A Partially Meshfree Galerkin Scheme for Representing Highly Anisotropic Fields

Samuel A. Maloney, Ben F. McMillan

Centre for Fusion, Space and Astrophysics, University of Warwick, Coventry, UK

sam.maloney@warwick.ac.uk

Motivation

Efficient representation and simulation of highly anisotropic fields:

- Structures elongated along one direction $\implies$ smaller gradients
- Require fewer degrees of freedom to represent variation
- Need not be aligned with any coordinate direction

Extremely difficult to create a standard mesh conforming to the anisotropy

- Different magnetic regions, e.g. core, SOL/walls, X-point(s), etc.
- Even in core, helical and ergodic field causes difficulties

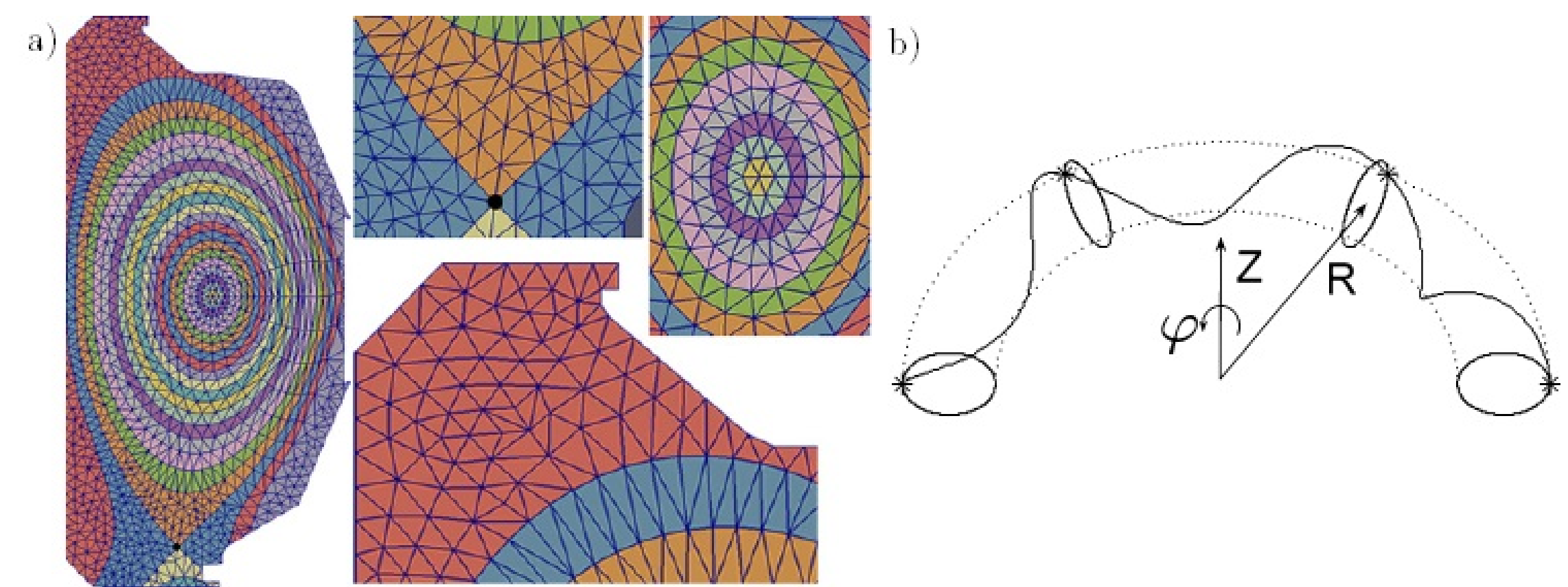Want method independent of flux coordinates[1] using unstructured mesh[2]

Figure 1: a) Standard mesh regions, b) helical magnetic field. Figs. from[3]

1) Finite Element (FEM) vs. Meshfree Methods

For a Galerkin formulation of either a FEM or meshfree method, one must:

