Maths for Scientists
Corrections to Common Mistakes

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These are some of the important points that people got wrong in the weekly assessment sheets.

There are some general methods given here to answer questions similar to some of those on the question sheets. The questions on the exam may be different!

In order to explain points of confusion, generally the maths here is quite rigorous. Unfortunately this may confuse some, and thus this sheet may not be useful for all. Also, as a quick disclaimer, this sheet is not authorized by the lecturer, or anyone marking the exam. If anything isn't clear feel free to e-mail or use the form on my webpage, http://go.warwick.ac.uk/mtcharemza.

Notation

Let P and Q be some statements. Then $P \implies Q$ means P implies Q, and $P \impliedby Q$ means Q implies P. Also $P \iff Q$ means both P implies Q and Q implies P. The symbol $\iff$ is pronounced "if and only if" and is also written "iff". Generally it is a good idea to use the appropriate symbol to make the logical flow of the maths clear. An example of the use of $\implies$ symbol is

$$\begin{aligned} & u(x,t) = e^{xt} \\ \implies & \frac{\partial}{\partial x}\Big(u(x,t)\Big) = te^{xt} \\ \implies & \frac{\partial^2}{\partial x^2}\Big(u(x,t)\Big) = t^2 e^{xt}. \end{aligned}$$

An example of the use of the $\iff$ symbol is

$$x = y \iff x + 3 = y + 3.$$

Vectors

- Remember that $\underline{\hat{\imath}}$, $\underline{\hat{\jmath}}$ and $\underline{\hat{k}}$ are vectors and not the coordinates in a particular direction. For example, you may be used to writing a vector $\underline{u} = (1, 2, 3)$. Using the vector notation presented in the course, this could be written as $\underline{u} = 1\underline{\hat{\imath}} + 2\underline{\hat{\jmath}} + 3\underline{\hat{k}}$.

Integration

- Remember the constant of integration if you're performing indefinite integration (i.e. without defined limits).
- Remember the absolute value signs when integrating $\frac{1}{x}$, i.e.

$$\int \frac{1}{x} = \ln|x| + C.$$

Limits

- Note ∞ is not just an ordinary number, so you cannot divide, multiply and add by it.[1] It's probably best to think of it as more of a shorthand way to say something else. For example

$$\lim_{x \to c} f(x) = \infty$$

  is just a shorthand way of saying, roughly speaking, that as x approaches c, $f(x)$ is arbitrarily large. More precisely, it means that for any real number M there is a small interval around c, that depends on M, such that for all $x \neq c$ in that interval we have $f(x) \geq M$. Note that in the definition is no reference to any sort of "infinity".

  Another example is

$$\lim_{x \to \infty} f(x) = \infty$$

  which is just a shorthand way of saying the more formal definition: for all real numbers M there is a real number N_M, such that for all $x \geq N_M$ we have $f(x) \geq M$. Again there is only reference to numbers being larger than other numbers - no mention of a place called "infinity".

  One sided limits, and limits with $-\infty$ are defined similarly.[2]

  The reason why $\frac{0}{\infty}$, ∞x, $\infty + x$ and similar are bad usage of notation is that those symbols are not defined (at least at this level of mathematics).

- Make sure you correctly use the notation for limits. Don't use

$$\lim_{n \to \infty} a_n \to c$$

  which actually means "the limit of a_n as n tends to infinity, itself tends to c". There may be times when you need to say something like this, but this did not happen in the assessed questions. What should have been written was

$$\lim_{n \to \infty} a_n = c$$

[1] Although there are mathematical constructions where you do treat it more as an ordinary number, but you probably won't meet them for a while

[2] If you are feeling especially keen, you could have a go at trying to define them formally, although for applied purposes usually an "intuitive" definition is enough. You could check your definitions in most Analysis books.

or

$$a_n \to c \text{ as } n \to \infty$$

or

$$a_n \xrightarrow{n\to\infty} c,$$

which all mean "the limit of a_n as n tends to infinity is equal to c".

