# Electrostatic wavepackets in the presence of superthermal (accelerated) electrons: modulational instability and envelope soliton modes

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## 1. Introduction

Nonthermal particles are omnipresent in various plasma environments, such as the solar wind [1, 2, 3] and the solar corona [4]. Plasmas containing these type of particles are generally characterised by a high energy trail non-Maxwellian distribution named *kappa distribution* [5, 6, 7]. The parameter $\kappa$ determines the high energy power law index and approaches a Maxwellian distribution when $\kappa \to \infty$ as shown in the following figure [8].

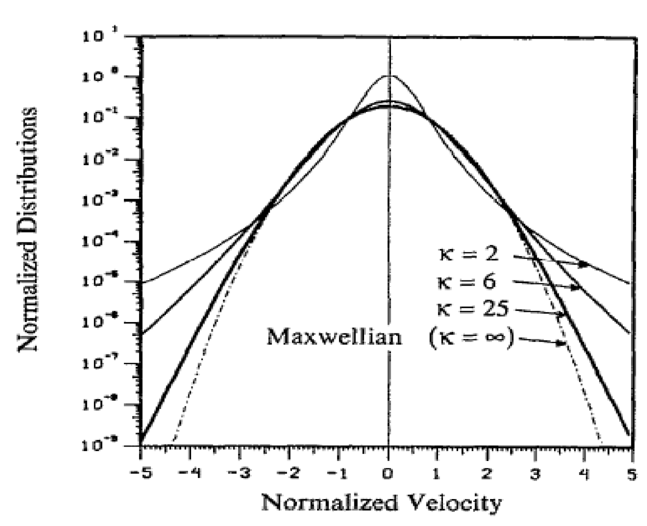


Figure 1. Comparison of generalized Lorentzian distributions for the spectral index of $\kappa = 2, 6$ and 25, with the corresponding Maxwellian distribution $\kappa = \infty$.

*Localized modulated-envelope structures* are generated due to nonlinear self interactions of the carrier wave in such plasmas.

The scope of this study is to trace, from first principles, the influence of superthermality on modulated-amplitude electron acoustic (EA) wavepackets propagating in a plasma containing superthermal electrons in the background.

## 2. A Fluid model for electron acoustic waves

We consider a three-component collisionless unmagnetized plasma consisting of:
* (species 1) inertial electrons (charge $-e$ and mass $m_e$),
* (species 2) kappa distributed inertialess hot electrons (charge $-e$ and mass $m_e$), and
* (species 3) stationary ions (charge $q_i = Z_i e$ and mass $m_i$).

The number density $n_c$, the mean velocity $u_c$ of cold electrons is governed by the continuity, momentum equation and the system potential $\Phi$ is obtained from Poisson's equation

$$\frac{\partial n_c}{\partial t} + \nabla\cdot(n_c\,\mathbf{u}_c) = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}_c}{\partial t} + (\mathbf{u}_c\cdot\nabla)\mathbf{u}_c = \frac{e}{m_e}\nabla\Phi\,, \quad (2)$$

$$\nabla^2\Phi = 4\pi e\left[n_c + n_{ho}\left(1 - \frac{e\Phi}{(\kappa-\frac{3}{2})k_B T_h}\right)^{-\kappa+\frac{1}{2}} - Z_i n_{i0}\right]. \quad (3)$$

We adopt a kappa distribution for the hot electrons

$$n_h = n_{h0}\left[1 - \frac{e\Phi}{(\kappa-\frac{3}{2})k_B T_e}\right]^{-\kappa+1/2};$$

where the index "0" denotes the equilibrium number density values.

The one dimensional normalized form of Eqs. (1) - (3) reads:

$$\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} = 0, \quad (4)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial \phi}{\partial x}, \quad (5)$$

$$\frac{\partial^2\phi}{\partial x^2} \approx \beta(n-1) + c_1\phi + c_2\phi^2 + c_3\phi^3, \quad (6)$$

*Normalization:*
$n = n_c/n_{c0}$, $u = u_c/v_0$, $\phi = \Phi/\Phi_0$, $t = t\,\omega_{peh}$, $x = x/\lambda_{Dh}$,
where $\lambda_{Dh} = \left(k_B T_h/4\pi n_{h0}e^2\right)^{1/2}$, $\omega_{peh}^{-1} = (4\pi n_{h0}e^2/m_e)^{-1/2}$,
$v_0 \equiv (k_B T_h/m_e)^{1/2}$.

