# Essential Graduate Math IV : Distributions / Fourier-Transformation

### 1. General properties of the “$\delta$-function”

Make yourself familiar with the definition of the $\delta$-distribution and its properties. What is the general meaning of distributions?

a) Find different representation of the $\delta$-distribution that involve series of exponentials and sin/cos functions.

b) Find the properties that allow you to solve the integral

$$I = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \, (E^3 - \sqrt{E}) \, \delta\left(E - \frac{p^2}{2m}\right) ,$$

where $p$ is of course the modulus of the vector $\mathbf{p}$.

### 2. More Properties

Sometimes it is stated as a defining equation that the $\delta$-distribution is the derivative of the step function. Show it rather follows from the definition as a distribution, that is $\int dx f(x)\delta(x-a)=f(a)$.

### 3. What is $\delta(0)$ ?

Given the fact that

$$\langle \mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}') \quad \text{and} \quad \langle \mathbf{p}|\mathbf{r}\rangle = \exp(-i\mathbf{r}\cdot\mathbf{p}/\hbar) \quad \text{and} \quad \langle \mathbf{r}|\mathbf{p}\rangle = \exp(i\mathbf{r}\cdot\mathbf{p}/\hbar)$$

what is

a) $\delta(\mathbf{p}-\mathbf{p}')$ and

b) $\delta(0)=\delta(\mathbf{p}-\mathbf{p})$?

### 4. Using distributions to obtain Relations in Fourier-space

Take the Fourier-transformation (and back-transformation)

$$g(\mathbf{p}) = \int d\mathbf{r} \exp(i\mathbf{r}\cdot\mathbf{p}) \, g(\mathbf{r}) \quad \text{and} \quad g(\mathbf{r}) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \exp(-i\mathbf{r}\cdot\mathbf{p}) \, g(\mathbf{p})$$

and

a) calculate $g(\mathbf{r})h(\mathbf{r})$ and

b) prove the integral relation

$$\int d\mathbf{r}_1 d\mathbf{r}_2 \, V(\mathbf{r}_1-\mathbf{r}_2) g(\mathbf{r}_1-\mathbf{r}_2) h(\mathbf{r}_1-\mathbf{r}_2) = \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} V(\mathbf{p}_1-\mathbf{p}_2) \, g(\mathbf{p}_1) \, h(\mathbf{p}_1) \, .$$