Continuous wavelet transform Python module

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1 Introduction

A continuous wavelet is a well-known fundamental tool that allows to filter data-sets such as to enhance localised features of a given shape (or periodicity) for a given scale, whilst diminishing features with scales far removed.

The module contains two Continuous Wavelet Transform (CWT) functions, a one-dimensional CWT `wavelet_1d` and a two-dimensional CWT `wavelet_2d`. Each has associated mother wavelet functions. Additionally, `wavelet_1d` has a plot function. There are two demo functions added to see the CWTs in action.

1.1 Prerequisites

This module has been designed for Python 2.7 and uses external standard Python modules:

- numpy

- scipy

- matplotlib

1.2 Using this module in publications

When using this module is published work, please cite the following papers: For the 1d CWT:

- Verwichte, E., Nakariakov, V. M., Ofman, L., & Deluca, E. E. 2004, Sol. Phys., 223, 77

- Torrence, C. & Compo, G. P.: 1998, Bull. Amer. Meteor. Soc. 79, 61.

For the 2d CWT:

- White, R. S., Verwichte, E., & Foullon, C. 2012, AA, 545, A129

- Witkin, A. P. 1983, in Proc. Int. Joint Conf. Artifcial Intell., IJCAI'83, San Francisco, CA, USA, 1019

2 Theory of the Continuous Wavelet Transform

2.1 Basics of a 2d wavelet

I will focus solely on the two-dimensional continuous wavelet transform as its use is much less common than the 1d wavelet. I refer the reader to the landmark paper by Torrence & Compo for the the 1d CWT.

The two-dimensional, continuous wavelet transform of an image $I(\vec{r})$ is defined as:

$$CWT(I)(\vec{b}, a, \theta) = \frac{1}{a^n} \iint\limits_{-\infty}^{+\infty} I(\vec{r}) \; \psi \left(\frac{1}{a} R_{-\theta}(\vec{r} - \vec{b}) \right) d\vec{r} \; . \tag{1}$$

It is a linear transformation of I where the function $\psi(\vec{r})$ is called the *mother wavelet*. It has the property that it is localised both in space and in its reciprocal space. Furthermore, the wavelet needs to fulfill an admissability condition:

$$\int \frac{| \hat{\psi}(\vec{k}) |^2}{| \vec{k} |^2} d\vec{k} < \infty \; , \tag{2}$$

where $\hat{\psi}$ is the Fourier transform of ψ and $\vec{k}$ is the reciprocal vector to the spatial coordinate vector $\vec{r}$. From this condition, it follows, when ψ is regular enough, that the wavelet has zero mean:

$$\iint\limits_{-\infty}^{+\infty} \psi(\vec{r}) d\vec{r} = 0 \; . \tag{3}$$

The following transformation are performed on the mother wavelet so that any signal can be represented as a linear combination of these wavelets: translation by a displacement $\vec{b}$, rotation over an angle θ by the rotation matrix $R_{-\theta}$ and a dilation by a scaling factor $a > 0$. The value of n in the factor a^{-n} in Equation (1) depends of the chosen norm as $n = 2/\text{norm}$ (e.g. for L_1-norm, n=2).

2.2 Mexican-hat wavelet

In this paper the *Mexican hat* mother wavelet is chosen. It is proportional to the Laplacian of a Gaussian function:

$$\psi(\vec{r}) \; = \; -\vec{\nabla}^2 \left(e^{-\frac{|\vec{r}|^2}{2}} \right) \; = \; \left(2 - | \vec{r} |^2 \right) e^{-\frac{|\vec{r}|^2}{2}} \; . \tag{4}$$

For the rest of the paper the continuous wavelet transform using the Mexican Hat wavelet is abbreviated by CWT. The Mexican hat wavelet is isentropic so that the dependency of the transform upon angle θ is superfluous. A wavelet transformation, using this motherwavelet, detects ridges (where $\vec{\nabla} I = 0$ and $| \vec{\nabla}^2 I |$ is large) in an image. It is easily seen by integrating Equation (1) by parts

$$
\begin{aligned}
CWT(I)(\vec{b}, a) \; &= \; -\frac{1}{a^{n-2}} \iint\limits_{-\infty}^{+\infty} I(\vec{r}) \, \vec{\nabla}^2 \left(\mathrm{e}^{-\frac{|\vec{r} - \vec{b}|^2}{2a^2}} \right) d\vec{r} \\
&= \; -\frac{1}{a^{n-2}} \iint\limits_{-\infty}^{+\infty} \vec{\nabla}^2 I(\vec{r}) \, \mathrm{e}^{-\frac{|\vec{r} - \vec{b}|^2}{2a^2}} d\vec{r} \; ,
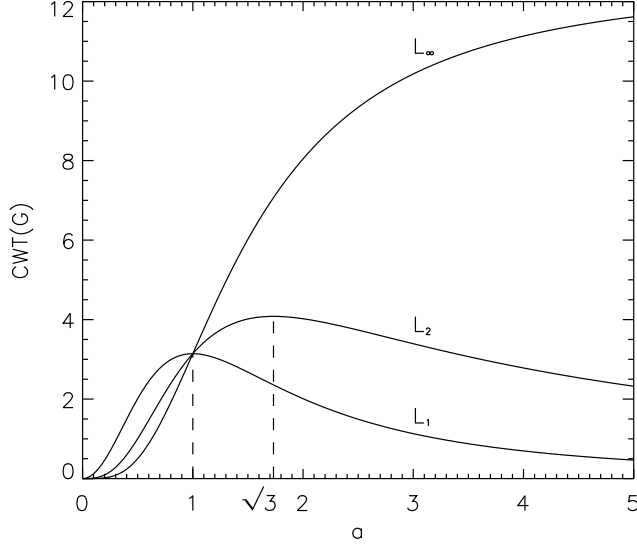\end{aligned} \tag{5}
$$

