Dissipative instability of MHD tangential discontinuity in magnetized plasmas with anisotropic viscosity and thermal conductivity.

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Abstract

The stability of the MHD tangential discontinuity is studied in compressible plasmas in the presence of anisotropic viscosity and thermal conductivity. The general dispersion equation is derived and solutions to this dispersion equation and stability criteria are obtained for the limiting cases of incompressible and cold plasmas. In these two limiting cases the effect of thermal conductivity vanishes and the solutions are only influenced by viscosity. The stability criteria for viscous plasmas are compared with those for ideal plasmas where stability is determined by the Kelvin-Helmholtz velocity V_{KH} as a threshold for the difference in the equilibrium velocities. Viscosity turns out to have a destabilizing influence when the viscosity coefficient takes different values at the two sides of the discontinuity. Viscosity lowers the threshold velocity V_c below the ideal Kelvin-Helmholtz velocity V_{KH}, so that there is a range of velocities between V_c and V_{KH} where the overstability is of a dissipative nature.

1. Introduction

The problem of the stability of the MHD tangential discontinuity has attracted the attention of scientists for a few decades. This problem arises in the study of the interaction of the solar wind with the interstellar plasma, the interaction of the solar wind with magnetospheres of the Earth and other planets, and different magnetic configurations in the solar atmosphere where equilibrium flows are present.

Syrovatski (1957) and Chandrasekhar (1961) studied the stability of the MHD tangential discontinuity for ideal incompressible plasmas. They showed that there is a critical value for the difference in the equilibrium velocity across the discontinuity. This critical value is called the Kelvin-Helmholtz (KH) threshold. The tangential discontinuity is stable for a difference in the equilibrium velocity below the KH threshold and unstable otherwise. The instability that arises is an overstable oscillation and is called the KH instability.

Fejer (1964) generalized the study of the KH instability to compressible plasmas. He derived the dispersion equation and studied the particular case of a plasma that is only slightly compressible. Subsequently many other particular cases were investigated (see, e.g., Gerwin, 1968, McKenzie, 1970, Duhau & Gratton, 1973, Myatnizkii, 1984, González & Gratton, 1994a,b, and Ruderman & Fahr, 1993, 1995). Duhau et al. (1970, 1971), Roy Choudhury & Patel (1985), and Roy Choudhury (1986) studied the KH instability of a tangential MHD discontinuity in plasmas with anisotropic pressure on the basis of the Chew, Goldberger and Low equations for collisionless strongly magnetized plasmas.

So far all studies of the KH instability were carried out for ideal plasmas. However an ideal plasma is an idealization. In real plasmas viscosity, finite resistivity and thermal conductivity are present. In addition an exact discontinuity in the equilibrium velocity cannot exist in a viscous plasma under general conditions. Therefore in general it does not make sense to consider the stability of MHD tangential discontinuity in viscous plasmas.

However there are two exceptions to this general observation. The first exception occurs when the coefficient of isotropic viscosity is much larger at one side of the surface of discontinuity than at the other side. This makes it possible to take the viscosity coefficient to be equal to zero at one side of the surface of discontinuity. A difference in the equilibrium velocity can exist across such a surface. Such an idealized situation was considered by Ruderman & Goossens (1995) when they studied the viscous instability of a tangential discontinuity in an incompressible plasma. The second exception is related to a more realistic situation. In space plasmas the magnetic field is very often sufficiently strong to make the ion gyrofrequency ω_{ci} much larger than the inverse mean collisonal time of ions τ_i^{-1}. If in addition the plasma can be considered as collisional (that is if all characteristic scales are much larger than the mean free path of the ions), then viscosity is described by Braginskii's tensorial expression (see Braginskii, 1965). This tensorial expression for viscosity containes five terms. The ratio of the sum of the four last terms to the first term is of the order of $\tau_i \omega_{ci}$. When $\tau_i \omega_{ci} \gg 1$ the first term of the Braginskii's tensorial expression very often gives a good approximation for viscosity. As a result the tensor of viscosity is highly anisotropic. A simple derivation of the highly anisotropic tensor of viscosity based on a qualitative physical analysis was given by Hollweg (1985). He also gives estimations of $\tau_i \omega_{ci}$ for typical conditions in the soalr corona. In accordance with these estimations $\tau_i \omega_{ci} \approx 3 \times 10^5$ in an active coronal region and $\tau_i \omega_{ci} \approx 7 \times 10^5$ near the base of a coronal hole. A property of the highly anisotropic tensor of viscosity is that it allows a jump in the velocity across a magnetic surface since a strong magnetic field causes charged particles to rotate around the magnetic field lines thus preventing particle diffusion across the magnetic field lines. This implies that there is not any momentum flux across the magnetic surfaces and different layers of plasma can slide with respect to each other along a magnetic surface without friction.

The thermal conductivity of plasmas is due mainly to the electrons. The expression for heat flux involves three coefficients, denoted $\kappa_{||}$, $\kappa_{\perp}$, and $\kappa_{\wedge}$ by Braginskii (1965).

The following estimates are valid: $\kappa_\perp/\kappa_{||} \sim (\tau_e\omega_{ce})^{-2}$, $\kappa_\wedge/\kappa_{||} \sim (\tau_e\omega_{ce})^{-1}$, where τ_e is the mean collisional time of electrons, and ω_{ce} is the electron gyrofrequency. If the Coulomb logarithm is taken to be 20, we have

$$\tau_e \approx 10^4 T_e^{3/2} n_e^{-1} \text{ s} , \tag{1}$$

where T_e is electron temperature and n_e is electron concentration (CI units will be used throughout). Following to Hollweg (1985) we take $T_e = 2 \times 10^6$ K, $n_e = 3 \times 10^{15}$ m^{-3}, and $B = 50$ G in an active region of the solar corona. Then $\tau_e \approx 10^{-2}$ s, and $\tau_e\omega_{ce} \approx 10^7$. Near the base of a coronal hole one might have $T_e = 10^6$ K, $n_e = 10^{14}$ m^{-3}, and $B = 10$ G. Then $\tau_e \approx 10^{-1}$ s and $\tau_e\omega_{ce} \approx 2 \times 10^7$.

Thus the parts of the heat flux vector that involves $\kappa_\perp$ and $\kappa_\wedge$ can often been neglected, which means that only the heat flux in the direction of magnetic field is taken into account. As a result we obtain strongly anisotropic thermal conductivity.

The relative importance of viscosity and thermal conductivity is characterized by the Prandtl number $Pr = \eta_0 k_B/m_p\kappa_{||}$, where η_0 is the largest coefficient of viscosity in the Braginskii's expression for the viscosity tensor, m_p is the proton mass, and k_B is the Boltzmann constant. Braginskii (1965) gives the following approximate expressions for η_0 and $\kappa_{||}$:

$$\eta_0 \approx k_B n_i T_i \tau_i , \quad \kappa_{||} \approx 3k_B^2 m_e^{-1} n_e T_e \tau_e , \tag{2}$$

where T_i is the temperature of ions and m_e is the mass of electron. Taking $n_e \approx n_i$ and $T_e \approx T_i$ and using (**??**) we obtain

$$Pr \approx \frac{m_e\tau_i}{3m_p\tau_e} \approx \frac{1}{3}\sqrt{\frac{m_e}{m_p}} \approx 10^{-2} . \tag{3}$$

(**??**) shows that in plasma which consists of electrons and protons with approximately equal temperatures $Pr \ll 1$. However this estimate leads to the conclusion that electron thermal conductivity is more important dissipative process than ion viscosity only if $\beta \gtrsim 1$, where β is the ratio of plasma pressure to magnetic pressure. In what follows we shall see that for problem of stability of MHD tangential discontinuity the relative importance of viscosity and thermal conductivity is characterized by the product $\beta^{-1}Pr$ rather then Pr if $\beta \lesssim 1$. With the use of the same values of parameters of plasma and magnetic field that we have taken in the previous discussion, we obtain that $\beta \approx 0.016$ in an active coronal region and $\beta \approx 0.007$ near the base of a coronal hole. Hence typically in the solar corona $\beta^{-1}Pr \sim 1$, so that viscosity and thermal conductivity are of the same inportance.

The relative importance of viscosity and resistivity is characterized by the magnetic Prandtl number $P_m = \eta_0/m_p n_i \lambda$, where λ is the coefficient of magnetic diffusion. We take the standard expression for λ in fully ionized plasmas to obtain

$$P_m \approx 3 \times 10^{-10} \tau_i \tau_e n_e T_i \, . \qquad (4)$$

Substituting into (**??**) the values of parameters for an active region of the solar corona and for the basis of a coronal hole, we get $P_m \approx 10^{10}$ in the both cases. This estimate shows that dissipation due to resistivity can be neglected in comparison with dissipation due to viscosity.

