Lecture 5: Phase coherence

Previous lecture: Below superfluid temperature quantum liquids develop a superfluid component. Its motion is described by irrotational velocity field related to phase

$$\mathbf{v}_{\mathrm{s}} = \frac{\hbar}{m}\nabla\Phi$$

This lecture

Absence of long range order in low dimensions due to thermal and quantum fluctuations (of phase)

Power law decay of correlations in 2D and 1D

Josephson effect and phase coherence

Bose Hubbard Hamiltonian and Mott Insulator – Superfluid Transition

Example in low dimensions

Consider total number of particles below BEC transition

$$N = N_0 + \int \mathrm{d}^D p \, n(p)$$

condensate

fluctuations

Bogoliubov theory predicts

$$n(p) = \langle a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \rangle = \langle (u_p b_{\mathbf{p}}^\dagger + v_p b_{-\mathbf{p}})(u_p b_{\mathbf{p}} + v_p b_{-\mathbf{p}}^\dagger) \rangle$$
$$= \langle (u_p^2 + v_p^2) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + v_p^2 \rangle$$

Divergence at low momenta

$$p \to 0 \qquad u_p^2,\ v_p^2 \simeq \frac{mc}{2p} \qquad \langle b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \rangle \simeq \frac{T}{cp}$$

Total number of particles **diverges**

2D: $$\int p\mathrm{d}p \left(\frac{mT}{p^2} + \frac{mc}{2p} \right)$$ at finite temperature

1D: $$\int \mathrm{d}p \left(\frac{mT}{p^2} + \frac{mc}{2p} \right)$$ even at $T = 0$

Long Range Order is destroyed by thermal ($T > 0$) or quantum fuctuations ($T = 0$)

Absence of Long Range Order in Low Dimensions

It is possible to generalise the above results beyond weakly interacting Bogoliubov picture. It can be done with the help of theorems of quantum statistical mechanics (Bogoliubov, Hohenberg-Mermin-Wagner)

Long Range Order is absent in 2D and 1D at finite temperatures due to thermal fluctuations of the order parameter

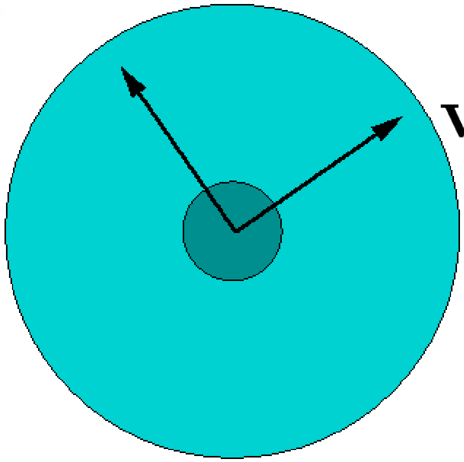
Quantum generalisation:

Long Range Order is absent in 1D at zero temperature due to quantum fluctuations of the order parameter

These theorems apply to any type of Long Range Order (ferro, anti-ferromagnetic, crystals etc.). In case of BEC the fluctuations are mainly fluctuatuations of the PHASE

Proofs of these theorems use general uncertainty type relations of creation and annihilation operators to demonstrate divergence of $n(p)$ at $p \longrightarrow 0$ in the presence of condensate

Time of flight experiment



$$\mathbf{v}(\mathbf{r},t) = \frac{\mathbf{r}}{t}$$

Uniform expansion

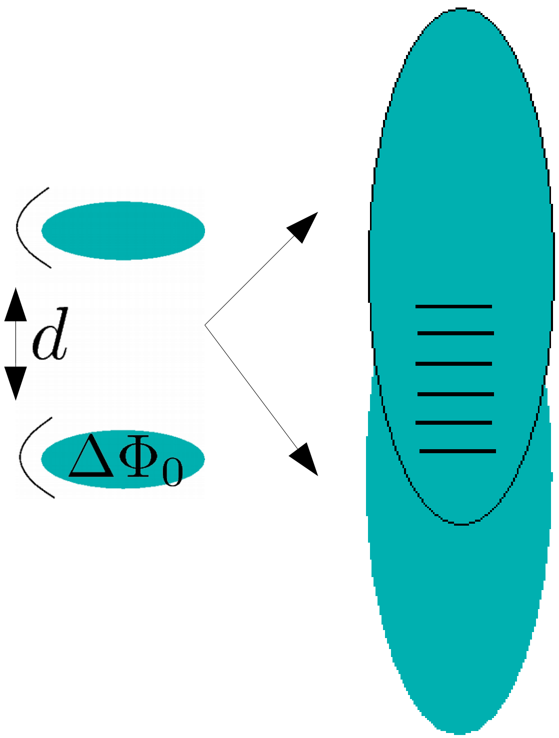
$$\Psi(\mathbf{r},t) = \frac{1}{t^{D/2}} |\Psi(\mathbf{r}/t)| e^{iS(\mathbf{r},t)}$$

$$S(\mathbf{r},t) = \frac{1}{2} \frac{mr^2}{\hbar t}$$

Two condensates a distance d from each other:

$$\Delta\Phi(\mathbf{r},t) = \Delta\Phi_0(\mathbf{r}) + S(x,y,z+d/2,t) - S(x,y,z-d/2,t)$$

