Illustrating how to Differentiate the Magnitude of a Function

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Introduction

Having been asked to explain how to differentiate the magnitude of a function, this document has been created to explain the general method of doing so and also attempts to explain why and how this method works.

Derivation

It is known that for $y(x) = f(x)$[1]

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

From this it can be seen that for $y(x) = |f(x)|$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{|f(x + \delta x)| - |f(x)|}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{f(x)}{|f(x)|} \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \frac{f(x)}{|f(x)|} \frac{df}{dx}. \end{aligned} \tag{1}$$

This result is true for all $f(x)$ and should be apparent if you closely consider what's happening. Don't worry though, that will be done for you further down. This result can also be obtained directly by using the chain rule and the result you were given in Webwork that $\frac{d|x|}{dx} = \frac{x}{|x|}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (|f(x)|) \\ &= \frac{d|f|}{df} \frac{df}{dx} \\ &= \frac{f}{|f|} \frac{df}{dx}. \end{aligned} \tag{2}$$

[1] An equation $y(x)$ may be given in terms of some expression containing functions and or terms of powers of x^n. This expression is usually referred to as $f(x)$.

Showing the Result Quoted in Webwork

Consider the graph of $y(x) = x$ in figure 1.

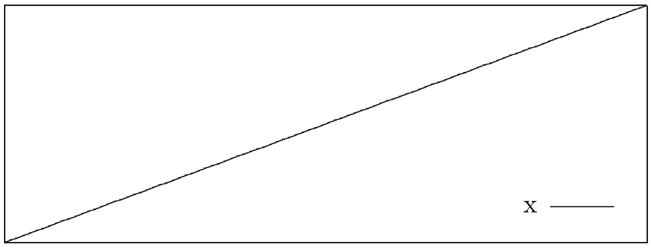

Figure 1: Origin (0:0) is at the center.

Consider the two regions $x \geq 0$ and $x < 0$. If we differentiate $y(x)$ in these regions we get

$$\left(\frac{dy}{dx}\right)_{x\geq 0} = \left(\frac{dy}{dx}\right)_{x<0} = 1.$$

This is expected since the gradient is unchanged for various values of x.

Now consider the function $y(x) = |x|$. This expression is not saying that we consider only positive values of x. It's saying that y will simply always be positive regardless of the value of x. So, this function is represented (for all x) by figure 2.

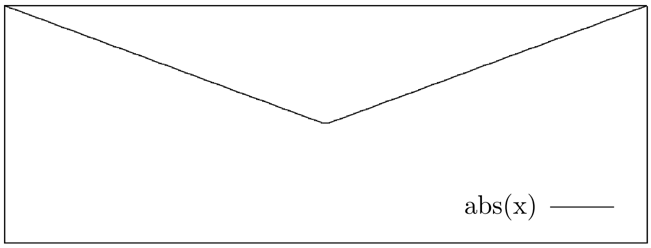

Figure 2: (0:0) at center.

Here it can be seen that differentiating $y(x) = |x|$ in the regions $x \geq 0$ and $x < 0$ gives

$$\left(\frac{dy}{dx}\right)_{x\geq 0} = 1 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{x<0} = -1.$$

By inspection, this derivative can be written more compactly as

$$\frac{dy}{dx} = \frac{x}{|x|}. \tag{3}$$

The graph for this derivative is in figure 3.

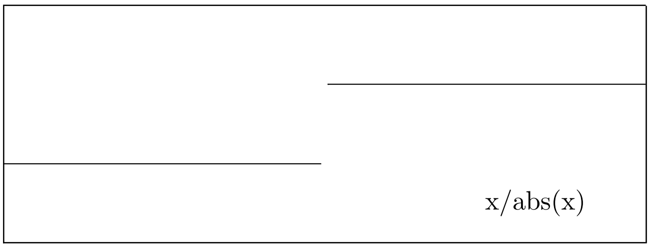

Figure 3: (0:0) at center and discontinuous at $x = 0$.

