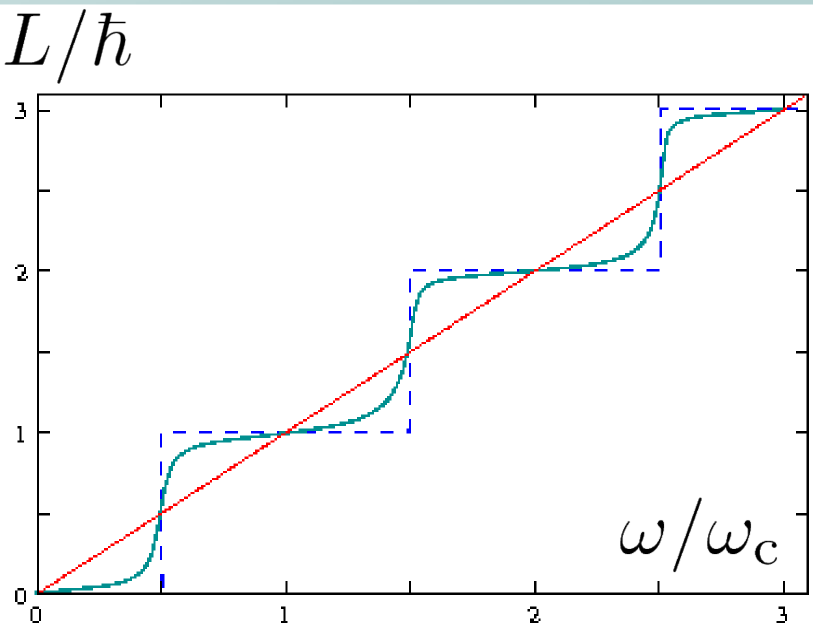
Below T_λ some part of density becomes superfluid $\rho = \rho_{\rm n} + \rho_{\rm s}$

If $\omega < \omega_{\rm c}/2$ the superfluid velocity is zero and all the angular momentum is carried by the normal component

$$L(T) = M_{\rm n} v_{\rm n} R = \frac{\rho_{\rm n}(T)}{\rho} M R^2 \omega = (f_{\rm n}(T) I_{\rm cl})\omega$$

- nonclassical moment of inertia

Angular momentum of superfluid

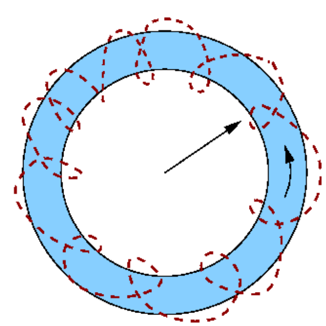


For $\omega > \omega_c/2$

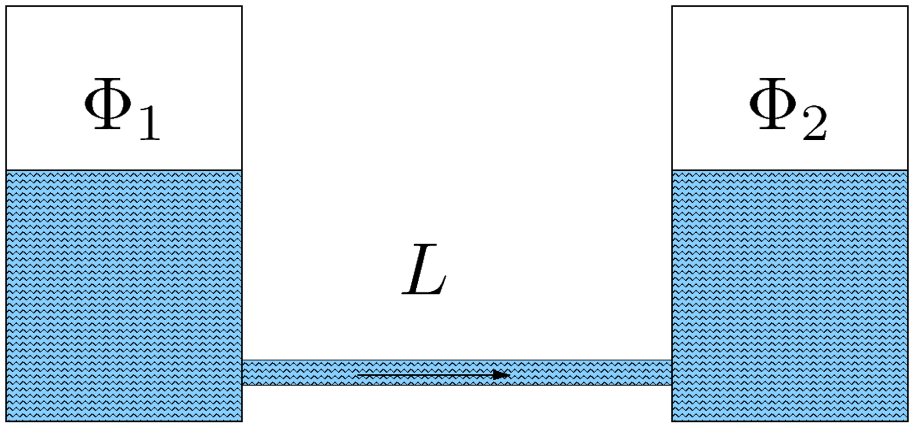
$$L(T) = I_{cl}\left[f_n(T)\omega + f_s(T)n\omega_c\right]$$