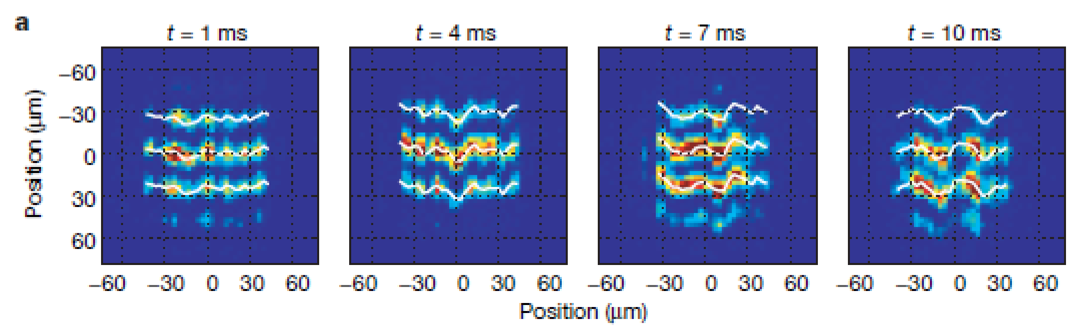
Interference of condensates. Relative Phase



Initial state $\Psi(\mathbf{r},0) = \Psi_a(\mathbf{r}) + e^{i\Delta\Phi_0}\Psi_b(\mathbf{r})$

Later $\Psi(\mathbf{r},t) = \Psi_a(\mathbf{r},t) + e^{i\Delta\Phi_0}\Psi_b(\mathbf{r},t)$

Interference



$$|\Psi(\mathbf{r},t)|^2 = |\Psi_a|^2 + |\Psi_b|^2$$

$$+2|\Psi_a||\Psi_b|\cos\left[\frac{md}{\hbar t}z + \Delta\Phi_0(\mathbf{r})\right]$$

defines position of fringes

Phase fluctuations in 1D

Density – phase representation of fields $\hat{\Psi}(x) = \sqrt{n_0 + \hat{\rho}(x)}e^{i\hat{\Phi}(x)}$

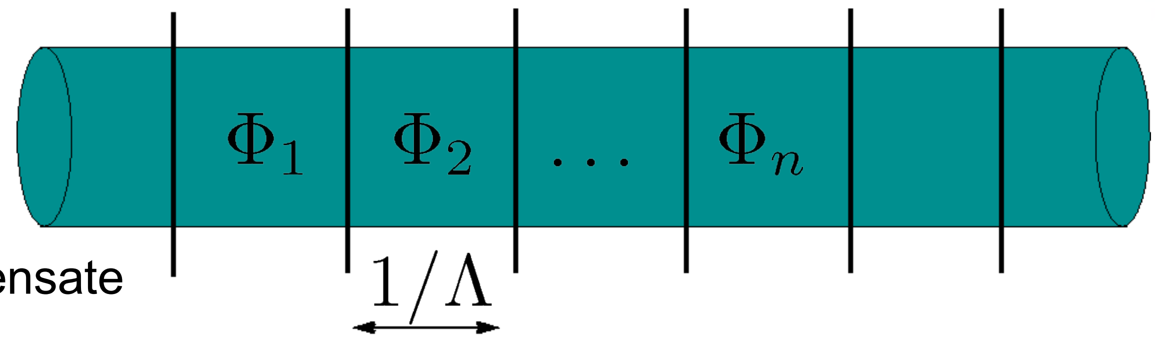
Fluctuations of the density $\langle\hat{\rho}(x)\hat{\rho}(0)\rangle$ decay fast (as $1/x^2$) for large distances. We can put

$$\hat{\Psi}(x) = \sqrt{n_0}e^{i\hat{\Phi}(x)}$$

If the phase does not fluctuate the operator $\hat{\Psi}$ can be replaced by c-number and we have BEC.

In 1D this can be done only *locally*

Quasicondensates



Assume the phase does not fluctuate too much at short scales. Then its fluctuations will only renormalise the condensate density:

$$n_0 \longrightarrow n_0(\Lambda)$$

One may then apply the Bogoliubov theory (density and phase representation)

$$\hat{\Psi} = \sqrt{n_0(\Lambda) + \hat{\rho}}\, e^{i\hat{\Phi}} = \sqrt{n_0} + \frac{\hat{\rho}}{2\sqrt{n_0}} + i\sqrt{n_0}\,\hat{\Phi}$$

*

$$\hat{\Phi}(x) = \frac{i}{\sqrt{2L}} \sum_{|k|<\Lambda} \left(\frac{m\epsilon(k)}{n_0 k^2}\right)^{1/2} (b_k - b^\dagger_{-k}) e^{ik\cdot r} \qquad \epsilon(k) = \sqrt{\frac{k^2}{2m}\left(2gn_0 + \frac{k^2}{2m}\right)}$$