Finally, the relative importance of the Hall effect to viscosity is characterized by the dimensionless parameter $\tau_e \omega_{ce} P_m^{-1}$. Using values of $\tau_e \omega_{ce}$ and P_m calculated previously, we obtain $\tau_e \omega_{ce} P_m^{-1} \approx 10^{-3}$ both in active region of the solar corona and near the basis of a coronal hole. This estimate implies that we can neglect the Hall effect in comparison with the effect of viscosity.

These observations based on dimensional analysis lead us to study the stability of an MHD tangential discontinuity in a viscous thermal conductive plasma. We only retain the first term in the Braginskii's expression for the tensor of viscosity and only take parallel termal conductivity into account.

The concept of negative energy waves turned out to be very fruitful for the study of dissipative instabilities. For instance, Ruderman & Goossens (1995) have used this concept to give an interpretation of the viscous instability of an MHD tangential discontinuity in terms of negative energy waves. The concept of negative energy waves is based on the energy equation

$$\frac{d\mathcal{E}}{dt} = -\mathcal{D}\,, \tag{5}$$

where $\mathcal{E}$ is the so-called linear part of wave energy and $\mathcal{D}$ is the dissipative function (a detailed discussion of the concept of negative energy waves in hydrodynamics can be found e.g. in Ostrovskii, Rybak & Tsimring, 1986). The functions $\mathcal{E}$ and $\mathcal{D}$ are not Galilei-invariant in the sense that they depend on the choice of coordinate system moving parallel to the discontinuity. If we choose a moving coordinate system such that $\mathcal{D} > 0$, then (??) shows that the linear part of the wave energy $\mathcal{E}$ decreases owing to dissipation. In case of a monochromatic perturbation $\mathcal{E}$ takes the form $\mathcal{E} = Ea^2$, where a is the wave amplitude. When $E > 0$ the wave is called a positive energy wave. In accordance with (??) its amplitude a decreases so that dissipation results in wave damping. When $E < 0$ the wave is called a negative energy wave. In accordance with (??) its amplitude a increases so that dissipation leads to instability.

When dissipation is only present at one side of the tangential discontinuity, the choice of the moving coordinate system in which $\mathcal{D} > 0$ is very simple. The unperturbed plasma must be at rest at the side of discontinuity where dissipation is present in this coordinate system. When dissipation is present at both sides of the discontinuity the choice of the moving coordinate system where $\mathcal{D} > 0$ is much more complicated as it depends on the ratio of the dissipative coefficients and on other plasma parameters. In fact the choice of the coordinate system turns out to be more complicated than the study of dissipative instability itself. In the present paper we study the stability of the MHD tangential discontinuity with dissipation present at both sides of the discontinuity. Therefore we do not use the concept of negative energy waves.

The paper is organized as follows. In §2 we present the set of dissipative MHD equations and boundary conditions that are used to study the stability of the MHD tangential discontinuity. In §3 we derive the dispersion equation governing the stability of MHD tangential discontinuity under the assumption that perturbations are only slightly damped during one wave period. In §4 and §5 we present solutions to the dispersion equation for incompressible plasmas and cold plasmas. In §6 we present physical consideration of ideal and dissipative instabilities in an incompressible plasma. In §7 we summarize our results.

2. Dissipative MHD equations and boundary conditions.

We consider a collisional one-fluid model of viscous thermal conductive plasmas. As explained in the Introduction we only retain the first term in Braginskii's expression for the tensor of viscosity and only take the parallel heat flux into account. The expressions for the tensor of viscosity $\hat{\pi}$ and the heat flux $\mathbf{q}$ take the simple form:

$$\hat{\pi} = \rho_0 \nu \left(\mathbf{b} \otimes \mathbf{b} - \frac{1}{3}\hat{I} \right) \{3\mathbf{b}.\nabla(\mathbf{b}.\mathbf{v}') - \nabla.\mathbf{v}'\} \; , \tag{6}$$

$$\mathbf{q} = -\kappa_{||}\mathbf{b}(\mathbf{b}.\nabla T') \; . \tag{7}$$

Here $\nu = \eta_0/\rho_0$ is the kinematic coefficient of viscosity, $\mathbf{B}$ is the magnetic induction, ρ is the density, $\mathbf{v}$ is the velocity, and T is the temperature. $\mathbf{b} = \mathbf{B_0}/B_0$ is the unit vector along the equilibrium magnetic field, $\hat{I}$ is the unit tensor, $\otimes$ denotes the tensor product, the subscript '0' refers the equilibrium quantity, and an accent denotes an Eulerian perturbation of any quantity. The coefficients ν and $\kappa_{||}$ depend on the equlibrium density and temperature. As the equlibrium density and temperature can be different at the two sides of the discontinuity, so can ν and $\kappa_{||}$. For the present investigation the fact that ν can differ on the two sides of the discontinuity will be important. Equations (**??**) and (**??**) are the linearized expressions for $\hat{\pi}$ and $\mathbf{q}$ as we only consider the linear stability of tangential discontinuities.

¿From a physical point of view the viscosity tensor (**??**) is characterized by the property that at any magnetic surface the viscous stresses are normal to the surface. The expression (**??**) means that the heat flux is directed along the magnetic field.

The unperturbed state is characterized by an MHD tangential discontinuity at $z = 0$, and all equilibrium quantities are constant at both sides of the discontinuity. The equilibrium magnetic field $\mathbf{B}_0$ and velocity $\mathbf{v}_0$ are parallel to the plane of the discontinuity.

With the aid of (??) and (??) the linear equations of viscous thermal conductive MHD can be written as

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla.\mathbf{v}' + \mathbf{v}_0.\nabla\rho' = 0 \,, \tag{8}$$

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}_0.\nabla)\mathbf{v}' = -\frac{1}{\rho_0}\nabla p' + \frac{1}{\mu\rho_0}(\nabla\times\mathbf{B}')\times\mathbf{B}_0$$
$$+\nu\{\mathbf{b}(\mathbf{b}.\nabla) - \frac{1}{3}\nabla\}\{3\mathbf{b}.\nabla(\mathbf{b}.\mathbf{v}') - \nabla.\mathbf{v}'\} \,, \tag{9}$$

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla\times(\mathbf{v}_0\times\mathbf{B}' + \mathbf{v}'\times\mathbf{B}_0) \,, \tag{10}$$

$$\frac{\partial p'}{\partial t} + \mathbf{v}_0.\nabla p' + \gamma p_0 \nabla.\mathbf{v}' = (\gamma - 1)\kappa_{\|}(\mathbf{b}.\nabla)^2 T' \,, \tag{11}$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} \,. \tag{12}$$

Here p is the pressure, μ the magnetic permiability, and γ the adiabatic index.

The perturbed surface of the discontinuity is defined by the equation $z = \eta(t, x, y)$. The kinematic boundary conditions and the condition of continuity of stresses have to be satisfied at this surface. In the linear approximation these boundary conditions can be imposed at the unperturbed surface $z = 0$.

The linearized kinematic boundary conditions are:

$$w_1 = \frac{\partial \eta}{\partial t} + \mathbf{v}_{01}.\nabla\eta \,, \quad w_2 = \frac{\partial \eta}{\partial t} + \mathbf{v}_{02}.\nabla\eta \,, \tag{13}$$

where w is the z component of the velocity, and the subscript '1' and '2' refer to quantities at $z < 0$ and $z > 0$ respectively.

Stresses at the discontinuity can be found with the use of (??) and take the form

$$\mathbf{F} = -\mathbf{n}\{P' + \rho_0\nu(\mathbf{b}.\nabla(\mathbf{b}.\mathbf{v}') - \frac{1}{3}\nabla.\mathbf{v}')\} \,, \tag{14}$$

where $\mathbf{n} = (0, 0, 1)$ is the unit vector normal to the unperturbed surface of the discontinuity, and $P' = p' + B_0(\mathbf{b}.\mathbf{B}')/\mu$ is the Eulerian perturbation of the total pressure. The condition of continuity of stresses can then be written as

$$\left[P' + \rho_0\nu\{\mathbf{b}.\nabla(\mathbf{b}.\mathbf{v}') - \frac{1}{3}\nabla.\mathbf{v}'\}\right] = 0 \,, \tag{15}$$

where the square brackets denote the jump in a quantity across the discontinuity.

Condition (**??**) does not contain derivatives of the x and y components of the perturbed velocity with respect to z. This implies that the x and y components of the perturbed velocity can have jumps across the discontinuity in agreement with the discussion in the Introduction.

The equations (**??**)–(**??**) and boundary conditions (**??**) and (**??**) are the basic equations for the study of the dissipative instability of the tangential discontinuity in the next Sections.

