Essential Graduate Math I : Basic Calculus

1. Analyse a function - basic points of interest

Give a full analysis of the function $f(x) = A\sin(x/\pi)\exp(-Bx)$, where x is real and B is real and positive. Your discussion should include:

    a) all roots,

    b) points of local and global maxima / minima,

    c) behaviour for $x \to 0$ and for $x \to \pm\infty$,

 and d) a drawing of a graph.

Moreover, express the function as a sum of an even and an odd function of x.

2. Differentiation, usually easy, but you need an idea here ...

Differentiate $f(x, y) = x^x \sin^{-1}(y/(y+1))$ with respect to x and y.

3. Taylor series

a) Expand the real function $f(z) = A\ln(Bz)$ with A and B being real and positive in a Taylor series. What would be the natural choice for the point to expand about?

b) Expand the function $f(x, y) = x^6 \exp(-x)\sin(y)/y$ in a Taylor series about $x=1$ and $y=\pi$.

c) Construct the plane which is tangential to the function $f(x, y)$ at the point $x=1$ and $y=\pi$.

4. Integration

Evaluate the indefinite integrals

    a) $\displaystyle\int \cos^{-1}(x)\, dx$

    and b) $\displaystyle\int \frac{1}{x^4 + x^3 + 4x^2 + x + 3}\, dx$.   (hint: easy roots!)

c) Prove that the following relation holds

$$\int_0^\infty x^n \exp(-x^2)\, dx = \frac{1}{2}(n-1)\int_0^\infty x^{n-2}\exp(-x^2)\, dx .$$

d) Use this result to calculate $\int_0^\infty x^5 \exp(-x^2)\, dx$.

5. Finally, all combined

a) Differentiate the following integral with respect to x

$$\int_{\sin x}^{x^5} \exp(x \cdot t)\, dt .$$

b) Expand the following integral for small x

$$\int_{x^2 - Ax}^{x^2 + Ax} t^{-2}\sin^2(t)\exp(-t^2)\, dt .$$

c) What is $\displaystyle\lim_{x \to 1}\left\{ [\ln(x-1)]^{-5}(x-1)^4 \tan(x-1)^2 \right\}$? Is there a difference if one approaches unity from $x < 1$ or from $x > 1$? When would that be the case?