Figure 1: Plot of the maximum in wavelet-power of a Gaussian signal of width $\sigma=1$, as a function of scale a for three choices of norm.

that the transform in fact corresponds to a Gaussian blurring of the Laplacian of the image signal. This leaves intensity ridges in the image more clearly visible. As a becomes smaller and smaller, the function $\exp(-(|\vec{r}-\vec{b}|^2)/(2a^2))/2\pi$ tends to the delta function $\delta(\vec{r}-\vec{b})$, pointing out to those points in the image where $|\vec{\nabla}^2 I|$ is large. This makes CWT is particularly suited for detecting discontinuities in an image. The bulk of the contribution to the integral in the CWT at point $\vec{b}$, comes from a circular area of radius 3-4 a around $\vec{b}$. Equation (5) also showes that the CWT is insensitive to signals, which are constant or linear, within a circle of radius 3-4 a of a particular point, i.e. $CWT(Axy + Bx + Cy + D) = 0$. Since the CWT is employed to enhance features of an image of interest at scale a, those contribution to the image signal that becomes negligable in the CWT can be denoted as the background intensity, $I_0(a)$ for that particular scale.

Let us examine the transformation of a Gaussian signal $G(\vec{r}) = A \exp(-(\vec{r}-\vec{c})/(2\sigma^2))$, i.e.

$$CWT(G)(\vec{b}, a) = \frac{1}{a^n} \iint\limits_{-\infty}^{+\infty} A \, e^{-\frac{|\vec{r}-\vec{c}|^2}{2\sigma^2}} \, \psi \left(\frac{\vec{r}-\vec{b}}{a} \right) \, d\vec{r} \, . \tag{6}$$

To solve this transform we introduce the following quantities:

$$X = \frac{x - c_x}{\sqrt{2}\,a} \, , \ Y = \frac{y - c_y}{\sqrt{2}\,a} \, , \ X_0 = \frac{b_x - c_x}{\sqrt{2}\,a} \, , \ Y_0 = \frac{b_y - c_y}{\sqrt{2}\,a} \, , \ \epsilon = \frac{a}{\sigma} \, . \tag{7}$$

Then,

$$\frac{|\vec{r}-\vec{c}|^2}{2\sigma^2} = \epsilon^2 \left(X^2 + Y^2 \right) \, , \ \frac{|\vec{r}-\vec{b}|^2}{2a^2} = X^2 + Y^2 - 2X_0X - 2Y_0Y + X_0^2 + Y_0^2 \, . \tag{8}$$

If we consider the contributions in the exponentials, note that for X-dependency we have

$$(1 + \epsilon^2)X^2 - 2X_0X = \left(\sqrt{1 + \epsilon^2}X - \frac{X_0}{\sqrt{1 + \epsilon^2}} \right)^2 - \frac{X_0^2}{1 + \epsilon^2} \, . \tag{9}$$

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It thus makes sense to change to new variables:

$$\xi = \sqrt{1+\epsilon^2}X - \frac{X_0}{\sqrt{1+\epsilon^2}} \ , \ \psi = \sqrt{1+\epsilon^2}Y - \frac{Y_0}{\sqrt{1+\epsilon^2}} \ . \tag{10}$$

Then for X (equivalently for Y):

$$X = \frac{\xi}{\sqrt{1+\epsilon^2}} + \frac{1}{1+\epsilon^2}X_0 \ , \ X - X_0 = \frac{\xi}{\sqrt{1+\epsilon^2}} - \frac{\epsilon^2}{1+\epsilon^2}X_0 \ , \epsilon^2 X^2 + (X-X_0)^2 = \xi^2 + \frac{\epsilon^2}{1+\epsilon^2}X_0^2 \ . \tag{11}$$

Also,

$$\begin{aligned} 2 - \frac{|\vec{r} - \vec{b}|^2}{a^2} &= 2 - 2(X-X_0)^2 - 2(Y-Y_0)^2 \\ &= 2 - 2\left(\frac{\xi}{\sqrt{1+\epsilon^2}} - \frac{\epsilon^2}{1+\epsilon^2}X_0\right)^2 - 2\left(\frac{\psi}{\sqrt{1+\epsilon^2}} - \frac{\epsilon^2}{1+\epsilon^2}Y_0\right)^2 \\ &= 2 - 2\left(\frac{\epsilon^2}{1+\epsilon^2}\right)^2 \left(X_0^2 + Y_0^2\right) + 4\frac{\epsilon^4}{(1+\epsilon^2)^{3/2}}\left(X_0\xi + Y_0\psi\right) - \frac{2}{1+\epsilon^2}\left(\xi^2 + \psi^2\right) \end{aligned} \tag{12}$$