3. Derivation of dispersion equation.

In order to derive the dispersion equation that governs the stability of the MHD tangential discontinuity we Fourier-analyze the perturbed quantities and take them to be proportional to $\exp\{i(\mathbf{k}.\mathbf{r} - \omega t)\}$, where $\mathbf{k} = (k_x,\, k_y,\, 0)$, $\mathbf{r} = (x,\, y,\, z)$, $\mathbf{k}$ is real, and ω is complex. This enables us to rewrite the equations (**??**)–(**??**) as

$$\rho_0 \frac{dw}{dz} + i\rho_0 \mathbf{k}.\mathbf{v}'_{\|} - i\Omega\rho' = 0\,, \tag{16}$$

$$\Omega\mathbf{v}'_{\|} = \frac{\mathbf{k}}{\rho_0}P' - \frac{B_0}{\mu\rho_0}(\mathbf{k}.\mathbf{b})\mathbf{B}'_{\|} - \nu\{\mathbf{b}(\mathbf{b}.\mathbf{k}) - \frac{1}{3}\mathbf{k}\}Q\,, \tag{17}$$

$$\Omega w = -\frac{i}{\rho_0}\frac{dP'}{dz} - \frac{B_0}{\mu\rho_0}(\mathbf{k}.\mathbf{b})B'_z - \frac{i\nu}{3}\frac{dQ}{dz}\,, \tag{18}$$

$$\Omega\mathbf{B}'_{\|} = B_0\mathbf{b}(\mathbf{k}.\mathbf{v}'_{\|}) - B_0\mathbf{v}'_{\|}(\mathbf{k}.\mathbf{b}) - iB_0\mathbf{b}\frac{dw}{dz}\,, \tag{19}$$

$$\Omega B'_z = -B_0 w(\mathbf{k}.\mathbf{b})\,, \tag{20}$$

$$\Omega p' + c_s^2\rho_0\left(i\frac{dw}{dz} - (\mathbf{k}.\mathbf{v}'_{\|})\right) = -i\chi\frac{(\mathbf{k}.\mathbf{b})^2}{\gamma - 1}(\gamma p' - c_s^2\rho')\,. \tag{21}$$

In these equations we have introduced the components of the perturbed velocity and the perturbed magnetic field that are parallel to the unperturbed plane of discontinuity

$$\mathbf{v}'_{\|} = (u', v', 0)\,, \quad \mathbf{B}'_{\|} = (B'_x, B'_y, 0)\,, \tag{22}$$

and the Doppler-shifted frequency

$$\Omega = \omega - \mathbf{k}.\mathbf{v}_0 \, . \tag{23}$$

The square of the sound speed c_s is determined as $c_s^2 = \gamma p_0/\rho_0$. The quantities Q and χ are given by

$$Q = 3i(\mathbf{k}.\mathbf{b})(\mathbf{b}.\mathbf{v}'_{||}) - i(\mathbf{k}.\mathbf{v}'_{||}) - \frac{dw}{dz} \tag{24}$$

$$\chi = \frac{m_p(\gamma-1)^2\kappa_{||}}{2\gamma k_B \rho_0} \, , \tag{25}$$

In these equations m_p is the proton mass and k_B is the Boltzmann constant. (**??**) was derived from (**??**) by use of (**??**) and the fact that the equilibrium pressure, density and temperature p_0, ρ_0, and T_0 are related by the ideal gas law for fully ionized plasmas

$$p_0 = \frac{2k_B}{m_p}\rho_0 T_0 \, . \tag{26}$$

The Prandtl number Pr measures the relative importance of viscosity and thermal conductivity, and the Reynolds number $Re = V_h/\nu k$ measures the importance of viscosity compared to inertia. Here V_h is a characteristic velocity. In what follows we assume that $Re \gg 1$ and $PrRe \gg 1$. These inequalities imply that we restrict the analysis to linear motions that are only slightly damped during a wave period. This enables us to use a perturbation method and to treat the terms proportional to ν and χ in equations (**??**), (**??**), and (**??**) as small in comparison to the terms that are already present in the ideal theory. The terms proportional to ν and χ are evaluated with the use of relations obtained in ideal MHD. All but two variables are eliminated from equations (**??**)–(**??**) and a set of two first-order linear ordinary differential equations are obtained for w and P'

$$w = -\frac{\Omega}{\rho_0 A}\left(i + \frac{\nu\Omega\{\Omega^2 - 3c_s^2(\mathbf{k}.\mathbf{b})^2\}}{3(c_s^2+v_A^2)C}\right)\frac{dP'}{dz} \, , \tag{27}$$

$$\frac{dw}{dz} = \frac{\Omega}{\rho_0(c_s^2+v_A^2)C}\left\{\frac{iD}{A} + \frac{\nu\Omega}{3}\{\Omega^2 - 3c_s^2(\mathbf{k}.\mathbf{b})^2\}\left(\frac{2c_s^2(\mathbf{k}.\mathbf{b})^2}{(c_s^2+v_A^2)C} - \frac{k^2-(\mathbf{k}.\mathbf{b})^2}{A}\right)\right.$$
$$\left. -\chi\frac{\Omega^3 c_s^2(\mathbf{k}.\mathbf{b})^2}{(c_s^2+v_A^2)C}\right\} P' \, . \tag{28}$$

The quantities A, C, and D take the form

$$\left.\begin{aligned} A &= \Omega^2 - v_A^2(\mathbf{k}.\mathbf{b})^2\,, \quad C = \Omega^2 - c_T^2(\mathbf{k}.\mathbf{b})^2\,, \\ D &= \Omega^4 - (c_s^2 + v_A^2)k^2\Omega^2 + c_s^2 v_A^2 k^2(\mathbf{k}.\mathbf{b})^2\,, \end{aligned}\right\} \tag{29}$$

and the squares of the Alfvén and cusp speeds are given by

$$v_A^2 = \frac{B_0^2}{\mu\rho_0}\,, \quad c_T^2 = \frac{c_s^2 v_A^2}{c_s^2 + v_A^2}\,. \tag{30}$$

In addition we obtain the approximate expression for Q:

$$Q = -\frac{i\Omega\{\Omega^2 - 3c_s^2(\mathbf{k}.\mathbf{b})^2\}}{\rho_0(c_s^2 + v_A^2)C}P'\,. \tag{31}$$

Fourier-analysis reduces the boundary conditions (**??**) and (**??**) to

$$w_1 = -i\Omega_1\eta\,, \quad w_2 = -i\Omega_2\eta\,, \tag{32}$$

$$\left[P' + \frac{\nu}{3}Q\right] = 0\,. \tag{33}$$

Elimination of P' from (**??**) and (**??**) leads to a single second-order ordinary differential equation for w

$$\frac{d^2w}{dz^2} - \Gamma^2 w = 0\,, \tag{34}$$

where

$$\left.\begin{aligned} &\Gamma^2 = \Gamma_0^2(1 + i\nu K_\nu + i\chi K_\chi)\,, \quad \Gamma_0^2 = -\frac{D}{(c_s^2 + v_A^2)C}\,, \\ &K_\nu = \frac{\Omega A\{\Omega^2 - 3c_s^2(\mathbf{k}.\mathbf{b})^2\}^2}{3(c_s^2 + v_A^2)CD}\,, \quad K_\chi = \frac{\Omega^3 A c_s^2(\mathbf{k}.\mathbf{b})^2}{(c_s^2 + v_A^2)CD}\,. \end{aligned}\right\} \tag{35}$$

To ensure that the perturbations vanish far away from the discontinuity we have to impose that $\Gamma_0^2 > 0$. The assumptions $Re \gg 1$, $PeRe \gg 1$ lead to $|\nu K_\nu| \ll 1$, $|\chi K_\chi| \ll 1$, and subsequently to $\Re(\Gamma^2) > 0$, where $\Re$ denotes the real part of a quantity. The solutions to (**??**) that vanish at $z \to -\infty$ and $z \to +\infty$ respectively and satisfy the boundary conditions (**??**) are

$$w_1 = -i\Omega_1\eta e^{\Gamma_1 z}\,, \quad w_2 = -i\Omega_2\eta e^{-\Gamma_2 z}\,, \tag{36}$$

where we impose that $\Re(\Gamma_{1,2}) > 0$. The expressions for the Eulerian perturbation of total pressure to the left ($z \to -0$) and the right ($z \to +0$) of the discontinuity immediately follow from (**??**). They are

$$\left.\begin{aligned} P_1 &= \eta\,\frac{\rho_{01}A_1}{\Gamma_{01}}\left(1 - \frac{i\nu_1}{2}K_{\nu 1} - \frac{i\chi_1}{2}K_{\chi 1}\right) - \frac{\nu_1}{3}Q_1\,, \\ P_2 &= -\eta\,\frac{\rho_{02}A_2}{\Gamma_{02}}\left(1 - \frac{i\nu_2}{2}K_{\nu 2} - \frac{i\chi_2}{2}K_{\chi 2}\right) - \frac{\nu_2}{3}Q_2\,. \end{aligned}\right\} \tag{37}$$