One-body density matrix

One body density matrix

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(0)\rangle \simeq n_0(\Lambda)\langle e^{-i\hat{\Phi}(x)}e^{i\hat{\Phi}(0)}\rangle$$

We use the identity for exponentials of linear combinations in creation/annihilation operators

$$\langle e^{-i\hat{\Phi}(x)}e^{i\hat{\Phi}(0)}\rangle = e^{-\frac{1}{2}\left\langle\left(\hat{\Phi}(x)-\hat{\Phi}(0)\right)^2\right\rangle}$$

Therefore we need to evaluate the phase fluctuations at distance $x \gg \Lambda$

$$\chi(x) = \left\langle\left(\hat{\Phi}(x) - \hat{\Phi}(0)\right)^2\right\rangle$$

Phase fluctuations

$$\hat{\Phi}(x) = \frac{i}{\sqrt{2L}} \sum_{|k|<\Lambda} \left(\frac{m\epsilon(k)}{n_0 k^2} \right)^{1/2} (b_k - b^\dagger_{-k}) e^{ik\cdot r}$$

At zero temperature only terms containing $\langle b_k b_k^\dagger \rangle = 1$ survive and yield

$$\chi(x) = \left\langle \left(\hat{\Phi}(x) - \hat{\Phi}(0) \right)^2 \right\rangle = \frac{mc}{\pi n_0} \int_0^\Lambda \mathrm{d}k \frac{\epsilon(k)}{ck^2} (1 - \cos(kx))$$ *

Let us separate the integral

$$\int_0^\Lambda \mathrm{d}k \left(\frac{\epsilon(k)}{ck^2} - \frac{1}{2mc} \right) (1 - \cos(kx)) + \frac{1}{2mc} \int_0^\Lambda \mathrm{d}k (1 - \cos(kx))$$

converges for large k

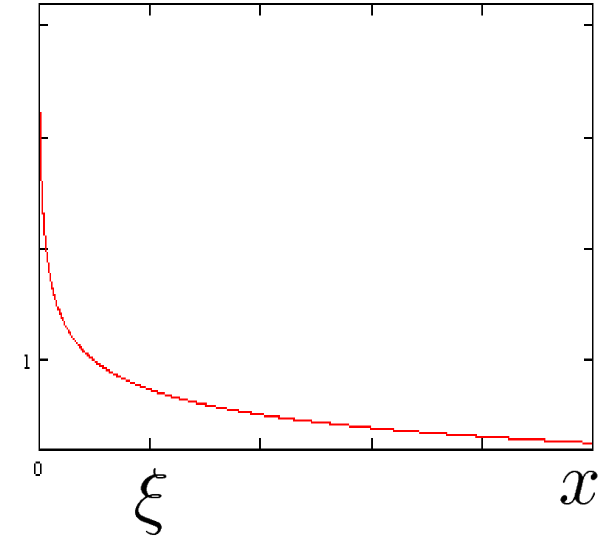
$\simeq \Lambda/2mc$

Power law decay

$$\left\langle \left(\hat{\Phi}(x) - \hat{\Phi}(0)\right)^2 \right\rangle \simeq \frac{mc}{\pi n_0} \left(\frac{\Lambda}{2mc} + \ln(mcx) \right)$$

$$\langle \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \rangle \simeq \left[n_0(\Lambda) e^{-\frac{\Lambda}{4\pi n_0}} \right] \left(\frac{\xi}{x} \right)^{\frac{mc}{2\pi n}}$$

constant



If the cutoff is chosen such that $1/x \ll \Lambda \ll n_0$

the long range behaviour does not depend on it

Beyond weak coupling

The power law decay of one body density matrix is due to phase fluctuations, i.e. absence of the phase coherence in 1D. It is not restricted to weak interactions and can always be written as

$$\langle \hat{\Psi}^\dagger(x)\hat{\Psi}(0)\rangle \sim \left(\frac{x_0}{x}\right)^{\frac{1}{2K}}$$

The parameter K is related to the compressibility (like sound velocity c) of the liquid and is called Luttinger parameter (Efetov & Larkin 1976, Haldane 1981)