<u>Persistent currents</u> appear since **winding number** n **is a topological invariant** and cannot be changed (at least easily – huge energy barrier)

For $\omega \gg \omega_c/2$ it will appear as the whole liquid is rotating



Open geometry (experiments)



$$v_s = \frac{\hbar}{m}\frac{\Phi_2 - \Phi_1}{L}$$

$$\Phi_2 - \Phi_1 < \pi$$

- Hess-Fairbank effect

$$\Phi_2 - \Phi_1 > \pi$$

- persistent current

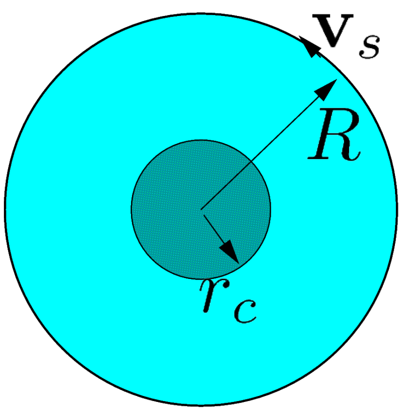
Vortices and critical rotation

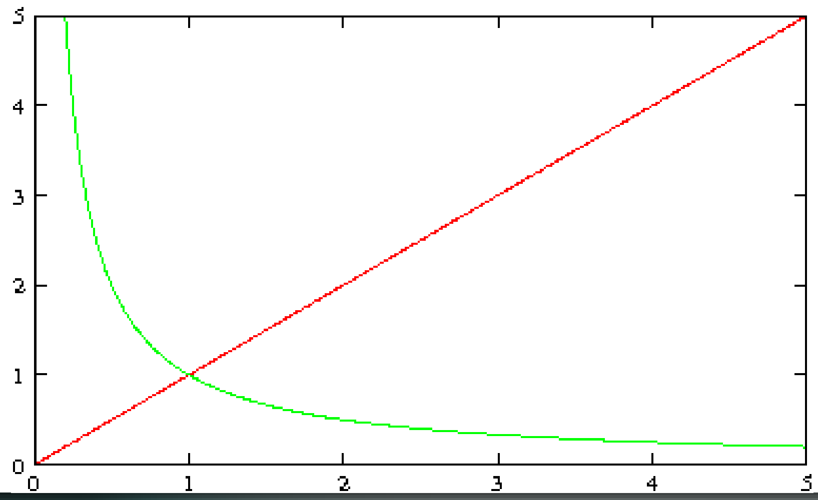
In order to allow rotation a superfluid “digs” a hole to have a nontrivial topology

$$n(r) = |\Psi_s(r)|^2 \sim r^{2s}$$

$$\oint \mathbf{v}_s \cdot \mathrm{d}\mathbf{l} = 2\pi s \frac{\hbar}{m}$$

$$\mathbf{v}_s = \frac{\hbar}{m}\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\theta} = \frac{\hbar}{m}\frac{s}{r}\hat{\theta}$$





Critical rotation

Energy of the vortex

$$E_v = 2\pi \int_{r_c}^{R} r dr \frac{1}{2} \rho_s v_s^2 = \pi \rho_s s^2 \left(\frac{\hbar}{m}\right)^2 \ln\left(\frac{R}{r_c}\right)$$

$$L_z = \int \rho_s v_s r \mathrm{d}^2 r = s \frac{\hbar}{m} \rho_s \pi R^2$$

Vortex core $r_c \sim \xi$

For single vortex $s = 1$

$$\omega_c = E_v / L_z = \frac{\hbar}{mR^2} \ln\left(\frac{R}{r_c}\right)$$

Two-fluid model. Dynamics

We are going to describe sound propagation in a superfluid.
We shall linearise where possible

