A Numerical Study of MHD Turbulence

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- "Extended Self-Similartiy" analysis performed on 3D decaying MHD DNS of *Biskamp and Muller, POP, 2000*
  - Unable to investigate refined similarity hypothesis satisfactorily ....needed more statsistics
- Developed 2D driven MHD DNS for further investigation
  - Can investigate refined similarity hypothesis
  - Present preliminary findings

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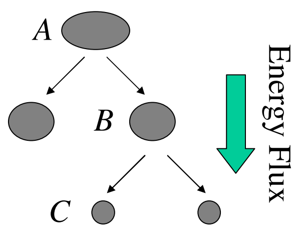
Scaling Laws

Energy Flux

- Branching Process (*Direct Cascade shown here)*
  - Large Scale eddies (pumping or driving scale)
  - *Cascade* to smaller scales unaffected by dissipation (*inertial range)*
  - Dissipate at small scales (dissipative range)
- At length scales far from the driving & dissipation scale *A*→*B* is a scaled version of *B*→*C*
- This is captured by scaling laws
  - Structure functions
  - Sensitive to correlations via l and to intensities via p

$$\langle \partial v_l^p \rangle = \langle |v(x+l) - v(x)|^p \rangle \sim l^{\phi_p}$$



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Elsasser Field Structure Functions $S^p{}_l$

$$z^{\pm} = v \pm \frac{B}{\sqrt{\rho\mu_0}}$$

•Describes Left and Right travelling Alfvenic disturbances

$$\frac{\partial z^{\pm}}{\partial t} + z^{\mp}\cdot\nabla z^{\pm} = -\frac{\nabla P}{\rho} + \mu_{+}\nabla^2 z^{\pm} + \mu_{-}\nabla^2 z^{\mp}$$

•Can describe incompressible equations of MHD in Elsasser symmetric form see *Biskamp, MHD Turbulence, Camb. Univ. Press 2003*

$$S_l^p = \left\langle \partial z_l^{(\pm)p} \right\rangle = \left\langle \left| z^{(\pm)}(x+l)\cdot\frac{l}{|l|} - z^{(\pm)}(x)\cdot\frac{l}{|l|} \right|^p \right\rangle$$

•Construct Elsasser field structure functions ($S^p{}_l$) to describe Magneto-kinetic fluid

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Fluid Interpretation of Structure Function Scaling Laws *(Fluid Phenomenology)*

$$S_l^{(\pm)1} = \left\langle \partial z_l^{(\pm)} \right\rangle \sim \varepsilon_l^{1/\alpha} l^{1/g}$$

ε_l = local rate of dissipation averaged over ball of radius l

Kolmogorov 1941 (K41):
- local in k space
- nonlinear process is random eddy scrambling
- g=3, α=3

Iroshnikov-Kraichnan (IK) *Kraichnon, POF, 1965* :
- non-local in k space
- nonlinear process is governed by Alfvenic collisions
- g=4, α=4

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Universal Scaling Laws In Turbulence

- Scaling laws attractive; exponents independent of flow detail provided homogenous and isotropic

• *Universal*

$$S_l^{(\pm)p} \sim l^{\xi_p} \qquad \chi_l^{(\pm)p} = \left\langle \left[\int \frac{\left(\partial_i z_i^{(\pm)}\right)^2}{V_l} \, dV_l \right]^p \right\rangle \sim l^{\tau_p}$$

• $S_l^{(\pm)p}$ and χ_l^p constructed from simulation to determine ξ_p and τ_p

• χ_l acts as a 1D surrogate to ε_l as is common in hydrodynamic numerics and experiments

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She-Leveque (1994) Intermittency Correction

• In practice $S_l^{(\pm)}$ does not behave as IK or K41

→ *Intermittent* eddy activity .... use theory of She-Leveque

• Level of intermittency (deviation from IK or K41) determined by geometry (co-dimension) of structures that are <u>most intensely</u> dissipating.

