Leggett Garg Inequalities: a Quasi-probability Approach to No-Signalling in Time

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September 28, 2016

Outline

1. EPRB experiment and the CHSH inequalities
2. The Leggett-Garg situation and its difficulties with no-signalling.
3. Improvement using quasi-probabilities
4. A non-invasive measurement protocol

General goal: to find a formulation of LG which parallels EPRB as closely as possible. In particular to what extent does there exist an analogue of Fine's theorem?

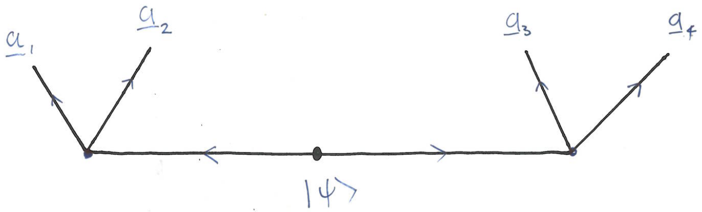
Main points: A *partial* parallel is possible and leads to an enriched perspective on macrorealism.

JJH, Phys Rev A93, 022123 (2016); arXiv:1605.09241

1.1 The EPRB Experiment

The EPRB experiment tests Local Realism.

Figure: Measurements are made of $p(s_1, s_3), p(s_1, s_4), p(s_2, s_3), p(s_2, s_4)$, where $s_i = \pm 1$.



- $p(s_1, s_3), p(s_1, s_4), p(s_2, s_3), p(s_2, s_4)$ satisfy no signalling (NS):

$$\sum_{s_1} p(s_1, s_3) = p(s_3) = \sum_{s_2} p(s_2, s_3), \quad etc.$$

- Seek a probability $p(s_1, s_2, s_3, s_4)$ such that

$$p(s_1, s_3) = \sum_{s_2, s_4} p(s_1, s_2, s_3, s_4), \quad etc.$$

- If such a probability exists then the correlation functions

$$C_{ij} = \sum_{s_1, s_2, s_3, s_4} s_i s_j \; p(s_1, s_2, s_3, s_4),$$

satisfy the eight CHSH inequalities

$$-2 \leq C_{13} + C_{14} + C_{23} - C_{24} \leq 2,$$

(plus six more).

1.3 Fine's Theorem

- Fine's theorem: The eight CHSH inequalities are also a sufficient condition for the construction of $p(s_1, s_2, s_3, s_4)$
- CHSH inequalities together with the NS conditions are a necessary and sufficient condition for Local Realism.
- Similarly for three spin measurements and the Bell inequalities.

2.1 The Leggett-Garg Inequalities and Macrorealism

The LG inequalities (Leggett-Garg, 1985) were introduced to investigate sequential measurements in time on a single system. They are designed to test Macrorealism (MR):

1. Macrorealism per se (MRps): the system is in a definite state at each moment of time;
2. Non-invasive measurability (NIM): the state can be measured without disturbing the subsequent dynamics;
3. Induction/Arrow of time (AoT): future measurements do not affect the present state.

Review: Emary, Lambert and Nori (2014)
Critique: Maroney and Timpson (2014)

2.2 The Leggett-Garg Inequalities

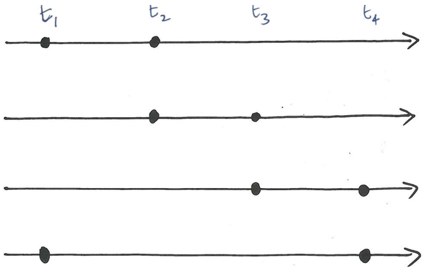


Figure: Measure a single dichotomic variable Q at pairs of times $t_1 < t_2 < t_3 < t_4$, to determine the four pairwise probabilities and C_{ij}

- MR implies an underlying probability exists and hence LG:

$$-2 \leq C_{12} + C_{23} + C_{34} - C_{14} \leq 2,$$

(plus six more).

2.3 Non-invasive Measurements

Classical models with invasiveness can explain the values of the correlation functions (Montina, 2012; Yearsley, 2013).