<u>Normal fluids</u>: pressure $P(\mathbf{r},t)$, temperature $T(\mathbf{r},t)$

mass density $\rho(\mathbf{r},t)$ and velocity $\mathbf{v}(\mathbf{r},t)$

Thermodynamics

$$\mathrm{d}\mu = \frac{m}{\rho}\mathrm{d}P + s\mathrm{d}T$$

Hydrodynamic equations

Continuity equations

mass conservation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \mathbf{j} = \rho \mathbf{v}$$

entropy conservation
$$\frac{\partial s}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \mathbf{J} = s \mathbf{v}$$

Linearised Euler equation
$$\frac{\partial \mathbf{j}}{\partial t} + \nabla P = 0$$

How to generalise them in the presence of superfluid component?

$$\rho = \rho_{\mathrm{s}} + \rho_{\mathrm{n}} \qquad \mathbf{j} = \rho_{\mathrm{s}} \mathbf{v}_{\mathrm{s}} + \rho_{\mathrm{n}} \mathbf{v}_{\mathrm{n}}$$

Superfluid hydrodynamics

1. Josephson relation: $$\frac{\partial \Delta\Phi}{\partial t} = -\frac{\Delta\mu}{\hbar}$$ (see e.g. lecture 2)

$$\mathbf{v}_\mathrm{s} = \frac{\hbar}{m}\nabla\Phi \qquad \Rightarrow \qquad \frac{\partial \mathbf{v}_\mathrm{s}}{\partial t} = -\frac{1}{m}\nabla\mu$$

2. Superfluid component carries *no entropy*

$$\mathbf{J} = s\mathbf{v}_\mathrm{n}$$

Zero temperature

Third law of thermodynamics: $s = 0$ at $T = 0$

$$\nabla P = \frac{\rho}{m} \nabla \mu$$

$$\frac{\partial \mathbf{v}_{\mathrm{s}}}{\partial t} = -\frac{1}{m} \nabla \mu$$ and $$\frac{\partial \mathbf{j}}{\partial t} + \nabla P = 0$$ imply

$$\mathbf{j} = \rho \mathbf{v}_{\mathrm{s}} \qquad \rho_{\mathrm{s}}(T = 0) = \rho$$

Finite temperature

$$\mathbf{j} = \rho_\mathrm{s}\mathbf{v}_\mathrm{s} + \rho_\mathrm{n}\mathbf{v}_\mathrm{n}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \frac{\partial \mathbf{j}}{\partial t} + \nabla P = 0$$

$$\frac{\partial \mathbf{v}_\mathrm{s}}{\partial t} + \frac{1}{m}\nabla \mu = 0 \qquad \frac{\partial s}{\partial t} + \nabla \cdot (s\mathbf{v}_\mathrm{n}) = 0$$

Supports steady state (superfluid counterflow)

$$\mathbf{j} = 0 \quad \mathbf{v}_\mathrm{n} = -\frac{\rho_\mathrm{s}}{\rho_\mathrm{n}}\mathbf{v}_\mathrm{s}$$

Entropy(temperature) flow

$$\mathbf{J} = s\mathbf{v}_\mathrm{n} \neq 0$$

First sound

$$\mathbf{v}_{\rm s} = \mathbf{v}_{\rm n} = \mathbf{v} \qquad \rho = \rho_{\rm n} + \rho_{\rm s} = \rho_0 + \delta\rho(\mathbf{r}, t)$$

$$T = \text{const} \qquad \nabla P \simeq \frac{\rho_0}{m} \nabla \mu$$

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Ordinary sound

$$\frac{\partial^2}{\partial t^2} \delta\rho = \left(\frac{\partial P}{\partial \rho_0} \right)_T \nabla^2 \delta\rho$$

Sound velocity

$$c_1^2 = \left(\frac{\partial P}{\partial \rho_0} \right)_T$$

Second sound

$$\mathbf{v}_{\mathrm{s}} = -\frac{\rho_{\mathrm{n}}}{\rho_{\mathrm{s}}}\mathbf{v}_{\mathrm{n}} \qquad \mathbf{j} = 0 \qquad P = \mathrm{const}$$

$$s = s_0 + \delta s(\mathbf{r}, t) \qquad \frac{\partial^2}{\partial t^2}\delta s = c_2^2 \, \nabla^2 \delta s$$