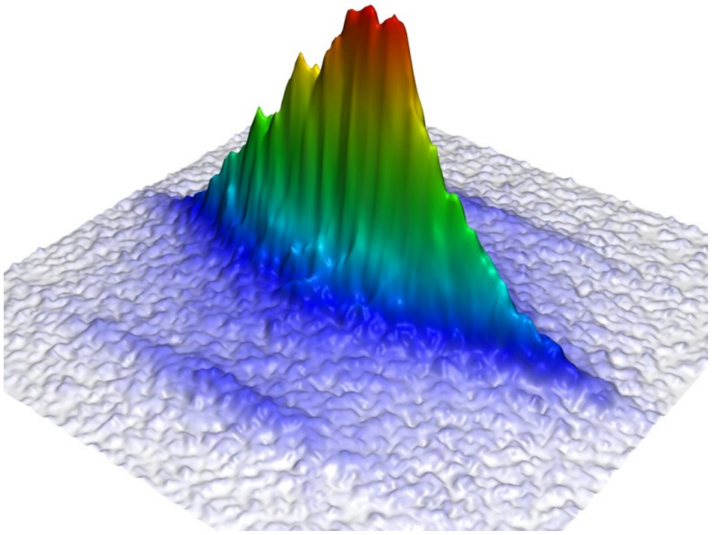
For weakly interacting bosons in 1D $K = \dfrac{\pi n}{mc} \rightarrow \infty$

For strongly interacting bosons in 1D $K \rightarrow 1$

Momentum distribution

$$n(p) = \int \mathrm{d}x e^{ipx/\hbar} \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \rangle \sim \int \frac{\mathrm{d}x}{x^{1/2K}} e^{ipx/\hbar}$$

$\sim p^{\frac{1}{2K}-1}$ for small p



Diverges at $p = 0$ in infinite system

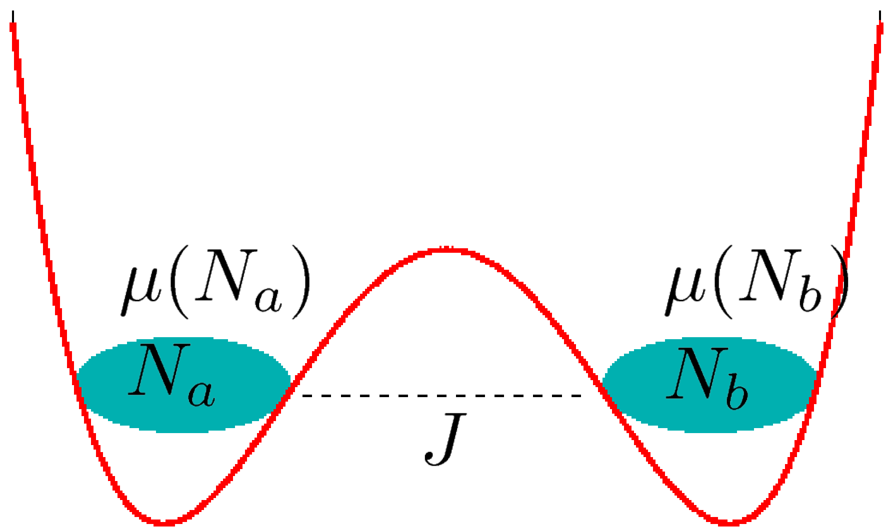
$\sim N^{1-\frac{1}{2K}}$ in finite system (trap)

can be measured using time of flight technique

Paredes et al., 2004

Simple model of phase dynamics

Consider 2 condensates (a double well configuration). $N_a, N_b \gg 1$



Relative difference

$$\Delta N = N_a - N_b \ll N_{\mathrm{tot}}$$

+ tunneling term $H_0 = J(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})$

Current of particles

$$i\hbar\dot{a} = [H_0, a] = -Jb \quad i\hbar\dot{b} = [H_0, b] = -Ja$$

$$I = \Delta\dot{N} = \frac{\mathrm{d}}{\mathrm{d}t}(a^\dagger a - b^\dagger b) = \frac{2J}{i\hbar}(b^\dagger a - a^\dagger b)$$

Assuming the number of particles is large, consider the classical ansatz

$$\hat{a} \to \sqrt{N_a}e^{i\Phi_a} \qquad \hat{b} \to \sqrt{N_b}e^{i\Phi_b} \qquad N_a, N_b \simeq \frac{N}{2}$$

Superfluid current (Josephson, 1962)

$$I = \frac{2JN}{\hbar}\sin(\Phi_a - \Phi_b) = \frac{E_J}{\hbar}\sin\Phi$$

Dynamics of the phase

$$\dot{\Phi} = -\frac{1}{\hbar}\left(\mu(N_a) - \mu(N_b)\right) = \frac{1}{\hbar}\left(\frac{\mathrm{d}\mu}{\mathrm{d}N}\right)\Delta N$$

This equation and equation for current can be derived from classical Hamiltonian

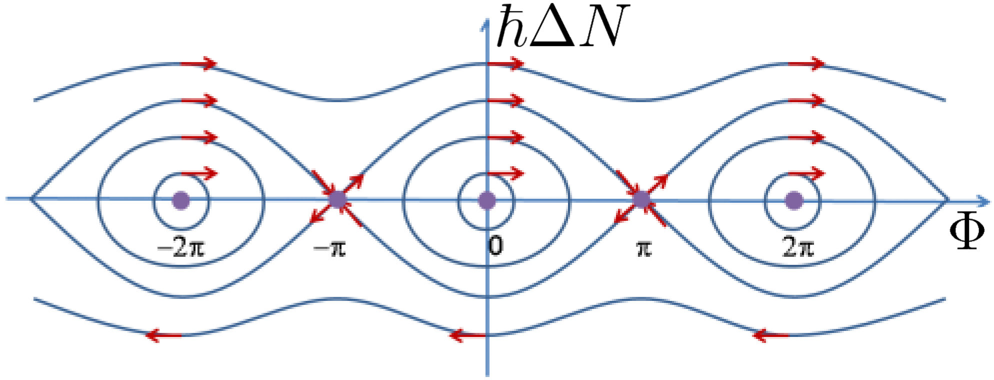
$$H_J = \frac{E_c}{2}\Delta N^2 - E_J \cos\Phi$$