- discretize the domain Ω with a set of N_n nodes $\mathbf{x}_i$
- define a set of basis functions ϕ_i associated with each node
- expand solution using basis $u(\mathbf{x}, t) \approx u_h(\mathbf{x}, t) = \sum_{i=1}^{N_n} u_i(t)\phi_i(\mathbf{x})$
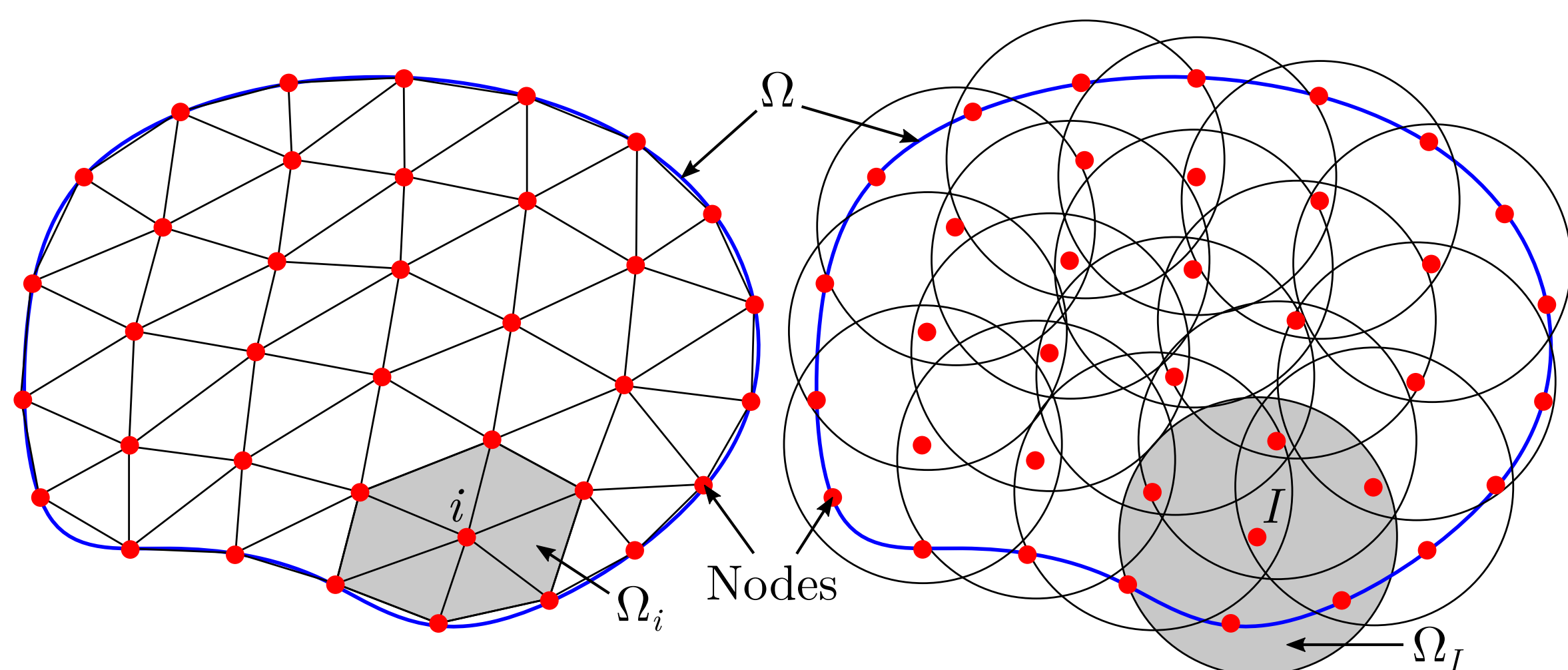

Figure 2: Standard FEM (left) and meshfree (right) discretizations.

2) Flux Coordinate Independent (FCI) FEM Scheme

Desired Method properties:

- Compact Support: $\phi_i(\mathbf{x}) = 0 \quad \forall \mathbf{x} \notin \Omega_i$
- Partition of Unity: $\sum_{i=0}^{N_n} \phi_i(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \Omega$
- Delta Property: $\phi_i(\mathbf{x}_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \implies u_h(\mathbf{x}_i, t) = u_i(t) \quad \forall \mathbf{x}_i$
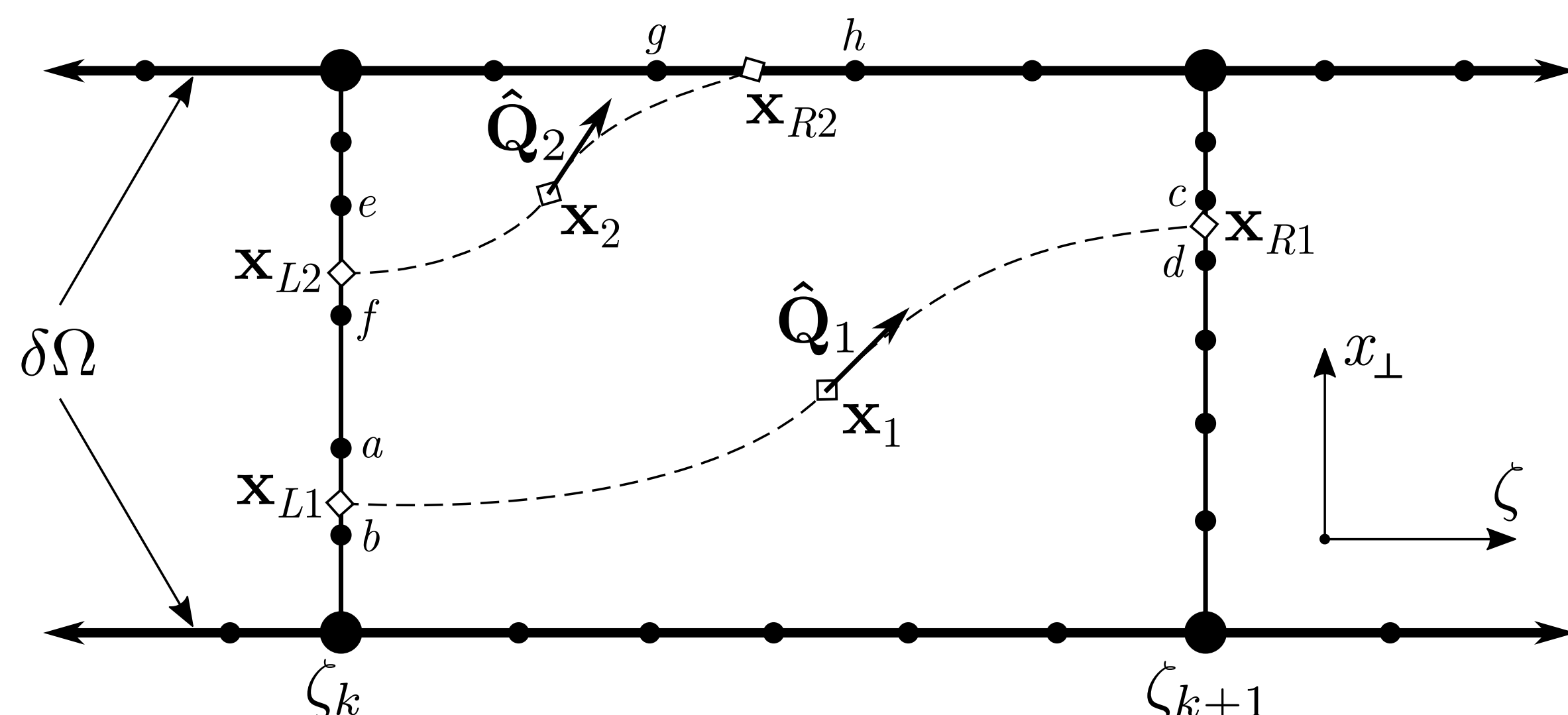

Figure 3: FCIFEM scheme discretizing a 2D domain Ω. Evaluation points are mapped to surrounding mesh on FCI planes and boundaries via a mapping function $Q: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, e.g. $Q(\mathbf{x}_1, \zeta_k) = \mathbf{x}_{L1} = \mathbf{x}_{\text{map}}$.