Curve Sketching

If asked to sketch the graph of $y = f(x)$, always remember to do the following (unless specified otherwise):

- State any symmetry the function may have, i.e. is it even, odd or neither.
- Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ (if they exist).
- Find any points of discontinuity, or state that there are none[3].
- For any points of discontinuity c, find $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^-} f(x)$ (if they exist - they may not).
- Find any turning points (both x and y coordinates)
- For the turning points, find whether they are maxima or minima.
- Show your working for all the above
- Make sure you put information you've found on your sketch.[4].

Complex Numbers

- If asked to solve an equation of the form

  $$z^n = a + bi \text{ where } z, c \in \mathbb{C} \text{ and } a, b \in \mathbb{R},$$

  or where it can be manipulated into such a form[5], there is a general method to follow, which is now explained in detail.[6]

  1. Clearly re-state the problem. The problem is that you are trying to find $z \in \mathbb{C}$ such that $z^n = a + bi$.

[3]It may have been said in class that the points where the function is not defined are technically not points of discontinuity, however it all depends on what you define as a point of discontinuity. The lecturer for Maths for Scientists seems to be saying that a point where a function is not defined is a point of discontinuity. However, there are contradictory definitions elsewhere. For this course you should stick to this lecturer's definition, but beware on other courses.

[4]Many people didn't do this in the weekly sheets. On the exam this could mean easy marks lost!

[5]Such as Sheet 2 Question 4, and Sheet 3 Question 1.

[6]Probably in more detail than would be required.

2. All $z \in \mathbb{C}$ can be written as

$$z = R(\cos\theta + i\sin\theta) \in \mathbb{C} \text{ where } R \in \mathbb{R}_{\geq 0} \text{ and } \theta \in [0, 2\pi),$$

so the problem is that we are trying to find $R \in \mathbb{R}_{\geq 0}$ and $\theta \in [0, 2\pi)$ such that $z = R(\cos\theta + i\sin\theta) \in \mathbb{C}$ satisfies $z^n = a + bi$.

3. Note that

$$\begin{aligned} & z = R(\cos\theta + i\sin\theta) && \text{where } R \in \mathbb{R}_{\geq 0} \text{ and } \theta \in [0, 2\pi) \\ \iff & z^n = R^n(\cos n\theta + i\sin n\theta) && \text{where } R^n \in \mathbb{R}_{\geq 0} \text{ and } n\theta \in [0, n2\pi) \end{aligned}$$

by De Moivre's Theorem. Thus (and this is the important bit) the problem is that we are looking for $R^n \in \mathbb{R}_{\geq 0}$ and $n\theta \in [0, 2n\pi)$ such that $R^n(\cos n\theta + i\sin n\theta) = a + bi$.

4. By comparing real and imaginary parts, we can say that the problem is that we are looking for $R^n \in \mathbb{R}_{\geq 0}$ and $n\theta \in [0, 2n\pi)$ such that that satisfy *both*[7] the following equations:

$$\begin{aligned} & R^n \cos n\theta = a \\ & R^n \sin n\theta = b. \end{aligned}$$

5. Solve the above two equations for R^n and $n\theta$. One way to do it follows.

   (a) Find R^n using the identity $\cos^2\Theta + \sin^2\Theta = 1$. This will result in $R^n = \sqrt{a^2 + b^2}$, remembering that R^n is a positive real number.

   (b) Solve each equation individually for $n\theta$, remembering that $n\theta \in [0, n2\pi)$.[8]

   (c) Pick which $n\theta$ are solutions to both equations.

6. Find R and θ to find the z that satisfy the original equation. Remember that unless the question specifies how to present the answer, the form $z = Re^{\theta}$ is usually acceptable.

Series

- When using the ratio test, make sure you're clear on what the nth term is in the series. Consider, for example the series

  $$\sum_{n=0}^{\infty} ax^n.$$

  The nth term in this series is ax^n. It is not a, nor is it some sort of partial sum, i.e. the sum of the first n terms.