Application of the boundary condition for the stresses (**??**) then gives the desired dispersion equation

$$F(\omega, k) \equiv F_I(\omega, k) + iF_D(\omega, k) = 0\,, \tag{38}$$

where

$$F_I(\omega, k) = \frac{\rho_{01}A_1}{\Gamma_{01}} + \frac{\rho_{02}A_2}{\Gamma_{02}}\,, \quad F_D(\omega, k) = S_\nu + S_\chi\,, \tag{39}$$

and

$$\left.\begin{aligned} S_\nu &= \frac{\rho_{01}\nu_1 A_1}{2\Gamma_{01}}K_{\nu 1} + \frac{\rho_{02}\nu_2 A_2}{2\Gamma_{02}}K_{\nu 2}\,, \\ S_\chi &= \frac{\rho_{01}\chi_1 A_1}{2\Gamma_{01}}K_{\chi 1} + \frac{\rho_{02}\chi_2 A_2}{2\Gamma_{02}}K_{\chi 2}\,. \end{aligned}\right\} \tag{40}$$

Here $\Gamma_{01} > 0$, $\Gamma_{02} > 0$. As already stated the dispersion equation (**??**) has been derived by using a perturbation method so that terms proportional to ν^2, $\nu\chi$, and χ^2 are systematically neglected. The notations in (**??**) are obvious. The indices I and D to F_I and F_D denote the ideal and dissipative parts of the left-hand side of the dispersion equation. When dissipation is absent so that $F_D = 0$, the dispersion equation (**??**) coincides with the dispersion equation obtained by Fejer (1964).

The objective of the paper is to show that dissipation can cause overstability of the MHD tangential discontinuity which is stable in the absence of dissipation. We are therefore interested in a situation where the discontinuity is stable in ideal MHD. The solution to the ideal dispersion equation is a real frequency, $\bar{\omega}$ ($F_I(\bar{\omega}) = 0$), which corresponds to an oscillation. Dissipation produces a small imaginary correction, $i\gamma$, to the real frequency $\bar{\omega}$ that leads to damping or growth of the oscillation. The approximate solution to (**??**)

can be written as

$$\omega = \bar{\omega} + i\gamma\,, \quad \gamma = \frac{F_D(\bar{\omega})}{\dfrac{\partial F_I}{\partial \omega}}\,. \tag{41}$$

This approximate solution and in particular the expression for the increment or decrement γ are obtained under the condition that $|\gamma| \ll |\bar{\omega}|$. The denominator $\dfrac{\partial F_I}{\partial \omega}$ in (**??**) is calculated at $\omega = \bar{\omega}$. $\gamma > 0$ corresponds to an overstable oscillation, i.e. oscillation of which the amplitude grows exponentially in time on a time scale γ^{-1}.

Now we can varify the *ad hoc* estimate given in Introduction, that the relative importance of viscosity and thermal conductivity is characterized by the dimensionless parameter $\beta^{-1}Pr$ when $\beta \lesssim 1$. It follows from (**??**) and (**??**) that contributions of viscosity and thermal conductivity to γ are proportional to S_ν and S_χ respectively. The relative importance of viscosity and thermal conductivity is characterized by the ratio S_ν/S_χ. With the use of (**??**) and (**??**) we obtain the estimate

$$\frac{S_\nu}{S_\chi} \sim \frac{\nu\Omega^2}{\chi c_s^2 k^2}\,. \tag{42}$$

When $\beta \lesssim 1$, so that $c_s^2 \lesssim v_A^2$, we have $\Omega^2 \sim v_A^2 k^2$. Substituting this estimate into (**??**) we finally obtain $S_\nu/S_\chi \sim \beta^{-1}Pr$, which means that the estimate taken *ad hoc* in Introduction is valid.

4. Incompressible plasma

In this section the general results of the previous section are applied to the study of the stability of the MHD tangential discontinuity in an incompressible plasma. The approximation of incompressibility is valid in plasmas where the sound velocity is much larger than the Alfvén velocity ($c_s^2 \gg v_A^2$). This approximation corresponds to taking the limit $c_s^2/v_A^2 \to \infty$, so that acoustic signals travel with infinite speed as it were. In particular we obtain for F_I, S_ν, and S_χ by means of this procedure

$$\left.\begin{aligned} kF_I &= \rho_{01}\Omega_1 + \rho_{02}\Omega_2 - \rho_{01}v_{A1}^2(\mathbf{k}.\mathbf{b}_1)^2 - \rho_{02}v_{A2}^2(\mathbf{k}.\mathbf{b}_2)^2 \; , \\ S_\nu &= -\frac{3}{4k^2}\{\rho_{01}\nu_1\Omega_1(\mathbf{k}.\mathbf{b}_1)^4 + \rho_{02}\nu_2\Omega_2(\mathbf{k}.\mathbf{b}_2)^4\} \; , \\ S_\chi &= 0 \; . \end{aligned}\right\} \tag{43}$$

It is not surprising that S_χ vanishes for an incompressible plasma since it describes dissipation related to thermal conductivity. This type of dissipation comes from the energy equation which is decoupled from the other MHD equations in an incompressible plasma.

In what follows we use the difference in velocity across the discontinuity surface $\mathbf{V} = \mathbf{v}_{02} - \mathbf{v}_{01}$ and the angles φ_1, φ_2, and ψ between $\mathbf{V}$ and $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{k}$ respectively. These angles are determined by the conditions $V^2\cos^2\varphi_{1,2} = (\mathbf{V}.\mathbf{b}_{1,2})^2$, $k^2V^2\cos^2\psi = (\mathbf{k}.\mathbf{V})^2$, $|\varphi_{1,2}| \le \frac{\pi}{2}$, and $|\psi| \le \frac{\pi}{2}$. An angle is considered positive when measured counterclockwise and negative otherwise (see figure 1). In figure 1 $\varphi_1 > 0$, $\varphi_2 < 0$, and $\psi > 0$.

We first consider the tangential discontinuity in ideal MHD since we want to start from a situation that is stable in ideal MHD and to see how dissipation can make it unstable. The ideal dispersion equation $F_I(\omega) = 0$ has two roots that can be written as

$$\omega_\pm = \frac{\mathbf{k}.(\rho_{01}\mathbf{v}_{01} + \rho_{02}\mathbf{v}_{02}) \pm k\cos\psi\sqrt{\rho_{01}\rho_{02}(V_{KH}^2 - V^2)}}{\rho_{01} + \rho_{02}} \; , \tag{44}$$

where the Kelvin-Helmholtz (KH) threshold velocity V_{KH} is determined as

$$V_{KH}^2 = \frac{(\rho_{01} + \rho_{02})\{\rho_{01}v_{A1}^2\cos^2(\varphi_1 - \psi) + \rho_{02}v_{A2}^2\cos^2(\varphi_2 - \psi)\}}{\rho_{01}\rho_{02}\cos^2\psi} \; . \tag{45}$$

If

$$V^2 > V_{KH}^2 \; , \tag{46}$$

$\Im(\omega_+) > 0$ ($\Im$ denotes the imaginary part) and the discontinuity suffers the ideal KH instability. When inequality (**??**) is satisfied the oscillations that propagate at the angles $\pm\psi$ with respect to $\mathbf{V}$ have an amplitude that grows in time and are thus KH unstable, so that (**??**) is a local criterion of KH instability.

In a real situation there are perturbations propagating in all directions. The tangential discontinuity becomes unstable as soon as the local instability criterion (**??**) is satisfied for at least one value of the angle ψ. This implies that the discontinuity is unstable whenever V^2 is larger than the minimal value of the right-hand side of (??) with respect to ψ. The condition for instability can then be written as

$$V^2 > \overline{V}_{KH}^2 \equiv \frac{(\rho_{01}+\rho_{02}) v_{A1}^2 v_{A2}^2 \sin^2(\varphi_1 - \varphi_2)}{\rho_{01} v_{A1}^2 \sin^2\varphi_1 + \rho_{02} v_{A2}^2 \sin^2\varphi_2} \,. \tag{47}$$

(??) is a global critetion of KH instability and $\overline{V}_{KH}$ is the global KH threshold velocity. When $\varphi_1 = \varphi_2$ or $\varphi_1 = \varphi_2 \pm \dfrac{\pi}{2}$ we get $\bar{V}_{KH} = 0$ and the tangential discontinuity is unstable for any value of V. The case $\varphi_1 = \varphi_2 = 0$ (vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{V}$ are collinear) is special. The right-hand side of (??) is now independent of ψ and equal to $\overline{V}_{KH}^2$ which is determined by

$$\overline{V}_{KH}^2 = \frac{(\rho_{01}+\rho_{02})(\rho_{01} v_{A1}^2 + \rho_{02} v_{A2}^2)}{\rho_{01}\rho_{02}} \,. \tag{48}$$

The criteria for local and global KH instabilities are the same and coincide with those given by Syrovatskii (1957) and Chandrasekhar (1961).