Charging energy

$$\hbar\Delta\dot{N} = -\frac{\partial H_J}{\partial \Phi} \qquad \dot{\Phi} = \frac{\partial\, H_J}{\partial(\hbar\Delta N)} \qquad E_c = \frac{\mathrm{d}\mu}{\mathrm{d}N} \sim \frac{\mu}{N}$$

Pendulum analogy

$$H_J = \frac{E_c}{2}\Delta N^2 - E_J \cos\Phi$$



has two regimes:

•vibrations, i.e. small oscillations around the origin with frequency $\omega_0 = \frac{\sqrt{E_c E_J}}{\hbar}$

- phase difference Φ is well defined

•rotations for $\Delta N > \sqrt{2E_J/E_c}$ phase difference not well defined

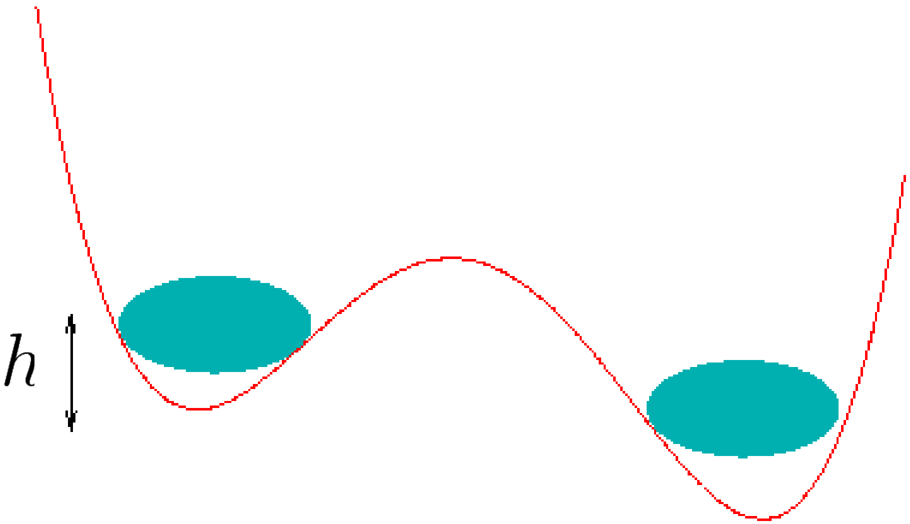
self trapping or AC Josephson effect

AC Josephson effect

External force (gravity)