Finally, with

$$\mathrm{d}x\mathrm{d}y = 2a^2\mathrm{d}X\mathrm{d}Y = \frac{2a^2}{1+\epsilon^2}\mathrm{d}\xi\mathrm{d}\psi \ , \tag{13}$$

the transform becomes:

$$\begin{aligned} CWT(G)(\vec{b}, a) = \ & \frac{2A\,\mathrm{e}^{-\frac{\epsilon^2\left(X_0^2 + Y_0^2\right)}{1+\epsilon^2}}}{a^{n-2}\left(1+\epsilon^2\right)} \iint\limits_{-\infty}^{+\infty} \left[2 - 2\left(\frac{\epsilon^2}{1+\epsilon^2}\right)^2 \left(X_0^2 + Y_0^2\right) + 4\frac{\epsilon^4}{(1+\epsilon^2)^{3/2}}\left(X_0\xi + Y_0\psi\right) \right. \\ & \left. - \frac{2}{1+\epsilon^2}\left(\xi^2 + \psi^2\right) \right] \mathrm{e}^{-\xi^2}\mathrm{e}^{-\psi^2}\,\mathrm{d}\xi\,\mathrm{d}\psi \ . \end{aligned} \tag{14}$$

We make use of the definite integrals

$$\int\limits_{-\infty}^{+\infty}\mathrm{e}^{-\alpha t^2}\mathrm{d}t = \sqrt{\frac{\pi}{\alpha}} \ , \ \int\limits_{-\infty}^{+\infty}t\,\mathrm{e}^{-\alpha t^2}\mathrm{d}t = 0 \ , \ \int\limits_{-\infty}^{+\infty}t^2\mathrm{e}^{-\alpha t^2}\mathrm{d}t = -\frac{\partial}{\partial\alpha}\int\limits_{-\infty}^{+\infty}\mathrm{e}^{-\alpha t^2}\mathrm{d}t = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \ . \tag{15}$$

Hence Eq. (14) becomes

$$\begin{aligned} CWT(G)(\vec{b}, a) &= \frac{2\pi A\,\mathrm{e}^{-\frac{\epsilon^2\left(X_0^2 + Y_0^2\right)}{1+\epsilon^2}}}{a^{n-2}\left(1+\epsilon^2\right)}\left[2 - 2\left(\frac{\epsilon^2}{1+\epsilon^2}\right)^2\left(X_0^2 + Y_0^2\right) - \frac{2}{1+\epsilon^2}\right] \\ &= \frac{2\pi A\,\epsilon^2\,\mathrm{e}^{-\frac{\epsilon^2\left(X_0^2 + Y_0^2\right)}{1+\epsilon^2}}}{a^{n-2}\left(1+\epsilon^2\right)^2}\left[2 - 2\frac{\epsilon^2}{1+\epsilon^2}\left(X_0^2 + Y_0^2\right)\right] \\ &= \frac{2\pi A}{a^{n-2}}\frac{\epsilon^2}{\left(1+\epsilon^2\right)^2}\left[2 - 2\frac{\epsilon^2}{1+\epsilon^2}\left(X_0^2 + Y_0^2\right)\right]\mathrm{e}^{-\frac{\epsilon^2\left(X_0^2 + Y_0^2\right)}{1+\epsilon^2}} \\ &= \frac{2\pi A}{a^{n-2}}\frac{\left(\frac{a}{\sigma}\right)^2}{\left(1+\left(\frac{a}{\sigma}\right)^2\right)^2}\,\psi\left(\frac{\vec{b} - \vec{c}}{\sqrt{a^2 + \sigma^2}}\right) \ , \end{aligned} \tag{16}$$

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where we used:

$$\frac{\epsilon^2}{1+\epsilon^2}\left(X_0^2 + Y_0^2\right) = \frac{a^2}{a^2+\sigma^2}\left(\frac{|\vec{b}-\vec{c}|^2}{2a^2}\right) = \frac{|\vec{b}-\vec{c}|^2}{2(a^2+\sigma^2)} \ . \tag{17}$$

For a fixed scale, the maximum value of $CWT(G)$ is reached at $\vec{b} = \vec{c}$. Let us consider the dependency of $CWT(G)$ as a function of scale a, at that position. This is illustrated in Figure 1. It is clear that for $n > 0$ (excluding L_∞-norm), $CWT(G)(\vec{c}, a)$ reaches a maximum at:

$$a_m = \sqrt{\frac{4}{n} - 1}\,\sigma \ , \ \ CWT(G)(\vec{c}, a_m) = \frac{\pi A}{4}n^2\left(\frac{4}{n} - 1\right)^{2-\frac{n}{2}} \tag{18}$$

For L_1-norm, $n = 2$, this means that the wavelet power has a local maximum, of value πA, at the scale of the signal, *i.e.* $a_m = \sigma$. For L_2-norm, $n = 1$, the wavelet power has a local maximum of value $(9/4\sqrt{(3)})\pi A$ at the scale $a_m = \sqrt{3}\sigma$. For L_∞-norm, $n = 0$, there is no local maximum in wavelet power.