The objective of the paper is to show that dissipation contrary to intuition can cause instabilities for differences in the equilibrium velocity that are smaller than the KH threshold velocity V_{KH}. For that reason we restrict the analysis to $V^2 < V_{KH}^2$ in an attempt to study the local instability (for fixed direction of $\mathbf{k}$) that is due to the presence of dissipation. With the use of (??) and (??) we get

$$\gamma_\pm = -\frac{3k^2}{8(\rho_{01}+\rho_{02})}\Bigg(\rho_{01}\nu_1\cos^4(\varphi_1-\psi) + \rho_{02}\nu_2\cos^4(\varphi_2-\psi) \\ \pm \frac{\rho_{01}\rho_{02} V\{\nu_1\cos^4(\varphi_1-\psi) - \nu_2\cos^4(\varphi_2-\psi)\}}{\sqrt{\rho_{01}\rho_{02}(V_{KH}^2 - V^2)}}\Bigg) \,, \tag{49}$$

where the subscripts $\pm$ correspond to $\omega_\pm$. It is easy to see that γ_+ and γ_- cannot be positive simultaneously. The condition that either $\gamma_+ > 0$ or $\gamma_- > 0$, so that one of the two waves that propagate in the directions $\pm\mathbf{k}$ is unstable, can be written as

$$V^2 > V_c^2 \equiv \Phi(\psi) V_{KH}^2 \,, \tag{50}$$

where

$$\Phi(\psi) = \frac{\{\rho_{01}\nu_1\cos^4(\varphi_1-\psi) + \rho_{02}\nu_2\cos^4(\varphi_2-\psi)\}^2}{(\rho_{01}+\rho_{02})\{\rho_{01}\nu_1^2\cos^8(\varphi_1-\psi) + \rho_{02}\nu_2^2\cos^8(\varphi_2-\psi)\}} \,. \tag{51}$$

The function Φ is important since it determines whether the threshold velocity for dissipative instability is smaller or larger than the ideal KH threshold velocity. It is easy to show that $\Phi \leq 1$, i.e. $V_c^2 \leq V_{KH}^2$ which implies that there is a range of velocities $]V_c, V_{KH}[$ in which the instability is caused by the action of viscosity. The function $\Phi(\psi)$ takes its minimal value $\{\min(\rho_{01}, \rho_{02})/(\rho_{01} + \rho_{02})\}$ for $\cos(\varphi_1 - \psi) = 0$ ($\mathbf{k}.\mathbf{b}_1 = 0$) if $\rho_{01} > \rho_{02}$, and for $\cos(\varphi_2 - \psi) = 0$ ($\mathbf{k}.\mathbf{b}_2 = 0$) if $\rho_{01} < \rho_{02}$. The function $\Phi(\psi)$ takes its maximal value 1 for $\cos^2(\varphi_1 - \psi)/\cos^2(\varphi_2 - \psi) = \sqrt{\nu_2/\nu_1}$.

In general we cannot find an analytical expression for the minimal value of the right-hand side of (**??**), and thus cannot obtain the global criterion for dissipative instability in a closed form. We consider two special cases. Let us first assume that dissipation is only present at one side of the discontinuity. Without loss of generality we can assume that viscosity is only present below the plane of discontinuity ($z < 0$), so that $\nu_2 = 0$. Φ is independent of ψ and takes the form

$$\Phi = \frac{\rho_{01}}{\rho_{01} + \rho_{02}} \,. \tag{52}$$

The global criterion of the dissipative instability is

$$V^2 > \overline{V}_c^2 \equiv \frac{\rho_{01}}{\rho_{01} + \rho_{02}} \overline{V}_{KH}^2 \,. \tag{53}$$

When the vectors $\mathbf{b}_{1,2}$ and $\mathbf{V}$ are collinear ($\varphi_1 = \varphi_2 = 0$) this result coincides with that obtained by Ruderman & Goossens (1995) who considered the dissipative instability of the MHD tangential discontinuity in an incompressible plasma with isotropic viscosity at one side of the discontinuity. This result supports the statement by Ruderman & Goossens (1995) that the threshold for the dissipative instability is independent of the type of viscosity that is present at one side of the discontinuity.

Let us now look at the case where the vectors $\mathbf{b}_{1,2}$ and $\mathbf{V}$ are collinear ($\varphi_1 = \varphi_2 = 0$). Again Φ is independent of ψ and the global criterion for the dissipative instability is

$$V^2 > \overline{V}_c^2 \equiv \frac{(\rho_{01}\nu_1 + \rho_{02}\nu_2)^2}{(\rho_{01} + \rho_{02})(\rho_{01}\nu_1^2 + \rho_{02}\nu_2^2)} \overline{V}_{KH}^2 \,, \tag{54}$$

where $\overline{V}_{KH}^2$ is determined by (**??**). The threshold velocity for global dissipative instability $\overline{V}_c$ satisfies the inequality $\overline{V}_c \leq \overline{V}_{KH}$ and depends on the ratio ν_1/ν_2. When $\nu_1 = \nu_2$ we obtain $\overline{V}_c = \overline{V}_{KH}$. For all other combinations of ν_1 and ν_2 $\overline{V}_c < \overline{V}_{KH}$, so that there is a range of differences in the equilibrium velocity V where the MHD tangential discontinuity is unstable when viscosity is present and stable in ideal MHD.

5. Cold plasma

In this section we consider the stability of the MHD tangential discontinuity in a cold plasma. For the sake of simplicity we assume that $\varphi_1 = \varphi_2 = 0$ (so that the vectors $\mathbf{b}_{1,2}$ and $\mathbf{V}$ are collinear), $\rho_{01} = \rho_{02} = \rho_0$, and $v_{A1} = v_{A2} = v_A$ (so that there is only a jump in $\mathbf{v}_0$). The expressions for F_I, S_ν, and S_χ are obtained by taking the limit $c_s^2/v_A^2 \to 0$ and reduce to

$$\left.\begin{aligned}
F_I &= \rho_0 v_A \left(\frac{\Omega_1^2 - k^2 v_A^2 \cos^2\psi}{\sqrt{k^2 v_A^2 - \Omega_1^2}} + \frac{\Omega_2^2 - k^2 v_A^2 \cos^2\psi}{\sqrt{k^2 v_A^2 - \Omega_2^2}} \right) , \\
S_\nu &= -\frac{\rho_0}{6 v_A} \left(\frac{\nu_1 \Omega_1 (\Omega_1^2 - k^2 v_A^2 \cos^2\psi)^2}{(k^2 v_A^2 - \Omega_1^2)^{\frac{3}{2}}} - \frac{\nu_2 \Omega_2 (\Omega_2^2 - k^2 v_A^2 \cos^2\psi)^2}{(k^2 v_A^2 - \Omega_2^2)^{\frac{3}{2}}} \right) , \\
S_\chi &= 0.
\end{aligned}\right\} \tag{55}$$

Here also the result $S_\chi = 0$ is what we expect since the temperature is equal to zero in a cold plasma and the finite thermal conductivity does not operate.

As in Section 4 the tangential discontinuity is first considered in ideal MHD. The equation $F_I(\omega) = 0$ can be transformed to the algebraic equation

$$(\Omega_1^2 - \Omega_2^2)\{\Omega_1^2\Omega_2^2 - k^2 v_A^2 \Omega_1^2 - k^2 v_A^2 \Omega_2^2 + k^4 v_A^4 \cos^2\psi (1 + \sin^2\psi)\} = 0. \tag{56}$$

(**??**) has been obtained from $F_I(\omega) = 0$ with the use of squaring. Therefore (**??**) can have spurious roots that do not satisfy $F_I(\omega) = 0$. The root of the first factor in (**??**) is spurious. The roots of the second factor are

$$X_\pm^2 = k^2 \left(v_A^2 + \frac{1}{4} V^2 \cos^2\psi \pm v_A \sqrt{v_A^2 \sin^4\psi + V^2 \cos^2\psi} \right) , \tag{57}$$

where

$$X = \omega - \frac{1}{2}\mathbf{k}.(\mathbf{v}_{01} + \mathbf{v}_{02})\,. \tag{58}$$

When

$$4v_A^2 < V^2 < 4v_A^2(1 + 2\tan^2\psi) \tag{59}$$

we have $X_-^2 < 0$. The root of (??) is then

$$\omega = \frac{1}{2}\mathbf{k}.(\mathbf{v}_{01} + \mathbf{v}_{02}) + i\sqrt{-X_-^2} \tag{60}$$

and has a positive imaginary part. This root (??) is not spurious and satisfies the equation $F_I(\omega) = 0$. The existence of the root (??) implies that (??) is the local criterion (for fixed $\mathbf{k}$ or ψ) for KH instability. The global criterion for KH instability is given by the left inequality (??). The right inequality (??) shows that for $V > \overline{V}_{KH} \equiv 2v_A$ the waves that propagate at angles ψ restricted by the inequality

$$|\psi| > \arctan\sqrt{\frac{V^2 - 4v_A^2}{8v_A^2}} \tag{61}$$

are unstable.