$$\mu_a - \mu_b \to E_c \Delta N + \Delta\mu$$

$$\Delta\mu = mg\Delta h$$



$$\dot{\Phi} = -\frac{E_c \Delta N}{\hbar} - \frac{\Delta\mu}{\hbar}$$

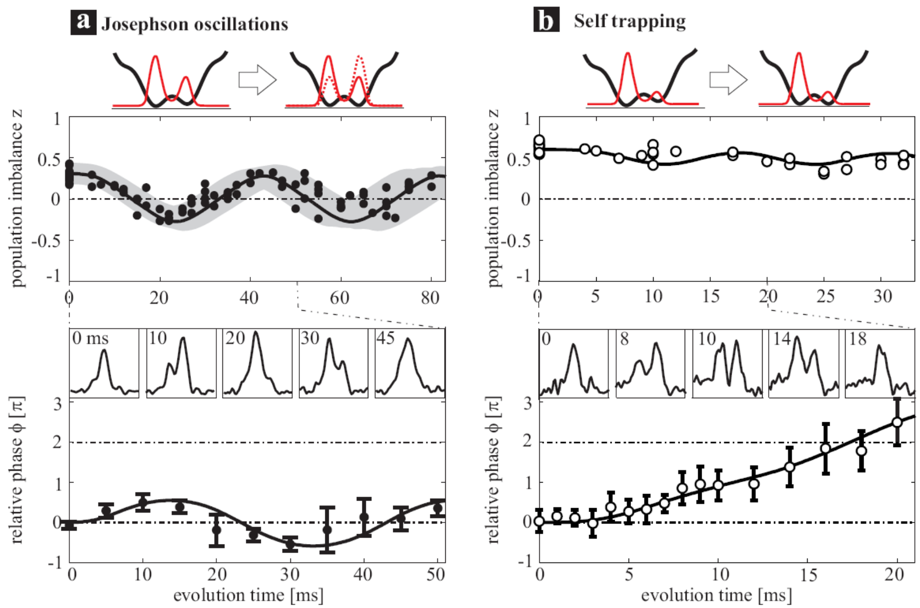
For $E_c \Delta N \ll \Delta\mu$

AC current

$$I = \frac{E_J}{\hbar} \sin \Delta\mu t$$

Experiments

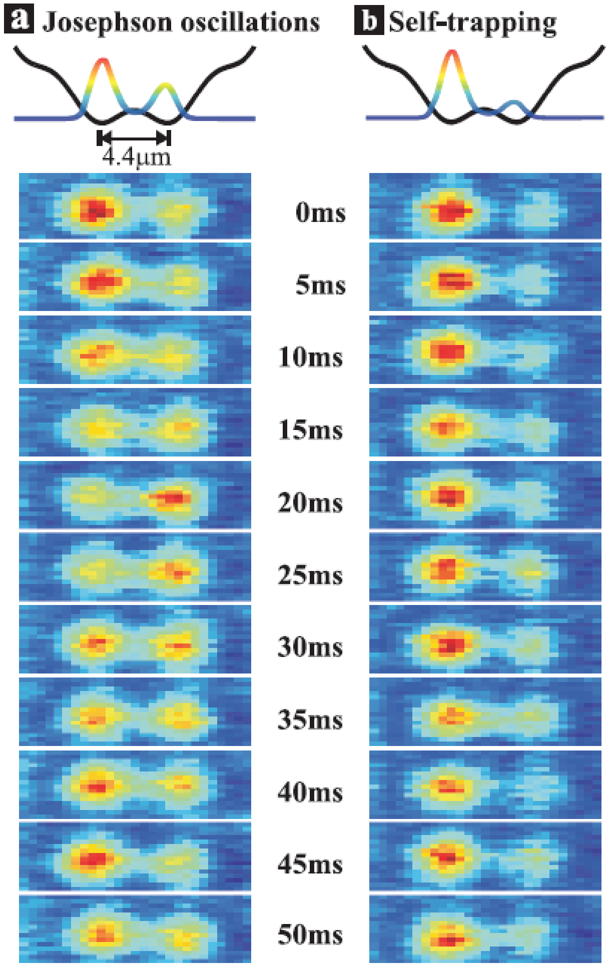
M. Oberthaler group, 2004



Technion group, 2007

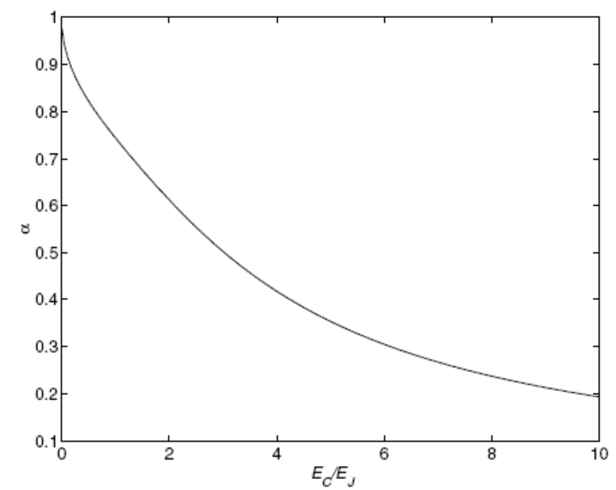
The a.c. and d.c. Josephson effects in a Bose–Einstein condensate

S. Levy[1], E. Lahoud[1], I. Shomroni[1] & J. Steinhauer[1]



Re-quantisation of Josephson equations

$$\Phi, N \to \hat{\Phi}, \hat{N} \qquad \left[\hat{\Phi}, \hat{N}\right] = i$$



In the "coordinate" representation $\hat{N} = \frac{1}{i}\frac{\partial}{\partial \Phi}$

coherence $\alpha = \langle \cos(\Phi - \Phi_0) \rangle$

$\frac{E_J}{E_c} \gg 1$ strong tunneling $\alpha \to 1 \quad \Psi(\Phi) \sim \delta(\Phi - \Phi_0)$

$\frac{E_J}{E_c} \ll 1$ weak tunneling $\alpha \to 0 \quad \Psi(\Phi) \sim e^{iN_0\Phi}$

Weak coupling regime in number representation

in “number representation

$$\tilde{\Psi}(N) = \int_0^{2\pi} \mathrm{d}\Phi \; e^{iN\Phi} \Psi(\Phi)$$

Random (delocalised) phase corresponds to well defined (relative) number of particles. This corresponds to so called Fock state

$$|F\rangle = \frac{1}{\sqrt{N_a! N_b!}} \left(a^\dagger\right)^{N_a} \left(b^\dagger\right)^{N_b} |\mathrm{vac}\rangle$$