Eq(3) is the expression that was quoted to you in webwork and you can see it's true since dividing a scalar by its magnitude gives unity with the original sign ($\pm$) left intact. So, if you were given $x = 6$ and $x = -3$ the respective gradients would be

$$\begin{aligned} \frac{dy}{dx} &= \frac{6}{|6|} = \frac{6}{6} = 1 \\ \frac{dy}{dx} &= \frac{-3}{|-3|} = \frac{-3}{3} = -1 \end{aligned}$$

In both cases, the magnitude of $\frac{dy}{dx}$ was 1 as expected, only the sign changed as it clearly should have (according to figure 3).

Another illustration

Consider now the function $y(x) = sin(x)$ which, as we all know, is given as figure 4.

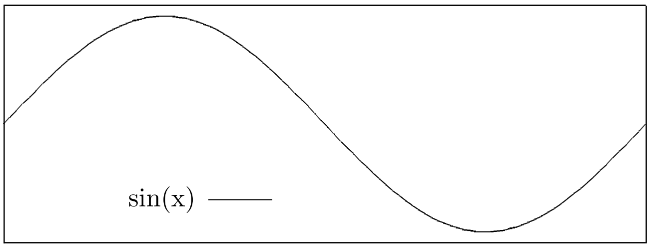

Figure 4: Origin is on the left i.e. $0 < x < 2\pi$.

Again, $y(x) = |sin(x)|$ does not preclude any values of x, it simply means y is always *positive* as shown in figure 5.

The function $sin(x)$ gives positive and negative values for various x:

$$\begin{aligned} y(x) &= sin(x) \quad \text{for} \quad 2n\pi < x < (2n+1)\pi \\ y(x) &= -sin(x) \quad \text{for} \quad (2n-1)\pi < x < 2n\pi \end{aligned}$$

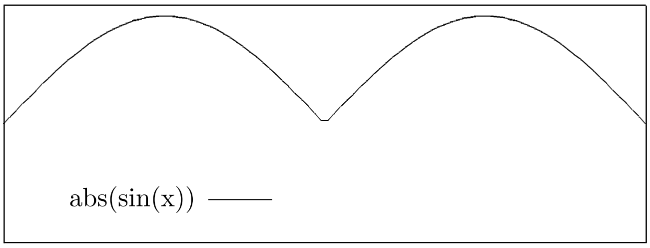

Figure 5: $0 < x < 2\pi$.

Differentiating these gives

$$\begin{aligned} \frac{dy}{dx} &= cos(x) \quad \text{for} \quad 2n\pi < x < (2n+1)\pi \\ \frac{dy}{dx} &= -cos(x) \quad \text{for} \quad (2n-1)\pi < x < 2n\pi \end{aligned}$$

The derivatives are both $cos(x)$ except for a difference in sign but since $\frac{dy}{dx}$ is negative for all x-values that y is negative, $\frac{sin(x)}{|sin(x)|}$ can be used to write the derivative more compactly as

$$\frac{dy}{dx} = \frac{sin(x)}{|sin(x)|} \cdot cos(x) \quad \text{for all } x. \tag{4}$$

Considering figure 5 and plotting eq(4) both give the same graph for the derivative of $|sin(x)|$. This should be obvious but just in case it's not, let's take this a step further. Eq(4) is basically just a $cos(x)$ curve with an extra factor $\frac{sin(x)}{|sin(x)|}$ giving a ± 1 where appropriate. The graph for $cos(x)$ is in figure 6.

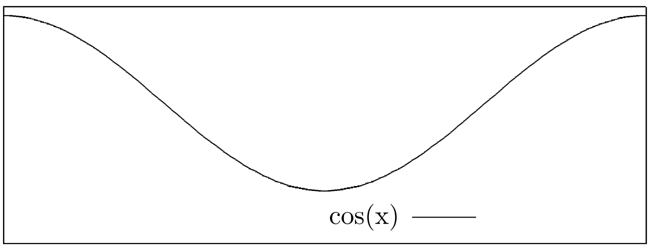

Figure 6: $0 < x < 2\pi$.

Multiplying this by -1 whenever $sin(x)$ is *negative*, i.e. whenever $\pi < x < 2\pi$, we get figure 7.