This result is in apparent contrediction with results obtained by Duhau & Gratton (1973). These authors studied the KH instability of the same magnetic plasma configuration but they took the finite plasma pressure into account. They in particular obtained that in case where $c_s^2 < v_A^2$ the MHD tangential discontinuity is unstable in two velocity ranges: $V > 2v_A$, and $J_1 v_A < V < J_2 v_A$, where $J_2 < 2$. $J_1 \to 1$ and $J_2 \to \sqrt{2}$ when $c_s \to 0$, which means that the MHD tangential discontinuity in a cold plasma is unstable not only for $V > 2v_A$, but also for $v_A < V < v_A\sqrt{2}$. This apparent contradiction is easily removed. The point is that the increment of the instability that is present for $J_1 v_A < V < J_2 v_A$ is of the order of $c_s k$, so that this increment vanishes when $c_s \to 0$. As a result the tangential discontinuity is stable for $v_A < V < v_A\sqrt{2}$ in complete agreement with the result of the present section.

However in reality a situation with $c_s = 0$ does not exist, so that from a physical point of view the approximation of a cold plasma implies that $c_s^2 \ll v_A^2$ rather than $c_s = 0$.

In what follows we shall study the dissipative instability of the tangential discontinuity that takes place for $V < 2V_A$. There is no ideal instability of the tangential discontinuity in the velocity range $v_A\sqrt{2} < V < 2v_A$ and the dissipative instability is important no matter how small its increment is. In the velocity range $v_A < V < v_A\sqrt{2}$ there is an ideal instability with a small increment of the order of $c_s k$, so that the dissipative instability is important only if its increment is much larger than $c_s k$.

In order to study the instability caused by dissipation we limit the analysis to the situation when the discontinuity is stable in ideal MHD. Hence we take $V < 2v_A$ so that the equation $F(\omega) = 0$ has four real roots that are determined by (**??**) and (**??**). However only the two roots

$$\omega_\pm^{(-)} = \frac{1}{2}\mathbf{k}.(\mathbf{v}_{01} + \mathbf{v}_{02}) \pm X_- \tag{62}$$

satisfy the equation $F(\omega) = 0$. The two remaining roots

$$\omega_\pm^{(+)} = \frac{1}{2}\mathbf{k}.(\mathbf{v}_{01} + \mathbf{v}_{02}) \pm X_+$$

are spurious. In what follows we only consider the roots $\omega_\pm^{(-)}$, so that we omit the superscript $(-)$. The expressions for the instability increment take the form

$$\gamma_\pm = \frac{k^2\cos^2\psi(\mathbf{k}.\mathbf{V})(4v_A^2 - V^2)\Psi_\pm}{24X_-^2(2v_A^2\sin^2\psi + V^2\cos^2\psi + 2v_A\sqrt{v_A^2\sin^4\psi + V^2\cos^2\psi})}\,, \tag{63}$$

where

$$\Psi_\pm = \nu_1\Omega_1(\Omega_2^2 - v_A^2k^2\cos^2\psi) - \nu_2\Omega_2(\Omega_1^2 - v_A^2k^2\cos^2\psi)\,. \tag{64}$$

The right-hand side of (**??**) is calculated at $\omega = \omega_\pm$, where $\omega_\pm$ are determined by (**??**). As $V^2 < 4v_A^2$, the condition $\gamma_\pm > 0$ is equivalent to

$$(\mathbf{k}.\mathbf{V})\Psi_\pm > 0\,. \tag{65}$$

This condition can be rewritten as

$$\begin{aligned} &\pm 2X_-(\mathbf{k}.\mathbf{V})(\nu_1 - \nu_2)\left\{X_-^2 - k^2\cos^2\psi\left(v_A^2 + \frac{1}{4}V^2\right)\right\} \\ &> k^2V^2\cos^2\psi(\nu_1 + \nu_2)\left\{X_-^2 + k^2\cos^2\psi\left(v_A^2 - \frac{1}{4}V^2\right)\right\}\,. \end{aligned} \tag{66}$$

The right-hand side of (??) is positive, so that the condition (??) cannot be satisfied for both signs in the left-hand side. With the aid of (??) the condition that (??) is satisfied for one choice of the sign in the left-hand side can be written as

$$M^2(1 - M^2 + \xi^2)^2 < \delta^2\xi^2(M^2 + 1 - \xi^2)\,, \tag{67}$$

where

$$\xi^2\cos^2\psi = 1 + M^2\cos^2\psi - \sqrt{\sin^4\psi + 4M^2\cos^2\psi}\,. \tag{68}$$

Here M is the Alfvén-Mach number and δ is a dimensionless quantity that measures the relative difference in viscosity at the two sides of the discontinuity. They are given by

$$M = \frac{V}{2v_A}\,, \quad \delta = \left|\frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}\right|\,. \tag{69}$$

The solution to (??) is

$$M^2 > M_c^2(\delta,\psi)$$
$$\equiv \frac{1}{2} + \frac{2\delta^2\tan^2\psi(\delta^2 - \cos 2\psi) + [1 - \delta^2(1 + 2\tan^2\psi)]\sqrt{(1-\delta^2)^2 + 4\delta^2\sin^4\psi}}{2(1-\delta^2)^2}\,. \tag{70}$$

In the particular case that $\delta \to 1$ ($\nu_1 \to 0$ or $\nu_2 \to 0$), (??) reduce to

$$M_c^2(1,\psi) = \frac{1 + \sin^2\psi}{4}\,. \tag{71}$$

When $\delta = 0$ ($\nu_1 = \nu_2$) $M_c(0,\psi) = 1$. This means that there is no instability due to the action of viscosity for a difference in equilibrium velocity below the KH threshold when the viscosity coefficients are equal at the two sides of the discontinuity. It can be shown that for $0 < \delta < 1$ the following inequalities hold

$$\frac{1 + \sin^2\psi}{4} < M_c^2(\delta,\psi) < 1\,. \tag{72}$$

Note the two other useful identities:

$$M_c^2(\delta, 90^o) = \frac{1}{1+\delta^2}\,, \quad M_c^2(\delta, 0) = 1\,. \tag{73}$$

It follows from (**??**) that the second identity (**??**) is only valid for $\delta < 1$. As a consequence the function $M_c(\delta, \psi)$ is discontinuous at $\delta = 1$, $\psi = 0$. The dependence of $M_c(\delta, \psi)$ on ψ for different values of δ is shown in figure 2. The right inequality of (**??**) shows that there is an interval of differences in the equilibrium velocity for which a wave propagating at the angle ψ with respect to the equilibrium magnetic fild is unstable in a viscous plasma and stable in an ideal plasma.

(**??**) is a local criterion for dissipative instability. Figure 2 shows that $M_c(\delta, \psi)$ is a non-monotonic function of ψ for fixed δ and that it attains its minimal value at $\psi = \psi_m$. The global criterion for the dissipative instability is

$$M > \overline{M}_c(\delta) \equiv M_c(\delta, \psi_m) \, . \tag{74}$$

where $\overline{M}_c$ is the global critical Alfvén-Mach number. The angle ψ_m is determined by the equation

$$\cos^2 \psi_m (1 + \sin^2 \psi_m \cos^2 \psi_m) = \delta^2 \, , \tag{75}$$

and $\overline{M}_c(\delta)$ can be expressed in terms of ψ_m as

$$\overline{M}_c = \frac{\delta}{\cos \psi_m (1 + \cos^2 \psi_m)} \, . \tag{76}$$

Figures 3 and 4 show how ψ_m and $\overline{M}_c$ depend on δ. Both quantities are monotonically decreasing functions of δ.

The present analysis shows that $\overline{M}_c < 1$ for $\delta < 1$ ($\nu_1 \neq \nu_2$). This means that there is an interval of differencies in the equilibrium velocities below the KH threshold velocity, for which the discontinuity is unstable owing to the presence of viscosity.

6. Physical discussion for an incompressible plasma

In this section we present a physical interpretation of the ideal and dissipative instabilities in an incompressible plasma. For the sake of mathematical simplicity we consider an unperturbed state with the equilibrium magneti field $\mathbf{B}_0$ and flow velocity $\mathbf{v}_0$ that are in the x direction. In addition we restrict the analysis to two-dimensional perturbations. Hence the perturbed magnetic field and velocity are in the xz plane, and all the perturbed quantities are independent of y.

In what follows we use a laboratory coordinate system in which the flow velocitites at the both sides of a tangential MHD discontinuity, $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$, are fixed. When the difference between the flow velocities at the two sides of the discontinuity, V, is below the Kelvin-Helmholtz threshold velocity, $V < V_{KH}$, two surface waves can propagate along the surface of the discontinuity. We introduce a new coordinate system that moves with the phase velocity of a surface wave in the x direction with respect to the laboratory coordinate system. We refer to this coordinate system as a concomitant coordinate system. In the concomitant coordinate system plasma motion perturbed by the presence of a surface wave is steady.