2.3 Image noise

The noise in the EIT images is due to two, assumed independent, contributions: photon noise and CCD read-out noise. The photon noise follows from Poisson statistics, for which the average value μ and variance σ_{pn}^2 of the photon density distribution are equal to the expectancy value, which in this case is equal to the part of the photon flux, Φ_{int}, interacting with the CCD camera. We express Φ_{int} and σ_{pn}, originally in units of number of photons, into Digital Number (DN) units by multiplying by a factor N_{ph}/d, where N_{ph} is the gain of electrons per photon, equal to 12398 / (3.65 λ) and d is the digitization constant, equal to 16.7 eV per DN for the EIT instrument (Dere *et al.*, 2000). λ is the wavelength bandpass, expressed in Angstroms. The average and variance of the CCD read-out noise, σ_{CCD}, are assumed constant. The latter is, for nominal EIT observation conditions, equal to 2.7 DN^2. The noise variance in the EIT images is written as:

$$\sigma_{noise}^2(\vec{r}) = \frac{N_{ph}}{d}I(\vec{r}) + \sigma_{CCD}^2 = \frac{203.4}{\lambda(A)}I(\vec{r}) + 2.7 \ , \tag{19}$$

where I corresponds to the image intensity, which is the photon flux in units of DN. The EIT instrument has four possible wavelength bandpasses: 171A, 195A, 284A and 304A. For these four values, the ratio $203.4/\lambda(A)$ is equal to 1.19, 1.04, 0.72 and 0.67 respectively.

How does the noise look like in a wavelet transformation? Consider $N(\vec{r})$ to be a function describing the noise contribution to the image intensity. The CWT of N may be approximated by two, essentially finite, sums:

$$CWT(N) \approx \sum_{i,j} \tilde{\psi}_{i,j}\, N_{i,j}\Delta A_{i,j} \ , \tag{20}$$

with $\tilde{\psi}_{i,j} = \psi(|\,\vec{r}_{i,j} - \vec{b}_{i,j}\,|\,/a)/a^n$ and $N_{i,j} = N(\vec{r}_{i,j})$, and where $\Delta A_{i,j} = \Delta x_{i,j}\Delta y_{i,j}$ is the interval of area, which is chosen small enough as to have the sum approximate the integral. We consider the photon noise. In this case each $N_{i,j}\Delta A_{i,j}$ has a Poisson distribution with an average and variance, both equal to $N_{ph}/d\ I_0(a)_{i,j}\Delta A_{i,j}$, where $I_0(a)$ is the background intensity at scale a. The CWT of the average, $< N >$, and the variance, σ_{pn}, of the noise fluctuations due to photon noise, are
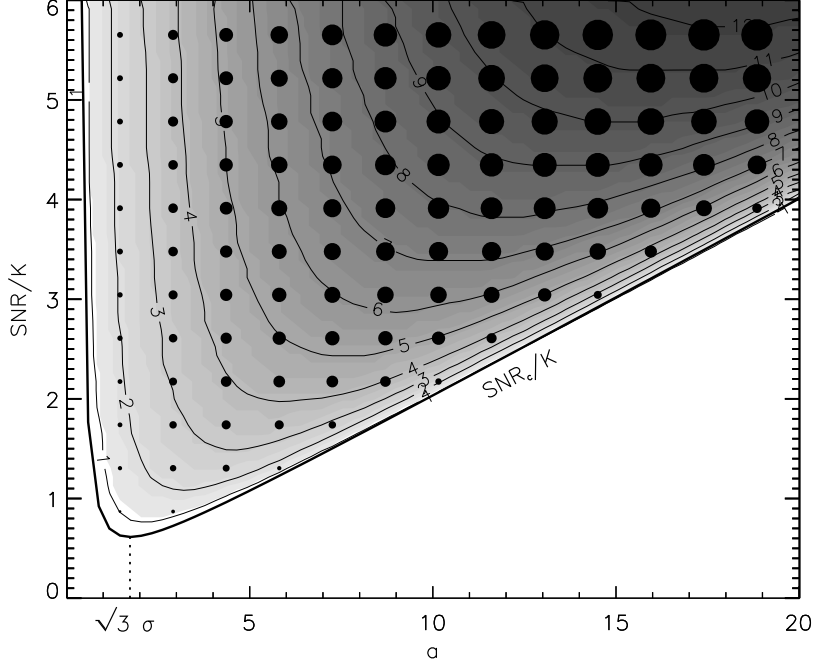
Figure 2: Size of detected source as a function of scale a and the ratio of the signal-to-noise in the original image and the signal-to-noise threshold, SNR/ K. $\sigma = 1$.

approximately equal to:

$$
\begin{aligned}
< N_{CWT} > \quad &= \quad \sum_{i,j} \tilde{\psi}_{i,j} < N_{i,j} \Delta A_{i,j} > \\
&\approx \quad \frac{N_{ph} I_0(a)}{d} \frac{1}{a^n} \int_{-\infty}^{+\infty} \psi(\vec{r}) \, d\vec{r} = 0 \quad (21) \\
\sigma_{CWT,pn}^2 \quad &= \quad \sum_{i,j} \tilde{\psi}_{i,j}^2 \, \sigma_{pn,i,j}^2 \Delta A_{i,j} \\
&\approx \quad \frac{N_{ph} I_0(a)}{d} \frac{1}{a^{2n}} \int_{-\infty}^{+\infty} \psi^2(\vec{r}) \, d\vec{r} = \frac{2\pi}{a^{2n-2}} \sigma_{pn}^2 \quad (22)
\end{aligned}
$$