This result is obtained consistently by various methods as you've now seen. Note, the derivative of $|sin(x)|$ has discontinuities at $n\pi$ for all integer n.

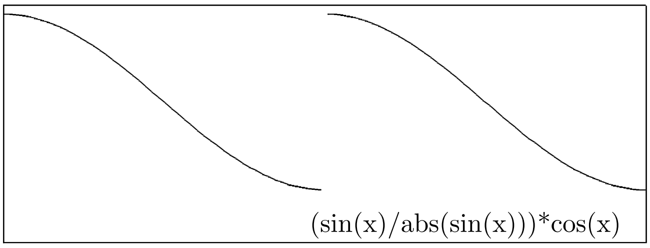

Figure 7: $0 < x < 2\pi$. Discontinuities at $n\pi$.

Examples

The following shows how to calculate the derivative of some functions analytically using eq(1).

Example 1, consider $y(x) = |x^3 + 2x^2 + 3|$.

$$\frac{dy}{dx} = \frac{x^3 + 2x^2 + 3}{|x^3 + 2x^2 + 3|} \cdot (3x^2 + 4x).$$

Example 2, consider $y(x) = e^{2x}|x^2 + 3x|$.

$$\frac{dy}{dx} = 2e^{2x}|x^2 + 3x| + e^{2x} \cdot \frac{x^2 + 3x}{|x^2 + 3x|} \cdot (2x + 3).$$

Example 3, consider $y(x) = x^2|cos(x)|$.

$$\frac{dy}{dx} = 2x|cos(x)| - x^2 \cdot \frac{cos(x)}{|cos(x)|} \cdot sin(x).$$

Some for you to try

Find the first and second derivatives of the following and try plotting them to see what they look like.

1. $y = e^{|x|}$
2. $y = cos|x|$
3. $y = (4x^3 - 3) \cdot tan|x^3|$
4. $y = a^{|x|}$ where a is some base greater than 0.
5. $y = a^{x^2 - |x|}$ where a is the same base.
6. $y = \frac{1}{ln|x+2|}$.

I have not evaluated these myself but if anyone actually bothers to do them and would like to check their answers, let me know and we'll arrange something.

An Aside

Someone asked about $y(x) = sin|x|$ specifically. This is considered here as a hint to the above questions. The graph is in figure 8.

Differentiating this using the chain rule gives

$$\frac{dy}{dx} = \frac{d|x|}{dx} \cdot cos|x|.$$

The function $cos(x)$ is symmetric i.e. $cos(-x) = cox(x)$. Similary, $cos|x| = cos(x)$. Also, it is known that $\frac{d|x|}{dx} = \frac{x}{|x|}$, so

$$\frac{dy}{dx} = \frac{x}{|x|} \cdot cos(x).$$

This graph is shown in figure 9.

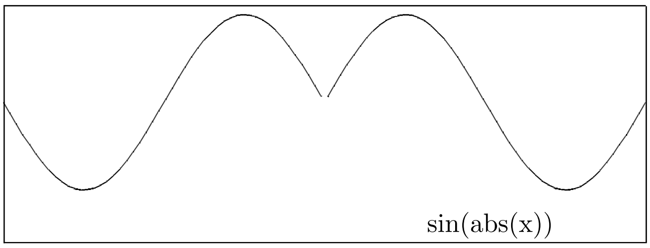

Figure 8: Origin (0:0) at center and $-2\pi < x < 2\pi$.

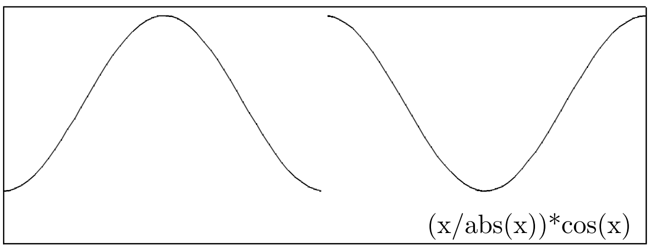

Figure 9: $-2\pi < x < 2\pi$. Discontinuos only at $x = 0$.

If you have any problems or find any glaring errors in this document, please let me know.