Before we embark on our physical discussion of dissipative instability, it is expedient to give a physical interpretation of the ideal KH instability and take $\nu_1 = \nu_2 = 0$. Let us turn to figure 5 where the plasma motion perturbed by the presence of a surface wave is shown. The solid line is the perturbed surface of the discontinuity. We focus on the balance of forces acting on a fluid volume that embraces the surface of the discontinuity (see figure 5). The centrum of curvature of the solid line that correspond to the centrum of the fluid volume is in a point C. The radius of curvature is R. The length of the fluid volume is $2R\,\delta\theta$, and the height is $2\,\delta z$. The surface of the dicontinuity devides the fluid volume in an upper and an lower part. The two parts of the fluid volume move approximately along the arc of the circle with raduis R and centrum at C. The upper part of the volume moves with the velocity $v_{02} - V_{\rm ph}$, while the lower part moves with the velocity $v_{01} - V_{\rm ph}$, where $V_{\rm ph}$ is the phase velocity of the surface wave. The centripetal acceleration of the upper part is $(v_{02} - V_{\rm ph})^2/R$, that of the lower part is $(v_{01} - V_{\rm ph})^2/R$.

It is well-known that a dynamical problem can be reduced to a static problem by the use of the inertial forces, which are equal to the product of the corresponding masses and accelerations and have the directions opposite to the directions of the accelerations. The dynamical equations are then reduced to the equations that reflet the balance of the inertial and the active forces. The masses of the upper and lower parts of the fluid volume equal $2\rho_{02}R\,\delta\theta\,\delta z$ and $2\rho_{01}R\,\delta\theta\,\delta z$ respectively. Therefore the projection of the inertial force, which in our case is the centrifugal force, on the normal to the surface of the discontinuity is given by

$$F_{\rm cen} = 2\,\delta\theta\,\delta z\{\rho_{01}(v_{01} - V_{\rm ph})^2 + \rho_{02}(v_{02} - V_{\rm ph})^2\}\ , \tag{77}$$

and is directed upwards.

Let us now calculate the projection of the active forces on the normal direction. When doing so we only calculate forces that are of the first order with respect to the three small quantities: δz, $R\,\delta\theta$, and R^{-1}. The quantity R^{-1} is small because we consider only small perturbations of the surface of discontinuity. The active force consists of two parts. The first part is due to the total pressure and the second part is due to magnetic tension. The contribution related to the total pressure can be splitted in two parts. The first part is due to the total equilibrium pressure P_0, while the second part is due to the Eulerian perturbation of the total pressure P'. It is straightforward to show with the use of Gauss theorem that the resulting force of the total equilibrium pressure P_0 on any closed surface equals zero. The projection of the force due to P' on the normal direction is due to the action of P' on the upper and lower boundaries of the fluid volume. However the quantity P' is continuous at the surface of the discontinuity and depends on z as $e^{-k|z|}$. Consequantly the values of P' at the upper and lower boundaries are the same and P' does not contribute in the projection of the active force on the normal direction. In summarizing the projection of the resulting force on the normal direction due to the total pressure is zero.

The active force that is due to the tension of the magnetic field acts on the end-walls of the fluid volume (see figure 5). The force of magnetic tension that acts on each end-wall is given by

$$F_{\rm t} = \delta z \left(\frac{B_{01}^2}{\mu} + \frac{B_{02}^2}{\mu} \right) , \tag{78}$$

and this force is perpendicular to this end-wall. We have used the equilibrium magnetic field instead of the perturbed magnetic field when calculating $F_{\rm t}$ since we want to be consistent with linear theory. There is a small inclination of the end-walls with respect to the normal direction to the surface of the discontinuity at the centrum of the fluid volume. As a result there is a non-zero projection of the magnetic tension force on the normal direction equal $F_{\rm t}\,\delta\theta$.

Now taking into account that the projection of the force of the magnetic tension on the normal is directed downwards, we write the balance of forces acting on the fluid volume in the normal direction as

$$F_{\rm cen} = 2F_{\rm t}\,\delta\theta\ . \tag{79}$$

With the use of (**??**) and (**??**) we obtain from (**??**) the equation that determines the phase velocity of a surface wave

$$\rho_{01}(v_{01} - V_{\rm ph})^2 + \rho_{02}(v_{02} - V_{\rm ph})^2 = \rho_{01} v_{A1}^2 + \rho_{02} v_{A2}^2\ . \tag{80}$$

It follows from (**??**) that the phase velocities of the two surface waves are given by

$$V_{\rm ph} = \frac{\rho_{01}v_{01} + \rho_{02}v_{02} \pm \sqrt{\rho_{01}\rho_{02}(V_{KH}^2 - V^2)}}{\rho_{01} + \rho_{02}}\ . \tag{81}$$

The same result can be obtained from (**??**) if to take $\mathbf{k}$ parallel to both $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$ and $\psi = 0$. We see that the phase velocities of the surface waves that can propagate on the surface of the discontinuity are determined by the force balance in the concomitant coordinate system. The centrifugal force related to the centripetal acceleration of a fluid volume moving along a curved surface of the discontinuity has to be balanced by the tension force due to the magnetic field.

The difference between the phase velocities of the two surface waves decreases when V is increased. This difference becomes equal zero for $V = V_{KH}$. If V is further increased, solutions in the form of a surface wave do not anylonger exist and the KH instability of the tangential discontinuity appears. From a physical point of view the development of KH instability can be explained as follows. Let us consider for $V_{\rm ph}$ as simply the speed of a moving coordinate system. The value of the centrifugal force $F_{\rm cen}$ is a squared function of $V_{\rm ph}$. It takes its minimal value $F_{\rm cen}^{\rm min}$ at $V_{\rm ph} = (\rho_{01}v_{01} + \rho_{02}v_{02})/(\rho_{01} + \rho_{02})$. When $V < V_{KH}$ we have $F_{\rm cen}^{\rm min} < 2F_{\rm t}\,\delta\theta$ and (**??**) possesses exactly two solutions. When $V > V_{KH}$ we have $F_{\rm cen}^{\rm min} > 2F_{\rm t}\,\delta\theta$, so that in all moving coordinate systems the centrifugal force cannot be balanced by the force of magnetic tension in any and there is not any coordinate sytem where the steady state can be reached.

Let us now consider the dissipative instability and take $\nu_1 + \nu_2 \neq 0$. As we are only interested in dissipative instability, we restrict our analysis to $V < V_{KH}$, so that there is no KH instability in ideal MHD. We use the concomitant coordinate system in which the plasma motion that is perturbed by a surface wave is steady when viscosity is absent. The presence of viscosity destroys an exact balance of the centrifugal force and the magnetic tension force, and thus causes the surface wave to damp or to grow. The growth of the surface wave means that dissipative instability is present. Our objective here is to obtain the criterion of the dissipative instability from a discussion of the forces that act on the fluid volume embracing the perturbed surface of the discontinuity.

The study of the ideal KH instability was carried out in general terms. In particular, we did not specify the shape of the perturbed surface of the discontinuity. In order to study the dissipative instability we have to be more specific and define the shape of the perturbed surface of the discontinuity in the concomitant coordinate system as $\eta = \eta_a \cos(kx)$. The Eulerian perturbation of total pressure modified by viscosity is

$$\tilde{P}' = P' + \rho_0\nu\{\mathbf{b}.\nabla(\mathbf{b}.\mathbf{v}') - \frac{1}{3}\nabla.\mathbf{v}'\}\,. \qquad (82)$$

It is the modification of the perturbed total pressure due to viscosity that causes the change in the shape of the perturbed surface of the discontinuity. As a result the amplitude of oscillation of the perturbed surface of the discontinuity depends on time: $\eta_a = \eta_a(t)$.

It is instructive to divide the total acceleration of the fluid volume a in the normal direction in two parts: $a = a_{\rm cen} + a_{\rm vis}$. The centripetal acceleration $a_{\rm cen}$ is caused by the magnetic tension force and is not related to the dependence of η_a on t. The viscous acceleration $a_{\rm vis}$ is related to the dependence of η_a on t. This acceleration is caused by the action of $\tilde{P}'$ on the lower and upper boundaries of the fluid volume. As we are interested in the dependence of η_a on t, we only calculate $a_{\rm vis}$.