The *cold-to-hot electron density ratio* is expressed by

$$\beta = n_{c0}/n_{h0}\,.$$

***The influence of the parameters $\beta$ (hot electron concentration) and $\kappa$ (superthermality) on EAW characteristics is the focus of the investigation that follows.***

The coefficients entering Poisson's equation [Eq. (6)] are

$$c_1 = \frac{\kappa-1/2}{\kappa-3/2}\,, \qquad c_2 = \frac{(\kappa-1/2)(\kappa+1/2)}{2(\kappa-3/2)^2}\,,$$

$$c_3 = \frac{(\kappa-1/2)(\kappa+1/2)(\kappa+3/2)}{6(\kappa-3/2)^3}\,.$$

## 3. Perturbative analysis

*Reductive perturbation* technique: consider small deviations from the equilibrium state $S^{(0)} = (0,0,1)^T$, i.e.

$$S = S^{(0)} + \Sigma_{n=1}^{\infty}\epsilon^n S^{(n)}$$

where $\epsilon \ll 1$ is a (real) smallness parameter. We assume that

$$S^{(n)} = \Sigma_{l=-\infty}^{\infty} S_l^{(n)}(X,T)e^{il(kx-\omega t)}\,,$$

where the condition $S_{-l}^{(n)} = S_l^{(n)*}$ holds, for reality. The wave amplitude is thus allowed to depend on the stretched (*slow*) coordinates of space and time as $X_n = \Sigma_n\epsilon^n x$ and $T_n = \Sigma_n\epsilon^n t$, respectively, where $n = 1, 2, 3, .....$, distinguished from the (*fast*) carrier variables $x\ (\equiv X_0)$ and $t\ (\equiv T_0)$.
Substituting the above expressions into Eqs. (4) - (6) provides the familiar EAW *dispersion relation* and the amplitude corresponding to the first harmonics in order $\epsilon$ for equations $n = 1$, $l = 1$,

$$\omega^2 = \frac{k^2\beta}{k^2 + c_1}\,, \qquad n_1^{(1)} = \frac{k^2 + c_1}{\beta}\phi_1^{(1)}\,, \qquad u_1^{(1)} = -\frac{k}{\omega}\phi_1^{(1)}\,.$$

## 4. Multi-harmonic solution up to order $\sim \epsilon^2$

The equations for $n = 2$, $l = 1$ provide the compatibility condition

$$\frac{\partial\phi_1^{(1)}}{\partial T_1} + v_g\frac{\partial\phi_1^{(1)}}{\partial X_1} = 0\,,$$

where the group velocity $v_g$ is defined as

$$v_g = \frac{d\omega}{dk} = \frac{\omega^3 c_1}{k^3\beta}\,.$$

We obtain the expression for the amplitudes corresponding to the first, second and zeroth harmonics to order $\sim\epsilon$, the solution:

$$\phi \simeq \epsilon\phi_1^{(1)}e^{i(kx-\omega t)} + \epsilon^2\left[\phi_0^{(2)} + \phi_2^{(2)}e^{2i(kx-\omega t)}\right] + \mathcal{O}(\epsilon^3)\,,$$

and similar expressions for $n, u$.

***→ Harmonic generation (in $n$, $u$, $\phi$)!***

## 5. Nonlinear Schrödinger (NLS) equation for $\phi_1^{(1)}$:

$$i\frac{\partial\psi}{\partial\tau} + P\frac{\partial^2\psi}{\partial\zeta^2} + Q\,|\psi|^2\,\psi = 0\,. \quad (7)$$

where

- $\psi \equiv \phi_1^{(1)}(\zeta,\tau)$.
- $\zeta = X_1 - v_g T_1$, $\tau = T_2$.
- Dispersion coefficient: $P = \frac{1}{2}\omega''(k) = -\frac{3}{2}\frac{\omega^5 c_1}{k^4\beta^2}$.
- Nonlinearity coefficient: $Q \to$ long expression omitted here.

## 6. Modulational instability (MI) of ES wavepacket

- Plane wave solution of (7): $\psi = \psi_0\exp(iQ|\psi_0|^2\tau)$;
- Linear analysis: set $\tilde\psi = \tilde\psi_0 + \epsilon\,\tilde\psi_{1,0}\cos(\tilde k\zeta - \tilde\omega\tau)$;
- Dispersion relation *(for an amplitude perturbation)*:

$$\tilde\omega^2 = P\,\tilde k^2\,(P\tilde k^2 - 2Q|\tilde\psi_{1,0}|^2)\,; \quad (8)$$

- If $PQ < 0$ then: stability;
- If $PQ > 0$ then instability occurs for $\tilde k_{cr} = (2Q/P)^{1/2}\mid\tilde\psi_{1,0}\mid$;
- Modulational instability growth rate: $\Gamma_\kappa = \alpha x\left(2\frac{\beta}{\alpha} - x^2\right)^{1/2}$;

  where $\Gamma_\kappa = \mathrm{Im}|\tilde\omega|/(Q_\infty|\tilde\psi_{1,0}|^2)$, $\alpha = P/P_\infty$, $\beta = Q/Q_\infty$, and $x = \tilde k/(Q_\infty/P_\infty)^{1/2}|\tilde\psi_{1,0}|$.
- *The sign of the product PQ determines the stability profile.*

## 7. Parametric analysis

### 7.1 The critical wavenumber

The wavenumber threshold $k_{cr}$ (for instability) is depicted versus
– *superthermality* (via $\kappa$) for different density ratio $\beta$ values (left panel), and
– the *density ratio* $\beta$, for different $\kappa$ (right panel).