  To use the ratio test for this series, you should define $b_n = ax^n$ and then find the limit of $|b_{n+1}/b_n|$ as n tends to infinity. If it's bigger that 1 the

[7] This is where many people went wrong - you must find $n\theta$ and R^n that satisfy *both*.

[8] Remember calculators are not allowed in the exam. You should remember the *sin*, *cos* and *tan* of $\frac{\pi}{3}$, $\frac{\pi}{6}$ and $\frac{\pi}{4}$, and of course 0 and π, or be able to quickly work them out in the exam.

series will diverge, and if it's less than 1 it will converge. This limit may depend on x, and so whether the series diverges or converges will depend on x.

Remember to include the absolute value signs when using the ratio test, and also remember that the ratio test does not give any information when the limit of $|b_{n+1}/b_n|$ is equal to 1.

- Understand all the terms in the definition of a Taylor series for a function $f(x)$. The Taylor series for such a function about a point a is formally written as

$$\sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a},$$

You may be used to seeing the Taylor series written as

$$\sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} f^{(r)}(a).$$

What is generally meant by $f^{(r)}(a)$ is

$$\left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a},$$

which means "$f(x')$ differentiated r times with respect to x', and then evaluated at $x' = a$". The variable x' is a "dummy" variable.

The use of the dummy variable x' in the expression is so that it is not confused with the variable x. These are different variables. No matter what x is chosen, it does not affect the differentiation where the dummy variable x' is used.

They are called "dummy" variables as in the expressions they appear they can be changed to any other variable (that isn't already used) and the expression will have the exact same value or meaning. For example the Taylor series for $f(x)$ could be written as

$$\sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^r f(\xi) \right) \right|_{\xi=a}.$$

You are probably already familiar with dummy variables, even if they have not been labeled as such. In $\sum$ notation, the index variable is a dummy variable. It can be changed to another letter (that isn't already used), without affecting the meaning of the expression. Thus the Taylor series of $f(x)$ can be written as

$$\sum_{m=0}^{\infty} \frac{(x-a)^m}{m!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^m f(\xi) \right) \right|_{\xi=a}.$$

- Note that a Taylor series, like any other power series, has a radius of convergence. Thus it is *not* always true that

$$f(x) = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a},$$

as the series on the RHS may may not be convergent. However it is acceptable to write

$$f(x) = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a} \quad \text{when convergent.}$$

- A truncated Taylor series is an approximation to $f(x)$ for appropriate values of x. Specifically, a truncated Taylor series of $f(x)$ about the point a is an approximation to $f(x)$. Thus it is acceptable to write, for example

$$f(x) \approx \sum_{r=0}^{5} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a} \quad \text{for } x \text{ close to } a.$$

- The entire Taylor series of $f(x)$ is not an approximation of $f(x)$. Thus it is wrong to write

$$f(x) \approx \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} \left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a}.$$

- When a question asks to find the Taylor series, usually what it wants you do to is to find evaluate the first few

$$\left. \left(\left(\frac{\mathrm{d}}{\mathrm{d}x'} \right)^r f(x') \right) \right|_{x'=a}$$

terms and "spot" a pattern to find the general rth term in the Taylor series.

- Remember the Taylor series is an infinite series, so if asked to find the Taylor series finding a few terms is not enough, although it is usually useful to do this to find the final answer. Generally you should find the rth term, and also make it clear that you are not writing down a truncated series unless the question asks you to.

  As a simple example a truncated Taylor series could be:

$$a + ax + ax^2 + ax^3 + ax^4,$$

but what is considered the entire Taylor series would be written

$$a + ax + ax^2 + ax^3 + ax^4 + \ldots + ax^r + \ldots,$$

or more concisely

$$\sum_{r=0}^{\infty} ax^r.$$

Partial Differentiation

- If f is a function of x, y and z, then

  $$\frac{\partial^3 f}{\partial x \partial y \partial z}$$

  is actually shorthand for

  $$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\left(f(x,y,z)\right)\right)\right).$$

  In order to evaluate the above expression, you must differentiate f first with respect to z, then with respect to y, and finally with respect to x. If you try to do it the other way around using the "longhand" version above you should get stuck!