Second sound velocity $$c_2^2 = s_0 \frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{n}}} \left(-\frac{\partial \mu}{\partial s} \right)_P = \frac{T s_0^2}{C_1} \frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{n}}}$$

Entropy (temperature) wave

Second sound in BEC

Low temperatures (Bogoliubov theory, phonons)

$T < \mu$

$$c_1^2 = \frac{gn}{m} \qquad c_2^2 = \frac{c_1^2}{3}$$

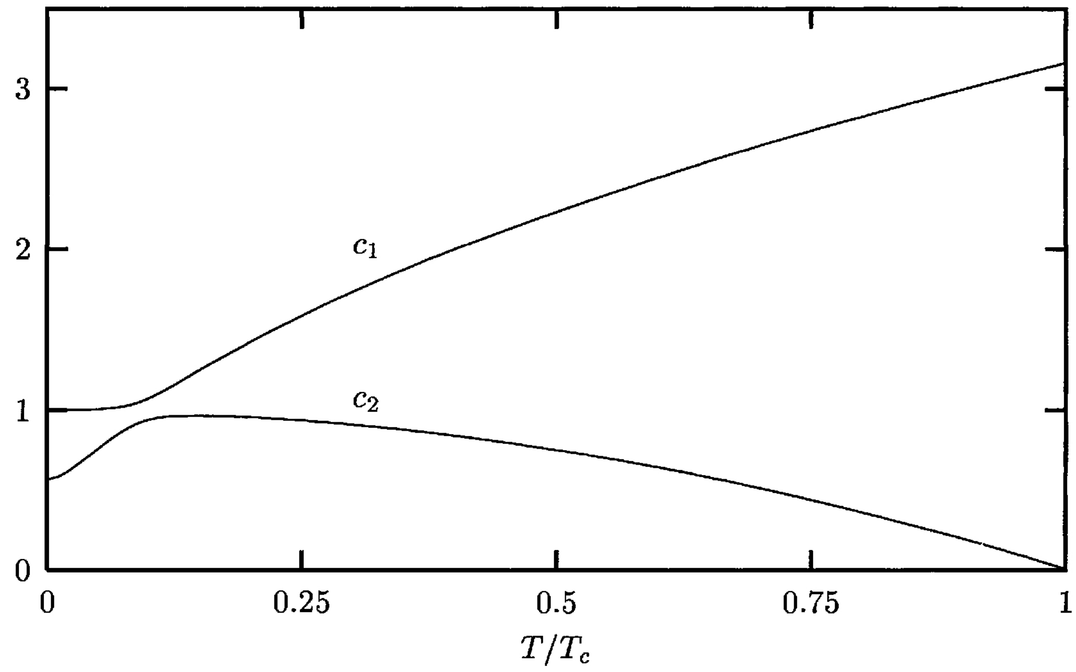
High temperatures

$$\rho_s = mn_0 = \rho\left(1 - (T/T_c)^{3/2}\right) \to 0$$

$T \to T_c$

$$c_1^2 = \frac{5}{3}\frac{g_{5/2}}{g_{3/2}}\frac{T}{m} \gg c_2^2 = \frac{gn_0(T)}{m}$$

Second sound in BEC



Conclusions of Lecture 4

- Superfluidity is a complex of phenomena associated with nonzero supefluid density. The motion of the latter has irrotational (topological) properties. The normal component (gas of excitations) does interact with walls but not with superfluid part below critical velocity.
- The combination of two components explains puzzling situation at thermal equilibrium (Hess-Fairbanks) and metastable (persistent current) states.
- Equation of hydrodynamics have to be modified to take into account superfluid velocity which carries no entropy and obeys Josephson relation. Two density components can oscillate in phase and out of phase which leads to existence of 2 sound modes
- In BEC(3D) the superfluid part is associated with the condensate and the normal part is associated with phonons. However superfluid fraction is different from condensate (2D,1D – next lecure)