The basis functions are formed as a product of two factors

$$\phi_i = \phi_{i,\text{FEM}}(\mathbf{x}_{\text{map}})\rho_i(\mathbf{x}).$$

The $\phi_{i,\text{FEM}}(\mathbf{x}_{\text{map}})$ are a standard FEM basis of reduced dimension defined on standard FEM meshes covering the FCI planes and domain boundaries. The ρ_i is a ramp function along the mapping given as

$$\rho_i(\zeta) = \frac{\zeta - \zeta_t}{\zeta_o - \zeta_t},$$

with ζ_o, ζ_t of the origin and terminus points $\mathbf{x}_L, \mathbf{x}_R$ along the mapping.
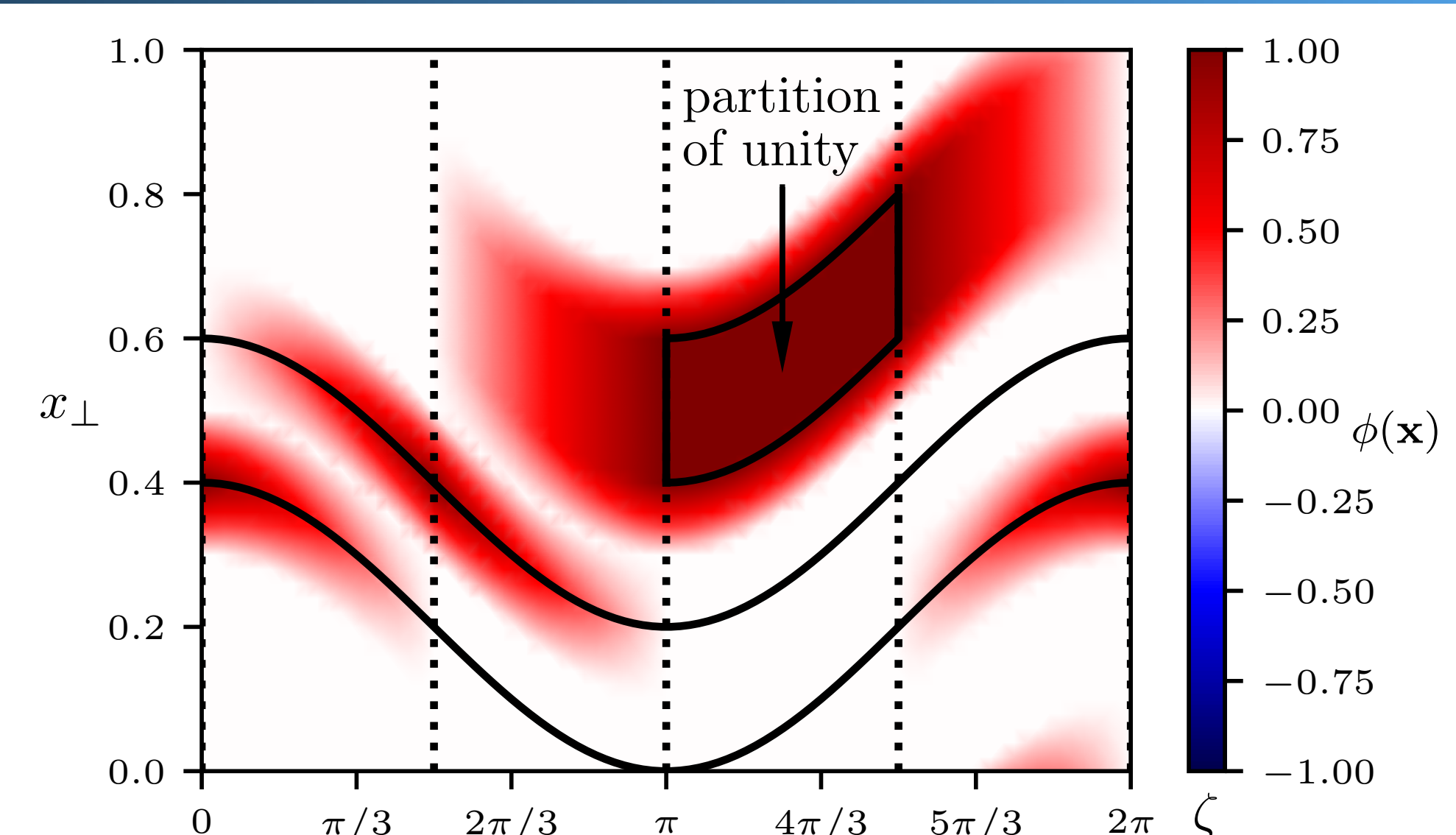
3) Implementation

Figure 4: Example of 2D basis functions in Python implementation with sinusoidal mapping showing alignment to mapping and partition of unity.