We made use of the fact that the background intensity I_0 is approximately constant in a circle of radius 3-4 a around the point we measure. For the case of the CCD-readout noise, we find $CWT(< N >) = 0$ and $CWT(\sigma_{CCD}^2) = 2\pi\sigma_{CCD}^2/a^{2n-2}$. Therefore, we find that the noise variance in the CWT is approximately equal to

$$
\sigma_{CWT,noise}^2 = \frac{2\pi}{a^{2n-2}} \sigma_{noise}^2 . \quad (23)
$$

For the detection of a feature in the image, we impose the rule that its signal-to-noise ratio $\text{SNR} = A/\sigma_{noise}$ in the original image exceeds a certain value, K, which is often set equal to 3-5, so that the chance that the signal is not due to noise is highly significant. Let us consider the signal-to-noise ratio of the CWT of the Gaussian signal, calculated in Expression (16). Its is equal

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to

$$\text{SNR}_{CWT} = \frac{CWT(G)}{\sigma_{CWT,noise}} = \text{SNR} \sqrt{2\pi}\sigma \frac{\left(\frac{a}{\sigma}\right)^3}{\left(1 + \left(\frac{a}{\sigma}\right)^2\right)^2} \psi\left(\frac{\vec{b} - \vec{c}}{\sqrt{a^2 + \sigma^2}}\right) . \tag{24}$$

The CWT signal-to-noise ratio SNR_{CWT} is proportional to SNR, but the strength of this proportionality is goverened by the scale a of the CWT and the width σ of the Gaussian signal. The expression for SNR_{CWT} is independent of the chosen norm. A feature is detectable if the SNR_{CWT} exceeds K in any point of the image. This occurs first where Expression (24) is maximal, *i.e.* $\vec{b} = \vec{c}$ and $\psi(0) = 2$. Thus we can say that a feature is detected if

$$\frac{\text{SNR}}{K} > \frac{\text{SNR}_c(a, \sigma)}{K} = \frac{1}{2\sqrt{2\pi}\sigma} \frac{\left(1 + \left(\frac{a}{\sigma}\right)^2\right)^2}{\left(\frac{a}{\sigma}\right)^3} . \tag{25}$$

We can say several things about Expression (25):

1) Detection depends on the size of the feature and the ratio between the scale and this width.
2) SNR_c becomes infinite at $a=0$ since it is impossible to detect a non-existing feature.
3) At $a = \sqrt{3}\sigma$, SNR_c has a local minimum of value $\text{SNR}_c(\sqrt{3}\sigma)/K = 8/(3\sqrt{6\pi}\sigma)$.
4) For $a \gg \sigma$, SNR_c is well described by

$$\frac{\text{SNR}_c}{K} = \frac{1}{2\sqrt{2\pi}\sigma} \left(\frac{a}{\sigma}\right) , \tag{26}$$

which can be seen as a function of σ to be the minimum detection curve.

3 1d CWT function

The main function to create a one-dimensional CWT is **wavelet_1d**:
Input:

- x: one-dimensional float array of size N_0 of the independent variable, e.g. time. If x is not equally spaced, the time-series will be resampled to the minimum step-size present in x.

- y: one-dimensional float array of size N_0 of the dependent variable, i.e. the signal time series.

- a0 (optional): one-dimensional float array of size $M_{0,0}$ of the first scale-space parameters. If not given, a set of scales is automatically generated. There is control over the resolution through the keyword resolv.

- a1 (optional): one-dimensional float array of size $M_{0,1}$ of the second scale-space parameters. This will be automatically passed on to the mother wavelet function.

- ...

Keywords:

- fit: scalar integer, if given will subtract a polynomial of order fit from y (default = average subtracted).

- kk: float array of reciprocal space variable as input (default = None)

- lag1: float scalar lag1 parameter (default = 0)

- `missing`: Boolean stating whether data is missing (default = False)

- `mother`: string of name of mother wavelet: `'morlet'` (default = `'morlet'`)

- `regular`: Boolean stating whether the intervals in x are regular (default = True) if not regular the data will be interpolated onto a regular grid

- `resolv`: Scalar of resolution of scales (default = 1.0)

- `siglvl`: Scalar of significance level between 0 and 1 (default = 0.99)

- `variance`: Scalar of variance to use in determining significance (default = variance(y)) Note that if fit is given, the default variance is that of the residual y where the polynomial has been substracted

Output:

C: object containing the following attributes

- `C.x0`: copy of original one-dimensional float array of size N_0 of independent variable x.

- `C.y0`: copy of original one-dimensional float array of size N_0 of dependent variable y.

- `C.x`: one-dimensional float array of size N of independent variable. Can be of different size than `C.x0`, especially when x0 is not equally-spaced and required resampling.

- `C.dxx`: float scalar of step-size of `C.x`

- `C.cwt`: the complex at least two-dimensional float array of dimension $N \times M_0 \times M_1 \times \ldots$ of the CWT.

- `C.cwt_real`: real part of CWT.

- `C.cwt_imag`: imaginary part of CWT.

- `C.cwt_abs`: absolute part of CWT

- `C.scale`: array of scales (first scale-space parameter) in units of indices, either bases on input or Choice of scales $s = s_0 2^{j\,\mathrm{d}j}$ with j=0..M_0-1, $M_0 = \log_2(N\mathrm{d}x/s_0)/\mathrm{d}j$. $\mathrm{d}j$ is chosen such that σ of reciprocal wavelet is larger than scale sampling difference.