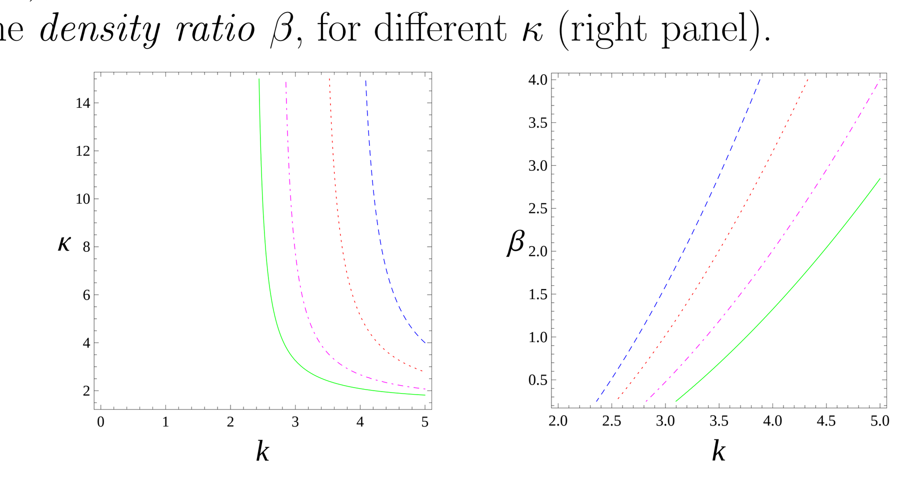


Figure 2. Left panel: $\beta$=0.25 (green); 1 (magenta); 2.5 (red); and 4 (blue).
Right panel: $\kappa$=3 (green); 4 (magenta); 8 (red); and 100 (blue).

### 7.2 Wavelength dependence of MI

The conditions for instability can be inferred from the diagrams:

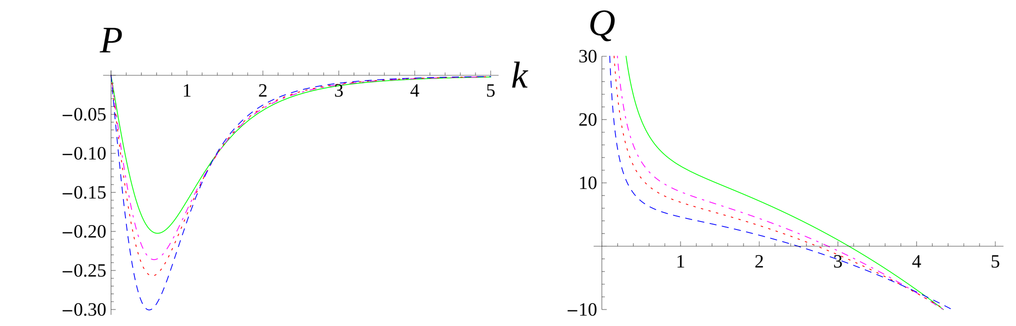


Figure 3. Left & right panels: green for $\kappa = 3.5$; magenta for $\kappa = 5$; red for $\kappa = 7$; and blue for $\kappa = 100$. Here, we have assumed $\beta = 0.5$.

### 7.3 Bright versus dark envelope solitons

(*Envelope solitons*): Two types of (analytical) localized solutions:
*Bright-type solitons* (for $P/Q > 0$) and
*dark solitons* (for $P/Q < 0$):

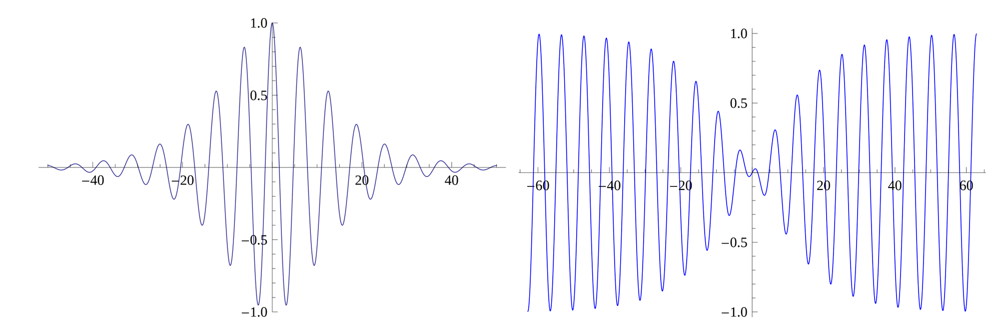

Figure 4. Left panel: Bright-type modulated wavepackets (for $P/Q > 0$). Right panel: Dark-type modulated wavepackets (for $P/Q < 0$).