Ideal negative measurements: the detector is coupled to $Q = +1$ at the first time and if $Q = +1$ is *not* detected we deduce, without interaction, that $Q = -1$.
(Knee et al, 2012; Robens et al. 2015; Katiyah et al 2016)

Weak measurements are frequently used. Although the disturbance can be made very small the effect measured is of the same order of magnitude.
(Palacios-Laloy et al, 2010)

2.4 No Signalling in Time (NSIT)

Brukner and Kofler (2013) introduced the NSIT condition:

$$\sum_{s_1} p_{12}(s_1, s_2) = p_2(s_2)$$

i.e. $p(s_2)$ is independent of earlier measurements.

- NSIT is an analogue of NS in EPRB.
- NSIT is not satisfied in general by QM. This is the key difference between EPRB and LG.
- Brukner and Kofler (2013) and Kofler and Clemente (2015) sought alternative definitions of MR using NSIT only.
- Here, we keep LG and find a way to make NSIT work.

2.5 QM Measurement Formula

- In QM, the one and two time measurement formulae are

$$
\begin{aligned}
p(s) &= \mathrm{Tr}\left(P_s(t)\rho\right) \\
p(s_1, s_2) &= \mathrm{Tr}\left(P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1}(t_1)\right)
\end{aligned}
$$

where $P_s = \frac{1}{2}(1 + s\hat{Q})$.

- QM respects AoT but not NSIT:

$$
\sum_{s_1} p(s_1, s_2) = \mathrm{Tr}\left(P_{s_2}(t_2)\rho_M(t_1)\right) \neq \mathrm{Tr}\left(P_{s_2}(t_2)\rho\right)
$$

where $\rho_M(t_1)$ denotes the measured density operator,

$$
\rho_M(t_1) = \sum_{s_1} P_{s_1}(t_1)\rho P_{s_1}(t_1)
$$

2.6 Key Issue

- Since $[\hat{Q}(t_1), \hat{Q}(t_2)] \neq 0$, the existence of a well-defined probability at the two-time level in the LG framework is not guaranteed. I.e., the two-time situation may or may not possess a macrorealistic description.

- If an MR description exists, there could be a number of different ways of assigning probabilities to such pairs of observables.

- Different probability assignments correspond to different measurement protocols.

3.1 Checking MR at the Two-Time Level

- Measure $\langle Q_1 \rangle$, $\langle Q_2 \rangle$ and C_{12} respecting NIM and AoT:
  - C_{12} measured using ideal negative measurement.
  - $\langle Q_2 \rangle$ measured in a separate set of runs.

  Attempt to construct a probability:

$$q(s_1, s_2) = \frac{1}{4}\left(1 + \langle Q_1 \rangle s_1 + \langle Q_2 \rangle s_2 + C_{12} s_1 s_2\right)$$

- In a MR theory, we must have,

$$(1 + s_1 Q(t_1))(1 + s_2 Q(t_2)) \geq 0,$$

  and averaging we obtain $q(s_1, s_2) \geq 0$. This holds if

$$-1 + |\langle Q_1 \rangle + \langle Q_2 \rangle| \leq C_{12} \leq 1 - |\langle Q_1 \rangle - \langle Q_2 \rangle|$$

- If $q(s_1, s_2) < 0$, MR fails at the two-time level.

3.2 A Quantum-mechanical Quasi-probability

- In QM the quasi-probability is

$$q(s_1, s_2) = \mathrm{Re}\ \mathrm{Tr}\left(P_{s_2}(t_2)P_{s_1}(t_1)\rho\right)$$

(c.f the discrete Wigner function).

- It satisfies generalized versions of NSIT and AoT:

$$\begin{aligned}\sum_{s_1} q(s_1, s_2) &= \mathrm{Tr}\left(P_{s_2}(t_2)\rho\right) = p(s_2) \\ \sum_{s_2} q(s_1, s_2) &= \mathrm{Tr}\left(P_{s_1}(t_1)\rho\right) = p(s_1)\end{aligned}$$

- In typical models $q(s_1, s_2) \geq 0$ for some parameter ranges; otherwise $q(s_1, s_2)$ has negative components.

- $q(s_1, s_2)$ can be measured either by measuring $\langle Q_1 \rangle$, $\langle Q_2 \rangle$ and C_{12} non-invasively, or by sequential measurements in which the first one is weak.