- `C.period`: floating array size M_0 of scales (first scale-space parameter) in units of independent variable. i.e.

- `C.coi`: float array of size N containing the scale for the cone of influence

- `C.signif`: array of size M_0 of the significance value as a function of scale

- `C.siglvl`: scalar float of significance level, between 0 and 1.

- `C.array_missing`: int array of size N indicating if data is missing (value=0) or not (value=1).

- `C.fft_theor`: array of size M_0 of statistics for significance calculation.

- `C.time_max`: float array of size of `C.count_max` of x-values of maxima in CWT.

- `C.period_max`: float array of size `C.count_max` of y-values of maxima in CWT

- `C.count_max`: number of maxima present

- `C.mother`: string name of mother wavelet used, e.g. `'morlet'`

- `C.fit`: scalar integer of order of polynomial fit (if set).

- `C.yfit`: float array of size N of polynomial fit.

An example usage syntax is:

$$C = \texttt{wavelet\_1d(t,y,mother='morlet',siglvl=0.99)}$$

which does the same as

$$C = \texttt{wavelet\_1d(t,y)}$$

The mother wavelet function is given as a function with the name construct `wvl_func_` + `mother` + 1, e.g. `wvl_func_morlet1`. Here, only one mother wavelet is given but others are possible. The function calculates the mother wavelet in reciprocal (Fourier) space as a function of `k`. The function **wvl_func_morlet1** represents the one-dimensional Morlet mother wavelet. The function in principle is never needed to be called separately. It has the following structure:

Input:

- `kx`: float array of size N_k of reciprocal space array

- `a0`: scalar float of first scale parameter

- `a1` (optional): scalar float of second scale parameter

- ...

Keywords:

- `norm`: Scalar integer of norm used (default = 1)

- `epsilon`: Scalar float controlling width of wavelet (default = 1.)

- `k0`: Scalar float controlling number of oscillations underneath envelope (default = 5.6)

Output:

- `wavelet`: FFT of mother wavelet for given scale as a function of `kx`

- `period`: Scalar of period associated with scale `a0`

- `coi`: float array containing the scale for the cone of influence

- `dofmin`: scalar integer of degree of freedom

In the module a plot function is included to easily visualise the result called `plot1`, with syntax:

Input:

- object as output from `wavelet_1d`.

Keywords:

- CWT:

  - `real`: Boolean, plot real part of CWT (default = False)
  - `imaginary`: Boolean, plot imaginary part of CWT (default = False)
  - `absolute`: Boolean, plot absolute part of CWT (default = True)
  - `zrange`: 2-element array of min-max of CWT to plot (default = [min,max])

- Scales:

  - `freq`: Boolean, plot versus frequency = $2\,\pi$ / `C.scale` (default = False)
  - `yrange`: 2-element array of scale-range (default = [min,max])
  - `ylog`: Boolean, plot logarithmic scale (defaull = True)

- Colors:

  - `theme`: theme of color scheme: `'blue'`,`'red'`,`''` (default = `''`)
  - `cmap`: colormap used for x-scale plot (default given by theme)
  - `color`: color of lines (default given by theme)
  - `coi_color`: color of COI hatches (default givem by theme)
  - `background_color`: color of background in time series and power spectrum plots (default given by theme)

- Titles:

  - `title`: title of x-scale plot (default = `'CWT'`)
  - `title_scale`: title of y-label in x-scale plot (default = `'period'` or `'frequency'`)
  - `title_time`: title of x-label of time series plot (default = `'time'`)
  - `title_signal`: title of y-label of time series plot (default = `'signal'`)