- State which rules you use when you use then when differentiating, i.e. the chain, product and quotient rules. Perhaps not strictly always necessary, but it is a good idea to show you know exactly what you're doing. If a question asks you to use a particular rule, then it probably would be necessary to state exactly where you use it.

- For a real function f of two real variables[9], the turning points are given by the solutions to

  $$\nabla f = 0,$$

  which is more formally written

  $$\nabla f(x,y) = \underline{0}.$$

  This makes it a bit clearer that the 0 on the right hand side is actually the zero vector, i.e. $0\underline{\hat{\imath}} + 0\underline{\hat{\jmath}}$. Thus, using the definition of gradient ∇ and by equating coordinates, the turning points are given by the solutions to

  $$\frac{\partial}{\partial x}\left(f(x,y)\right) = 0 \qquad\qquad \frac{\partial}{\partial y}\left(f(x,y)\right) = 0.$$

  It is not correct to find the solutions to

  $$\frac{\partial}{\partial x}\left(f(x,y)\right) = \frac{\partial}{\partial y}\left(f(x,y)\right),$$

  as you may find solutions that are not turning points.

- If asked to find show that a particular point, say (x_1, y_1), is a turning point of $f(x,t)$, you do not have to solve $\nabla f(x,y) = \underline{0}$. You only have to find $\nabla f(x,y)$, substitute in (x_1, y_1), and show that it equals $\underline{0}$.

- Try to be clear what is the variable and what is constant when partially differentiating. Getting this wrong would result in not using certain rules when you have to, or applying rules when you shouldn't.

[9]This is formally written $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

- Let $u = \Phi(x,t)$, where $\Phi(x,t)$ is some given function. Say that you are asked to show that this $u = \Phi(x,t)$ satisfies a given PDE. A good general method to do this follows.

  1. Re-state the problem. Showing that $u = \Phi(x,t)$ satisfies satisfies a given PDE is the same as showing

     $$u = \Phi(x,t) \implies u \text{ satisfies the PDE.}$$

  2. Find the partial derivatives of u that are in the PDE.
  3. Substitute what you've found into *one side* of the PDE, and show that it equals the other side.

  For example, let $u = e^{xt}$. Say that you are asked to show that $u = e^{xt}$ satisfies the PDE $\partial u/\partial x = (t/x)\,(\partial u/\partial t)$. Following the method above results in the following.

  1. We re-state the problem. Showing that u satisfies the PDE $\partial u/\partial x = (t/x)\,(\partial u/\partial t)$ is the same as showing

     $$u = e^{xt} \implies \frac{\partial u}{\partial x} = \frac{t}{x}\frac{\partial u}{\partial t}.$$

  2. We find the partial derivatives of u that are in the PDE.

     $$\begin{aligned} & u = e^{xt} \\ \implies & \frac{\partial u}{\partial x} = te^{xt} \text{ and } \frac{\partial u}{\partial x} = xe^{xt} \end{aligned}$$

  3. Substituting in what we've found, and writing the working out in full gives

     $$\begin{aligned} & u = e^{xt} \\ \implies & \frac{\partial u}{\partial x} = te^{xt} \text{ and } \frac{\partial u}{\partial t} = xe^{xt} \\ \implies & \frac{\partial u}{\partial x} = te^{xt} = \frac{t}{x}\left(xe^{xt}\right) = \frac{t}{x}\frac{\partial u}{\partial t}. \end{aligned}$$

     The above clearly shows how $u = e^{xt} \implies \partial u/\partial x = (t/x)\,(\partial u/\partial t)$, which is the same thing as saying $u = e^{xt}$ satisfies the PDE $\partial u/\partial x = (t/x)\,(\partial u/\partial t)$.