4) Conservation and Convergence Results

Conservation: Hughes et al.[4] showed the standard continuous Galerkin method is locally conservative w.r.t. point-wise fluxes at the nodes

- We can show that using integration by parts allows a similar result for FCIFEM w.r.t. point-wise fluxes at the quadrature points

Convergence: $\nabla^2 u = f(x, y)$ on $\Omega : (0, 1) \times (0, 1)$ with Dirichlet boundaries, where f is forcing term giving $u(x, y) = x\sin(2\pi n[y - ax^2 - bx])$ with $n = 3, a = 0.95, b = 0.05$, giving quadratically aligned anisotropy.
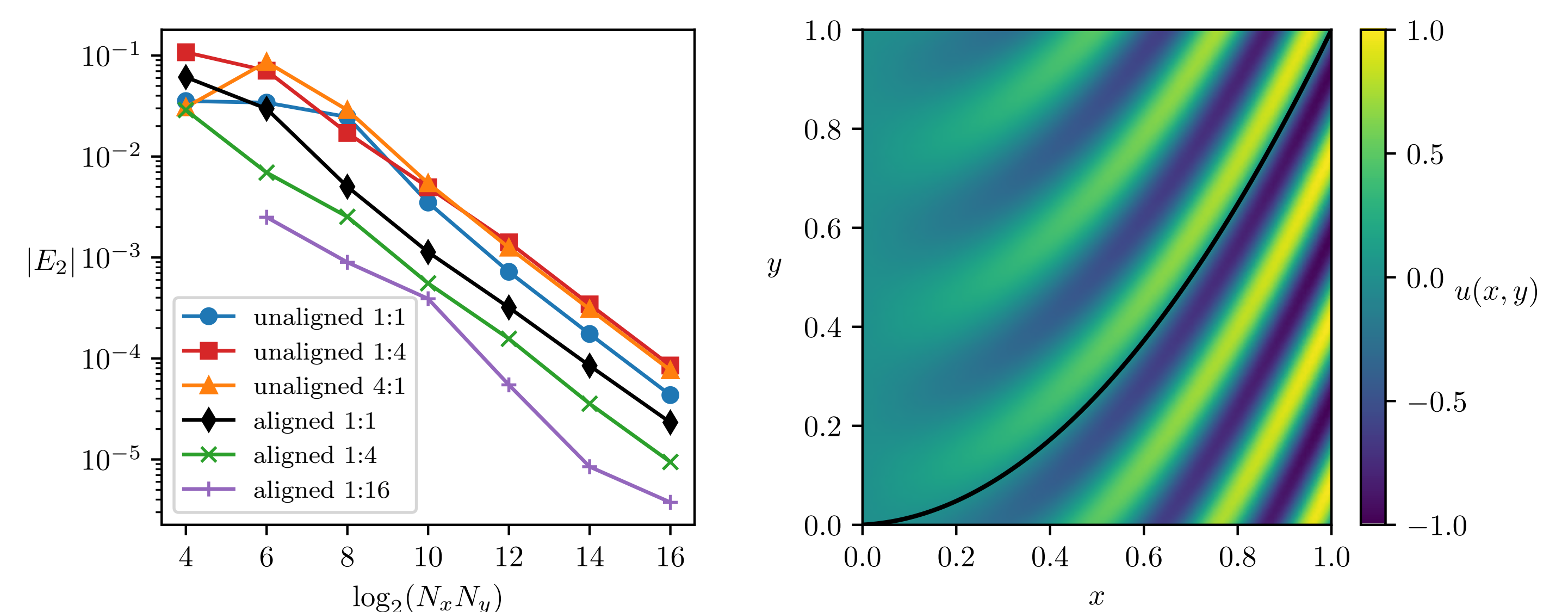

Figure 5: (left) L_2 error norms for different ratios of grid divisions N_x:N_y, w/ and w/o mapping aligned to anisotropy, otherwise coordinate aligned (right) example of well-converged simulation showing anisotropy in $u(x, y)$

- Essentially 2nd order convergence in all cases
- Alignment reduces error somewhat on its own
- More importantly, it allows for larger ratios, giving lower error for same number of nodes, whereas unaligned error much worse for 1:4 ratio

References and Acknowledgements

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(3) F. Zhang, R. Hager et al., *Eng. Comput.*, 2016, **32**, 285–293.

(4) T. J. R. Hughes, G. Engel et al., *J. Comput. Phys.*, 2000, **163**, 467–488.