3.3 Comparison with Sequential Measurements

- The sequential measurement probability p and quasi-probability q are related by

$$p(s_1, s_2) = q(s_1, s_2) + \frac{1}{8}\langle[\hat{Q}_1, \hat{Q}_2]\hat{Q}_1\rangle s_2$$

Same $\langle Q_1\rangle$ and C_{12} but different $\langle Q_2\rangle$.

- NSIT for $p(s_1, s_2)$ means zero interference.

- NSIT for $p(s_1, s_2)$ implies $q(s_1, s_2) \geq 0$, but not conversely.

- $q(s_1, s_2) \geq 0$ requires only that the interferences are bounded.

- The switch from $p(s_1, s_2)$ satisfying NSIT to $q(s_1, s_2) \geq 0$ is a weakening of classicality conditions.

3.4 Fine's Theorem Restored

- $q(s_i, s_j) \geq 0$ for $i, j = 12, 23, 34, 14$ tests MR at the two-time level. It is satisfied under (reasonably weak) restrictions on the parameter space.

- These positivity constraints together with the eight LG inequalities, supply a natural parallel with the EPRB system and therefore a necessary and sufficient condition for MR.

$$MR \Leftrightarrow MR_{12} \wedge M_{23} \wedge M_{34} \wedge M_{14} \wedge LG_{1234}$$

- LG alone misses some violations of MR, namely those in which LG are satisfied but some $q(s_i, s_j) < 0$.

- This suggests it would be of interest in current experiments to examine $q(s_i, s_j)$ in addition to the LG inequalities.

3.5 Summary So Far

- A formulation of LG paralleling EPRB is possible – for a non-trivial range of parameters, but not in general.

- It arises from an exploration of MR at the two-time level using the quasi-probability $q(s_i, s_j)$ which automatically incorporates generalized versions of NSIT and AoT.

- Experiments should check both LG and $q(s_i, s_j) \geq 0$ for violations of MR.

- Of particular interest is the connection of these results to the MR conditions of Clemente and Kofler (2015).

For further details see JJH, Phys Rev A93, 022123 (2016)

4.1 Measurement of C_{12} with a "Waiting Detector"

- Seek new ways of measuring C_{12} non-invasively.
- First note that

$$C_{12} = \langle Q_1 Q_2 \rangle = 1 - \frac{1}{2} \langle [Q_2 - Q_1]^2 \rangle$$

- Assume there exists a velocity $v(t) = \dot{Q}(t)$.

$$Q_2 - Q_1 = \int_{t_1}^{t_2} dt \ v(t).$$

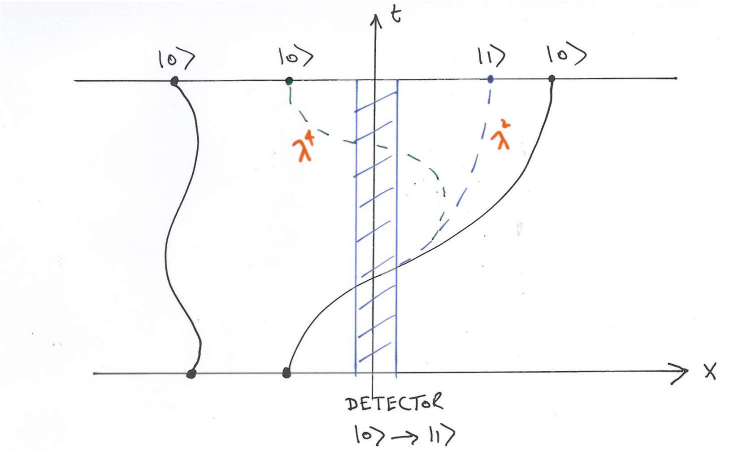
RHS can be measured using a weak coupling λ to a detector continuous in time.

- Assume that $Q(t)$ changes sign at most once during $[t_1, t_2]$. This is reasonable in some models and includes regimes in which there is substantial LG violation.

JJH, arXiv:1605.09241

4.2 The Waiting Detector

For illustrative purposes suppose $Q = \text{sign}(X)$.



The effect of interest, $p(|1\rangle)$, is of order λ^2 but the back-action disturbance is order λ^4, so is approximately non-invasive for $\lambda \ll 1$.