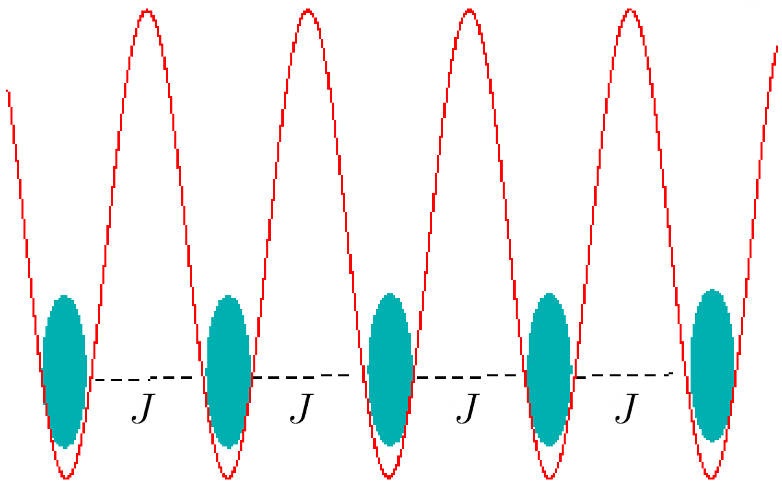
which corresponds to “fragmented condensate” discussed in Lecture 3. We see that in the present case the condensate wave functions are well separated and not overlapping due to the smallness of tunneling

Optical Lattices and Bose-Hubbard model

Generalise the Josephson model to many wells forming a lattice

The lattice potential is made of standing light wave

$$V(x) = \alpha E^2(x) = V_0 \sin^2\left(\frac{2\pi x}{\lambda}\right)$$



Tight-binding approximation

$$H = -J\sum_{<ij>}(a_i^\dagger a_j + \text{h.c.}) + \frac{U}{2}\sum_i n_i(n_i - 1)$$

M.P.A. Fisher et al. 1989

Deep insulator phase

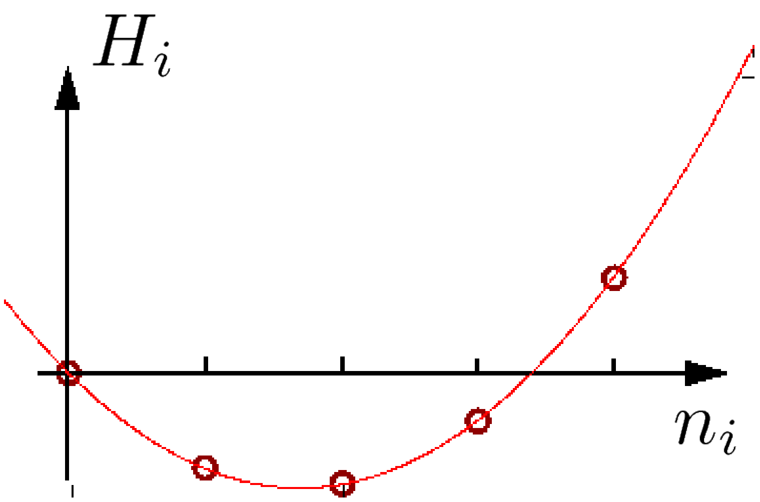
Neglect tunneling $J = 0$

The Hamiltonian is a sum of independent terms on each lattice site.

$$H_i = \frac{U}{2} n_i(n_i - 1) - \mu n_i$$

Eigenstates are just Fock states $|n\rangle$ with definite number of particles fixed by chemical potential

$$n = \text{int}\left[\frac{\mu}{U} + \frac{1}{2}\right]$$



Tunneling

Treat tunneling in the mean field approximation

$$-J\sum_{<ij>}(a_i^\dagger a_j + a_j^\dagger a_i) \rightarrow -Jz\psi(a_i + a_i^\dagger)$$

coordination number

Order parameter (condensate)

$\langle a_j \rangle = \langle a_j^\dagger \rangle = \psi$ is found self-consistently

Note that $\langle a_j \rangle$ is calculated in the eigenstates of Hamiltonian which is modified by tunneling