- Let $u = \Phi_a(x,t)$, where $\Phi_a(x,t)$ is some given function that depends on parameter a. Say that you are asked to find a value of a such that $u = \Phi_a(x,t)$ satisfies a given PDE. A general method to do this follows

  1. Re-state the problem. Finding a value of a such that $u = \Phi_a(x,t)$ satisfies a given PDE is the same as finding a λ such that

     $$\left.\begin{aligned} u &= \Phi_a(x,t) \\ a &= \lambda \end{aligned}\right\} \implies u \text{ satisfies the PDE.} \tag{1}$$

2. Find the partial derivatives of u that are in the PDE.
3. Substitute what you've found into *both sides* of the PDE, and rearrange to find a.

   You must be careful with the rearrangement, as you are trying find an a such that (1) is true, and not the converse. The aim of the substitution and the rearrangement for a is to find a λ such that, if we assume $u = \Phi_a(x,t)$, then

$$\begin{aligned}
&u \text{ satisfies the PDE} \\
\Longleftarrow\; &[\text{the PDE with the substitution } u = \Phi_a(x,t)\ ] \\
\Longleftarrow\; &[\text{rearranged equation}] \\
&\vdots \\
\Longleftarrow\; &[\text{rearranged equation}] \\
\Longleftarrow\; &a = \lambda.
\end{aligned}$$

   Note the direction of the $\Longleftarrow$. The reason for this is that is that with the above working shows that the assumption of $u = \Phi_a(x,t)$ and $a = \lambda$ implies that u satisfies the PDE. This is exactly what re-statement of the problem asked to do!

   You must be sure that when you rearrange the equation, each "rearrangement" satisfies the $\Longleftarrow$ condition. To illustrate, consider the statement $x = y$. Clearly $x = y \quad \Longleftarrow \quad x + 3 = y + 3$ but $x = y \not\Longleftarrow \cos x = \cos y$. All rearrangements that only involve addition, subtraction, multiplication (by non-zero numbers) and division do satisfy the $\Longleftarrow$ condition.

An example that uses this method follows. Let $u = e^{axt}$. Say that you are asked to find a value of a such that $u = e^{axt}$ satisfies $\partial u / \partial x = \left(t/x^2\right)\left(\partial^2 u / \partial t^2\right)$. Using the method above gives the following.

1. We re-state the problem. Finding a value of a such that $u = \Phi_a(x,t)$ satisfies a given PDE is the same as finding a λ such that

$$\left.\begin{aligned} u = e^{axt} \\ a = \lambda \end{aligned}\right\} \implies \frac{\partial u}{\partial x} = \frac{t}{x^2}\frac{\partial^2 u}{\partial t^2}$$

2. We find the partial derivatives of u that are in the PDE. We have that

$$\begin{aligned}
&u = e^{axt} \\
\implies &\frac{\partial u}{\partial x} = ate^{axt},
\end{aligned}$$

   and

$$\begin{aligned}
&u = e^{axt} \\
\implies &\frac{\partial u}{\partial t} = axe^{axt} \\
\implies &\frac{\partial^2 u}{\partial t^2} = a^2x^2e^{axt}.
\end{aligned}$$

3. We now substitute what we've found into *both sides* of the PDE, and rearrange to find a, which, assuming $u = e^{axt}$, gives

$$
\begin{aligned}
& u \text{ satisfies the PDE} \\
\Longleftarrow\ & \frac{\partial}{\partial x}\left(e^{axt}\right) = \frac{t}{x^2}\frac{\partial^2}{\partial t^2}\left(e^{axt}\right) \\
\Longleftarrow\ & ate^{axt} = \frac{t}{x^2}a^2x^2e^{axt} \\
\Longleftarrow\ & at = \frac{t}{x^2}a^2x^2 \\
\Longleftarrow\ & a = a^2 \\
\Longleftarrow\ & a = 1.
\end{aligned}
$$

We define $\lambda = 1$. Therefore we have found a value of λ such that assuming $u = e^{axt}$ and $a = \lambda$ implies that u satisfies the PDE $\partial u/\partial x = \left(t/x^2\right)\left(\partial^2 u/\partial t^2\right)$. This is exactly what the re-statement of the problem asked us to do.