In accordence with (**??**) $\tilde{P}'_1 = \tilde{P}'_2 = \tilde{P}'_0$ at $z = 0$. In the concomitant coordinate system

$$\Omega = k(V_{\rm ph} - v_0)\,. \tag{83}$$

When $c_s \to \infty$ (an approximation of an incompressible plasma) we have the following limiting expressions for Γ_0, K_ν, and K_χ

$$\Gamma_0 = k\,, \quad K_\nu = \frac{3k(v_0 - V_{\rm ph})}{(v_0 - V_{\rm ph})^2 - v_A^2}\,, \quad K_\chi = 0\,. \tag{84}$$

We use (**??**) and (**??**) to get from (**??**)

$$\tilde{P}'_0 = \rho_{01} k \eta_a \{[(v_{01} - V_{\rm ph})^2 - v_{A1}^2]\cos(kx) + \frac{3}{2}\nu_1 k (v_{01} - V_{\rm ph})\sin(kx)\}\,. \tag{85}$$

The dependence of $\tilde{P}'_1$ and $\tilde{P}'_2$ on z is given by the functions $e^{\Gamma_1 z}$ and $e^{-\Gamma_2 z}$ respectively. Taking into account that δz, ν_1, and ν_2 are small, it is straighforward to obtain the following approximate expressions that determine $\tilde{P}'_1$ at the lower boundary and $\tilde{P}'_2$ at the upper boundary of the fluid volume

$$\left.\begin{aligned}
\tilde{P}'_1 &= \rho_{01} k \eta_a \{(1 - k\,\delta z)[(v_{01} - V_{\rm ph})^2 - v_{A1}^2]\cos(kx) + \tfrac{3}{2}\nu_1 k (v_{01} - V_{\rm ph})\sin(kx)\}\,, \\
\tilde{P}'_2 &= \rho_{01} k \eta_a \Big\{(1 - k\,\delta z)[(v_{01} - V_{\rm ph})^2 - v_{A1}^2]\cos(kx) + \tfrac{3}{2}\nu_1 k (v_{01} - V_{\rm ph})\sin(kx) \\
&\quad + \tfrac{3}{2}k^2\,\delta z\left[\nu_2 (v_{02} - V_{\rm ph})\frac{(v_{01} - V_{\rm ph})^2 - v_{A1}^2}{(v_{02} - V_{\rm ph})^2 - v_{A2}^2} - \nu_1 (v_{01} - V_{\rm ph})\right]\sin(kx)\Big\}\,.
\end{aligned}\right\} \tag{86}$$

The force that causes the acceleration $a_{\rm vis}$ consists of three contributions. The first contribution is related to the difference in $\tilde{P}'_1$ and $\tilde{P}'_2$ at the lower and the upper boundary of the fluid volume. The second contribution is due to a small difference in the lengths of the upper and the lower boundaries. The third contribution is due to a small inclination of the two end-walls of the fluid volume with respect to the normal direction. It is easy to see that the two latter contributions are of the order of η_a^2. In linear theory these two contributions can be neglected. With the use of (**??**) it is straightforward to obtain from (**??**) the following expression for $a_{\rm vis}$

$$a_{\rm vis} = \frac{3k^3}{4\rho_{01}}\{\rho_{01}\nu_1(v_{01} - V_{\rm ph}) + \rho_{02}\nu_2(v_{02} - V_{\rm ph})\}\sin(kx)\,. \tag{87}$$

This acceleration is directed upwards.

On the other hand the upward accelerations of the upper and lower parts of the fluid volume are given by

$$a_2 = \frac{\partial w_2}{\partial t} + v_{02}\frac{\partial w_2}{\partial x}\,, \quad a_1 = \frac{\partial w_1}{\partial t} + v_{01}\frac{\partial w_1}{\partial x}\,. \tag{88}$$

With the use of (**??**) we then obtain for the upward acceleration of the centre of mass of the fluid volume

$$a = \frac{\partial^2\eta}{\partial t^2} + 2\frac{\rho_{01}(v_{01} - V_{\rm ph}) + \rho_{02}(v_{02} - V_{\rm ph})}{\rho_{01} + \rho_{02}}\frac{\partial^2\eta}{\partial t\partial x} + \frac{\rho_{01}(v_{01} - V_{\rm ph})^2 + \rho_{02}(v_{02} - V_{\rm ph})^2}{\rho_{01} + \rho_{02}}\frac{\partial^2\eta}{\partial x^2}\,. \tag{89}$$

The last term in (**??**) represents $a_{\rm cen}$. As we only consider small viscosity, the characteristic time of changing η_a is much larger than the wave period in the laboratory coordinate system, so that $|d\eta_a/dt| \ll kV_{\rm ph}\eta_a$. This enables us to neglect the first term in (**??**) in comparison with the second term. As a result we arrive at the following approximate expression for $a_{\rm vis}$

$$a_{\rm vis} = -2k\frac{\rho_{01}(v_{01} - V_{\rm ph}) + \rho_{02}(v_{02} - V_{\rm ph})}{\rho_{01} + \rho_{02}}\frac{d\eta_a}{dt}\sin(kx)\,. \tag{90}$$

We compare expressions (**??**) and (**??**) for $a_{\rm vis}$ to obtain an equation for $\eta_a(t)$

$$\frac{d\eta_a}{dt} = \frac{3k^2(\rho_{01} + \rho_{02})}{8\rho_{01}}\mathcal{F}\eta_a\,, \tag{91}$$

where

$$\mathcal{F} = -\frac{\rho_{01}\nu_1(v_{01} - V_{\rm ph}) + \rho_{02}\nu_2(v_{02} - V_{\rm ph})}{\rho_{01}(v_{01} - V_{\rm ph}) + \rho_{02}(v_{02} - V_{\rm ph})}\,. \tag{92}$$

The amplitude of the surface wave decreases if $\mathcal{F} < 0$ and increases if $\mathcal{F} > 0$. Hence the criterion for the dissipative instability is given by the inequality $\mathcal{F} > 0$.

It is straightforward to show that $\mathcal{F} > 0$ for one choise of sign in (??) when the criterion of dissipative instability (??) found in Section 4 is satisfied. If (??) is not satisfied, $\mathcal{F} < 0$ for the both choises of sign in (??). Hence we obtain the criterion of dissipative instability from a physical discussion of the plasma motion. The present discussion shows that the cause of the dissipative instability is the breakdown of the balance of forces acting on the fluid volume embracing the surface of the discontinuity caused by anisotropic viscosity.

7. Conclusions

In the present paper the dissipative instability of the MHD tangential discontinuity has been studied. The dissipative mechanisms considered here are viscosity and thermal conductivity. Both viscosity and thermal conductivity have been assumed to be strongly anisotropic, so that the viscous stresses are normal to the magnetic surfaces, and the heat flux is 1-dimensional along the magnetic field lines. The general dispersion equation determining the stability of the MHD tangential discontinuity has been derived. This equation has been studied for the two limiting cases of an incompressible plasma and of a cold plasma. In these two limiting cases viscosity is the only dissipative process that affects the stability. In an incompressible plasma thermal conductivity does not affect stability because the energy equation is decoupled from the other MHD equations. In a cold plasma thermal conductivity does not affect stability because the temperature and, consequently, the internal energy of the plasma is equal to zero.

In the case of an ideal plasma the stability of the MHD tangential discontinuity is determined by the Kelvin-Helmholtz (KH) threshold velocity V_{KH}. When the difference in the equilibrium velocity V is smaller than V_{KH} the discontinuity is stable, while it is unstable when V is larger then V_{KH}. The instability which is present in that case is called the Kelvin-Helmholtz (KH) instability.

The present paper was concentrated on a situation when the MHD tangential discontinuity is stable in ideal MHD and intended to find out whether dissipation, contrary to the intuitive expectation, can lead to instability. The main result is that viscosity introduces a new threshold velocity V_c for overstability which is lower than the ideal KH threshold velocity V_{KH} at least when the viscosity coefficient ν takes different values at the two sides of the discontinuity. There is a range of differencies in the equilibrium velocity between V_c and V_{KH} when the overstability is due to dissipation, so that we would like to call it the dissipative instability. In general we can state that dissipation destabilizes the MHD tangential discontinuity.

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Figure captions

FIGURE 1. The scetch of introduction of the angles φ_1, φ_2, and ψ. The counterclockwise direction is considered as positive. On this figure $\varphi_1 > 0$, $\varphi_2 < 0$, and $\psi > 0$.

FIGURE 2. The dependence of the local critical Alfvén-Mach number M_c on the angle ψ between the direction of the equilibrium magnetic field and the direction of the propagation of the perturbation for different values of the relative difference in the viscosity coefficient δ in the case of a cold plasma. The values of δ are shown under corresponding curves.

FIGURE 3. The dependence of the angle ψ_m at which the local critical Alfvén-Mach number M_c attains the minimum value on the relative difference in the viscosity coefficient δ.

FIGURE 4. The dependence of the global critical Alfvén-Mach numbem $\bar{M}_c$ on the relative difference in the viscosity coefficient δ.

FIGURE 5. The scetch of the flow in the vicinity of the perturbed surface of the tangential discontinuity (shown by solid line). The fluid volume (shaded) embracing the surface of the discontinuity is shown together with quantities used in the physical consideration.