The demo `demo1` shows an example of the usage of $\mathrm{wavelet}_1 d\,and\,\mathrm{plot1}$:

```
import numpy as np
import random
nx = 128
t = np.arange(nx)/(nx-1.) * 10.
w1 = 2.*np.pi/1.9
w2 = 0.5*w1 + t/t.max()*(2.5*w1)
y = np.cos(w1*t) + 2*np.cos(w2*t)
r = np.zeros(nx)
for i in np.arange(nx): r[i] = random.random() * 1.5
y = y + r
C = wavelet_1d(t,y,mother='morlet',siglvl=0.9)
fig = plt.figure(num=0,figsize=[8,6],dpi=100,facecolor='White',edgecolor='White')
plot1(C,freq=False,theme='Blue')
```
the result of which is shown in Fig. 3, and can be reproduced by running the function `demo1`.
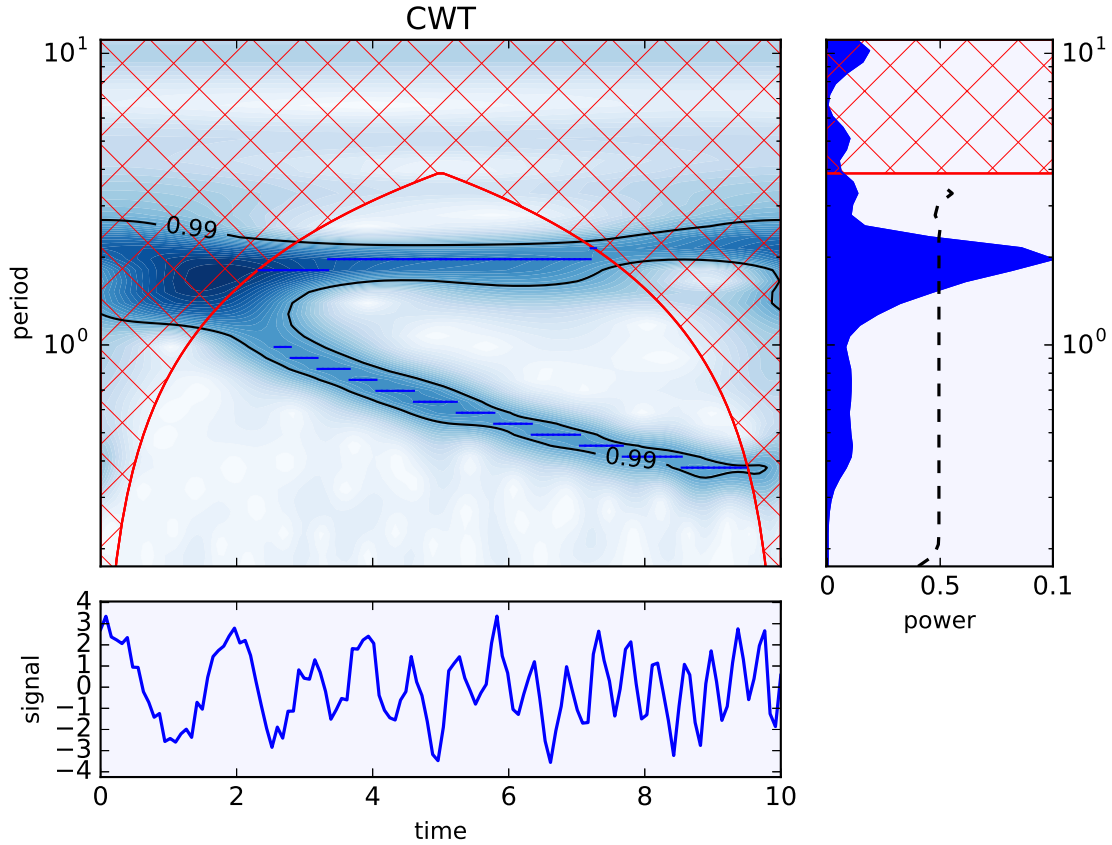
Figure 3: Example of the Continuous Wavelet Transform of a time series using a Morlet wavelet.

4 2d CWT function

Similarly to the 1d version, the 2d CWT function is called **wavelet_2d**:

Input:

- `img`: two-dimensional float array of image of dimensions $N_x \times N_y$.

- `scale1`: scalar float or one-dimensional float array of size M_0 of parameter values related to first parameter taken by mother wavelet.

- `scale2`: (optional): scalar float or one-dimensional float array of size M_1 of parameter values related to second parameter taken by mother wavelet]

- ...

Keywords:

- `absolute`: boolean if set outputs absolute part of CWT (default: False)

- `edge`: scalar of introducing extra edge pixels (default: 0)

- `imaginary`: boolean if set outputs imaginary part of CWT (default: False)

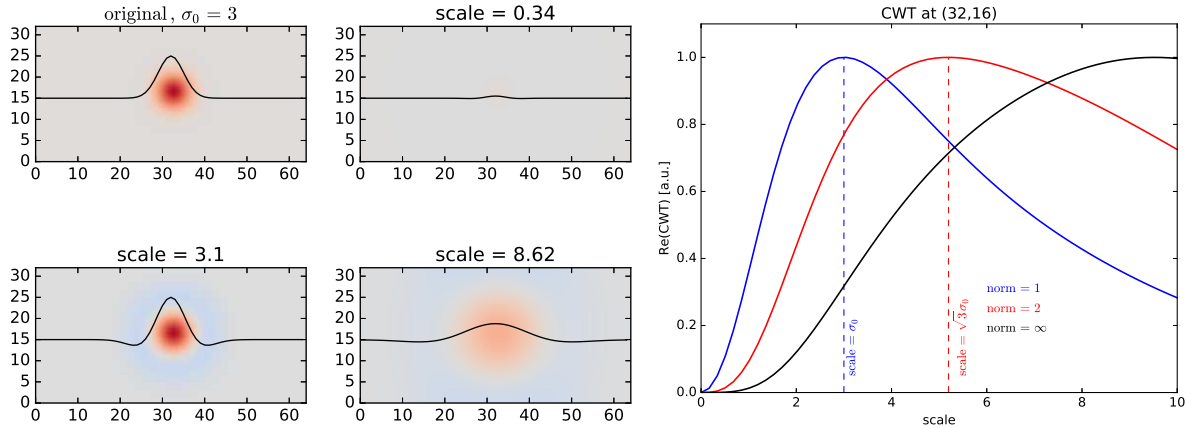- `kk`: 2d float array of reciprocal space variable as input (default = None)

Figure 4: Left: Original image and CWT for various scales, demonstrating how the signal is reduced when the scale of the wavelet is far from the scale of the signal. Right: Real part of CWT at location of peak in Gaussian signal as a function of wavelet scale, for three norms. This is the same figure as shown earlier. For L_1 norm, which is the default, the CWT is maximal at the scale of the Gaussian signal.