Self-consistency equation

Perturbation theory

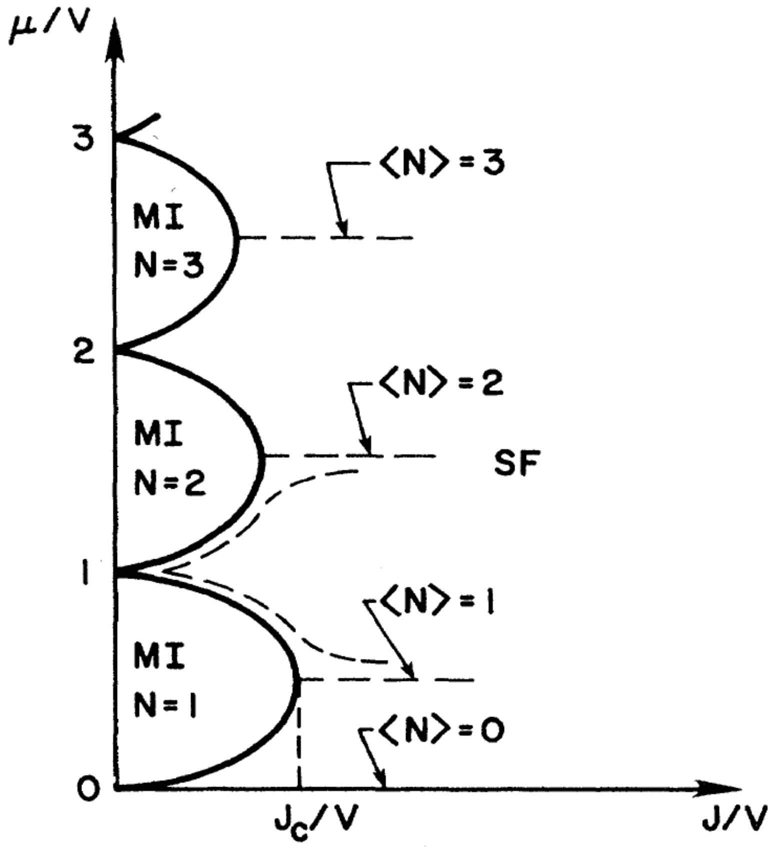
$$|n\rangle \to |n\rangle - \sum_{m\neq n} \frac{\langle m|Jz\psi(a^\dagger + a)|n\rangle}{E_n - E_m}|m\rangle$$

mixes with states $|m\rangle = |n \pm 1\rangle$ only *

$$\psi = \langle a\rangle = z\psi\frac{J}{U}\underbrace{\left(\frac{n\left(\frac{\mu}{U}\right)+1}{n\left(\frac{\mu}{U}\right)-\frac{\mu}{U}} + \frac{n\left(\frac{\mu}{U}\right)}{\frac{\mu}{U}-n\left(\frac{\mu}{U}\right)+1}\right)}$$

response function $\chi\left(\frac{\mu}{U}\right)$

Mott Insulator – Superfluid transition



NB: $n \neq$ integer – always superfluid

The boundary of the transition with $\psi = 0$ (Mott Insulator phase)
and $\psi \neq 0$ (Superfluid phase)
are determined from the condition

$$\chi\left(\frac{\mu}{U}\right) = \frac{U}{Jz}$$

Of course the prediction power of the mean field approach is limited.
Monte Carlo simulations give $(n = 1)$

$$U/J = 34.8$$ in 3D

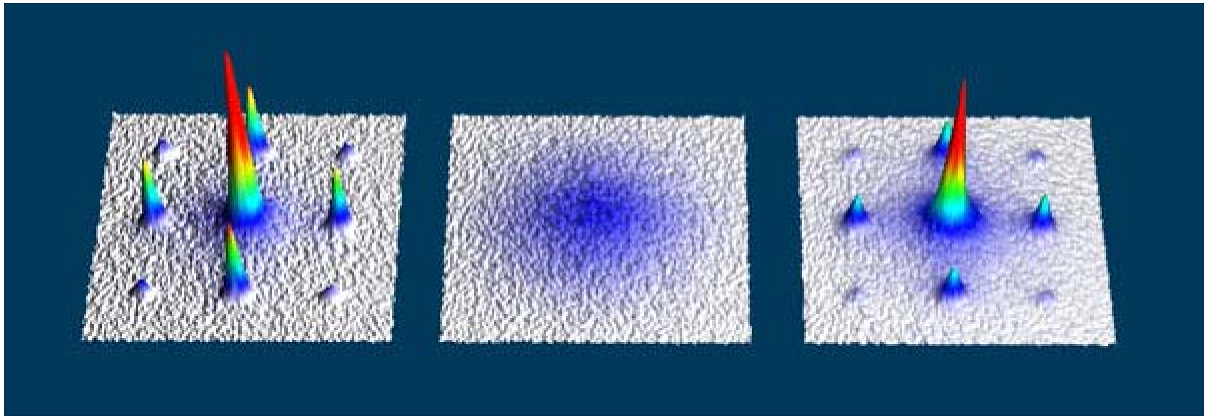
Experimental observation of MI-SF

The value of interaction parameter U is determined by scattering length

The value of tunneling J is controlled by optical lattice depth (laser intensity)

SF MI SF

Momentum distribution
(time of flight)
measures coherence



Greiner et al , 2002

Conclusions of Lecture 5

- Relative phase between condensates can be measured in interference experiment
- Pure BEC: the phase doesn't fluctuate and results in well defined interference fringes
- In lower dimension phase fluctuations are large on long distance scales. They are responsible for power law decay of one body density matrix and absence of BEC (quasicondensates)
- Quantum dynamics of phase is crucial for understanding of Josephson model. Depending on tunneling we have 2 regimes: phase coherent and Fock states
- The same phase fluctuations are responsible for Mott Insulator – Superfluid transition in optical lattices (Bose-Hubbard)