We depict the ratio $P/Q$ versus k for $\beta = 0.5$ (left panel), and versus $\beta$ for $k = 3$ (right panel).

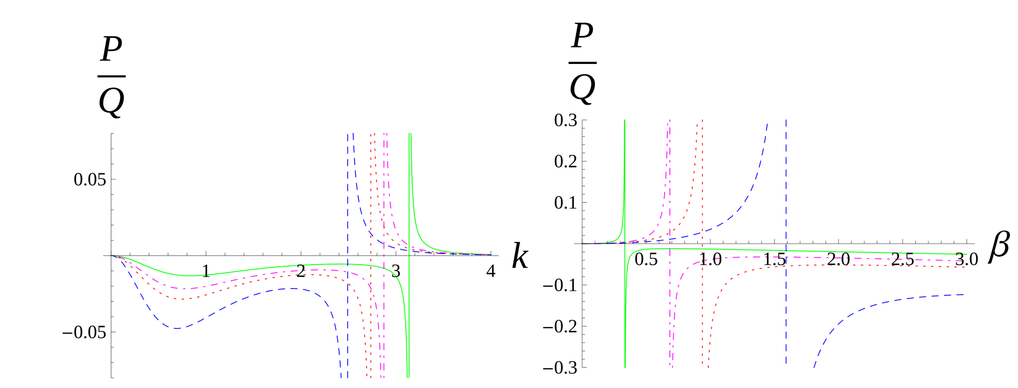


Figure 5. $P/Q > 0$ ($P/Q < 0$) provide the region where bright (dark, respectively) type solitons may occur. Also, modulational instability occurs for $P/Q > 0$ only (see above). Curves: green for $\kappa = 3.5$, magenta for $\kappa = 5$, red for $\kappa = 7$, and blue for $\kappa = 100$ (Maxwellian!).

### 7.4 Superthermality (via $\kappa$) and electron composition (via $\beta$) effect

The effect of superthermality on the MI growth rate has been analysed by considering $k = 3.2$ and $\beta = 0.5$ (left panel). The effect of $\beta$ on the MI growth rate for representative parameter values $k = 4.5$, $\kappa = 8$ is depicted in the region $0.25 \le \beta \le 4$ (right panel), where EAW may survive Landau damping [9]

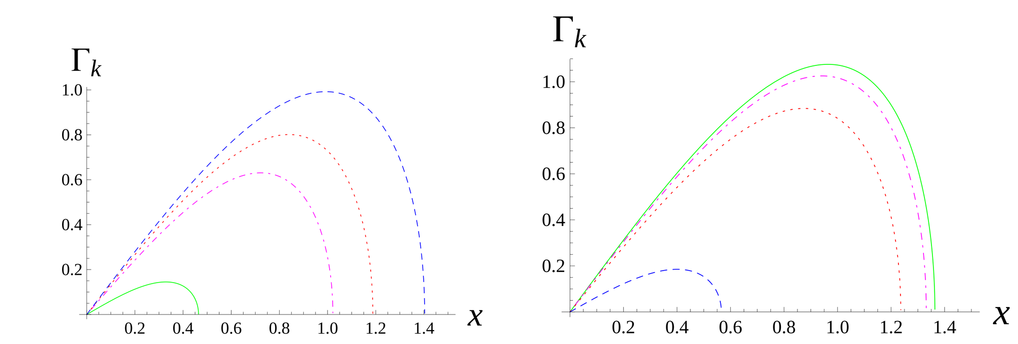


Figure 6. Left panel: $\kappa = 3.5$ (green); 5 (magenta); 7 (red); and 100 (blue) for $\beta = 0.5$. Right panel: green for $\beta = 0.5$; magenta for 1; red for 2; blue for 4.

## 8. Summary

- The MI wavenumber *threshold $k_{cr}$ decreases* (leading to a wider stability window and bright solitons) in the presence of more superthermal electrons (via $\beta$) for fixed $\kappa$.
- Short carrier wavelengths (wavenumber above the MI threshold $k_{cr}$) lead to a bright type envelope solitary structures, while on the other hand large wavelengths lead to a dark type soliton;
- A higher concentration of superthermal electrons (above a $\beta$ threshold, see Fig. 5b) gives rise to bright solitons (and enables modulational instability).
- Increasing superthermality provides a wider stable region (for fixed $\beta$; see Fig. 5a);
- The instability growth rate depends on $\beta$ and $\kappa$ dramatically.

**References**

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