- mother: string of name of mother wavelet: 'mexicanhat' (default = 'mexicanhat')

- norm: scalar integer of norm used in transform, i.e. 1, 2 or np.inf (default: 1)

- real: boolean if set outputs real part of CWT (default: False)

Output:

- cwt: multi-dimensional float array of CWT of dimension $N_x \times N_y[\times M_0][\times M_1][\times ...]$.

The two mother wavelets available are the 2d mexican hat and a 2d Gaussian (not a wavelet of course). Other more advanced mother wavelets are possible. The name construct wvl_func_ + mother + 2, e.g. wvl_func_mexicanhat2. The mother wavelet, which will normally not be called directly, is structured as follows:

Input:

- kx: two-dimensional floating array of x-component of reciprocal coordinate.

- ky: two-dimensional floating array of y-component of reciprocal coordinate.

- scale1: scalar float or one-dimensional float array of size M_0 of parameter values related to first parameter taken by mother wavelet.

- scale2: (optional): scalar float or one-dimensional float array of size M_1 of parameter values related to second parameter taken by mother wavelet]

- ...

Keywords:

- norm: scalar integer of norm used in transform, i.e. 1, 2 or np.inf (default: 1)

Output:

- `wavelet`: two-dimensional float array of scaled wavelet function in reciprocal space.

The demo `demo2` shows an example of the usage of `wavelet_2d` and reproduces the analytical exercise of calculating the Mexican Hat CWT of a two-dimensional Gaussian intensity pulse. The resulting plots are shown in Fig. 4. Below, we show a demonstration of the application of the 2d CWT to solar data.
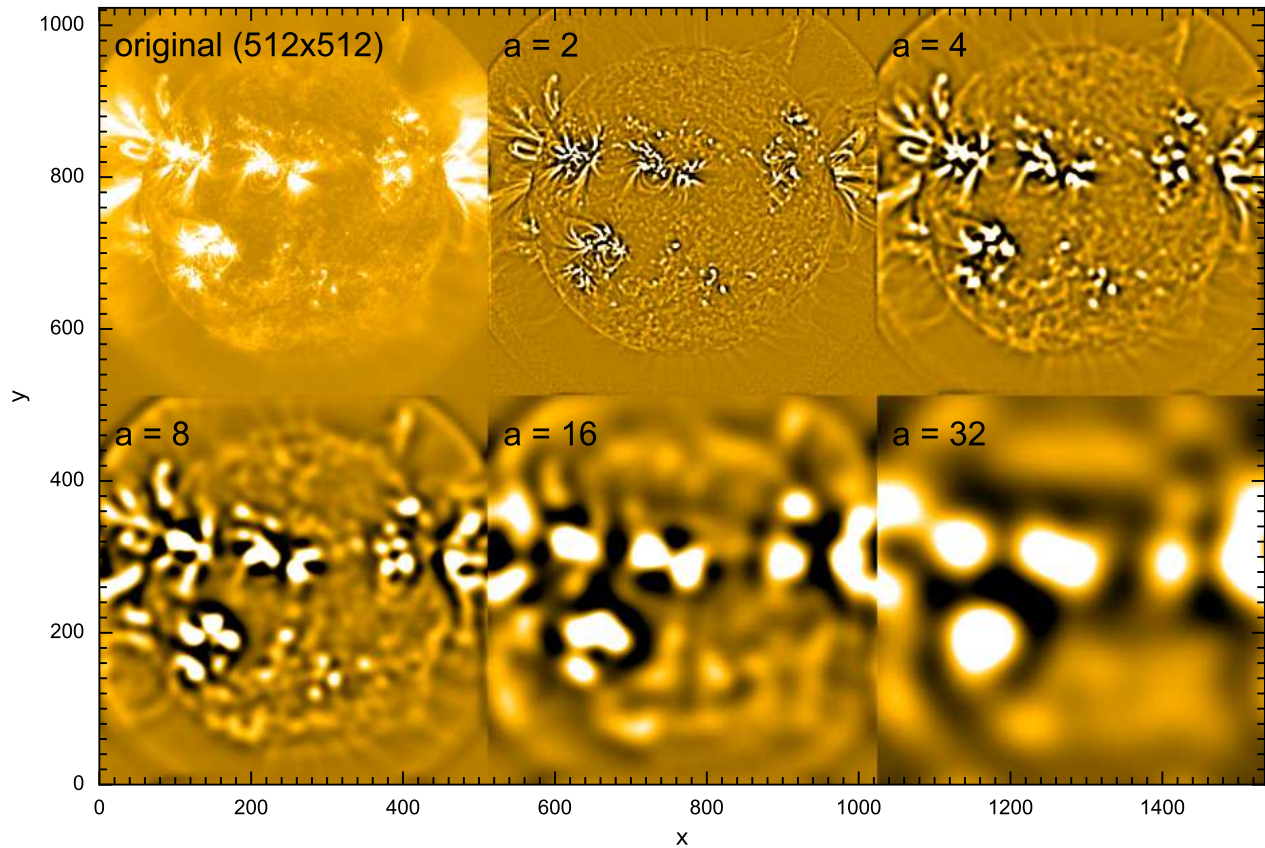

Figure 5:   Demonstration of 2d wavelet CWT for various scales on an AIA 171Åimage.