• Link made by refined similarity hypothesis

$$S_l^{(\pm)p} \sim \chi_l^{(\pm)p/3} l^{p/3} \qquad \zeta_p = \tau_{p/3} + 1/3 \qquad \text{K41}$$

$$S_l^{(\pm)p} \sim \chi_l^{(\pm)p/4} l^{p/4} \qquad \zeta_p = \tau_{p/4} + 1/4 \qquad \text{IK}$$

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Extended Self-Similarity (ESS)

- These scaling laws not seen in numerics because of resolution contraints
- Find scaling laws from DNS via Extended Self Similarity (ESS), see *Benzi et al., PRE, 1993*

$$S_l^p \sim S_l^{q\left(\xi_p/\xi_q\right)}$$

- These hold in the inertial and dissipative range (above ≅ 5x the Kolmogorov dissipation length scale)

•Kolmogorov scale is scale on which flows become diffusions dominated

- Scaling law is *extended* into the dissipative range.

•Physically means all structures of all intensity of the same length scale are affected by viscosity by the same amount.

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Extended Self-Similarity & Refined Similarity

- Refined Similarity re-written for consistency with ESS

$$S_l^{(\pm)p} \sim \chi_l^{p/3} l^{p/3} \rightarrow S_l^{(\pm)p} \sim \chi_l^{p/3} \left(S_l^{(\pm)3}\right)^{p/3} \quad \text{K41}$$

$$S_l^{(\pm)p} \sim \chi_l^{p/4} l^{p/4} \rightarrow S_l^{(\pm)p} \sim \chi_l^{p/4} \left(S_l^{(\pm)4}\right)^{p/4} \quad \text{IK}$$

•She Leveque interpretation of ESS exponents requires the appropriate above relation to hold

•ESS in χ is also implied

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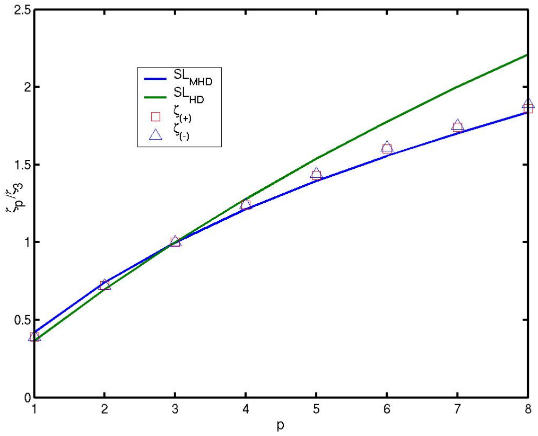
Numerical Results: *Biskamp and Muller, POP, 2000* (3D incompressible decaying DNS)



She Leveque model

•Cascade by random eddy scrambling (K41)

•Structures that dissipate most intensely are sheet-like

Need to check scaling of χ for consistency with theory

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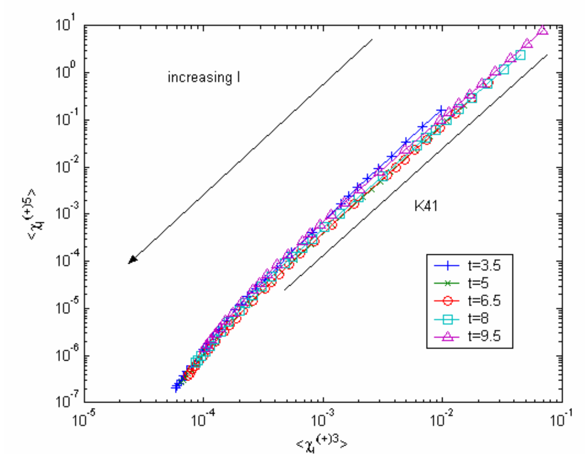
New - ESS in χ: Merrifield et al PoP 2005

$$\chi_l^p \sim \chi_l^{q\left(\tau_p / \tau_q\right)}$$



SL scaling of *Biskamp and Muller, POP, 2000* requires this self similarity in χ_l^p

Break of scaling at large l could be finite size effect

Thus, scaling recovered relies mostly on small l measurements → dissipative range

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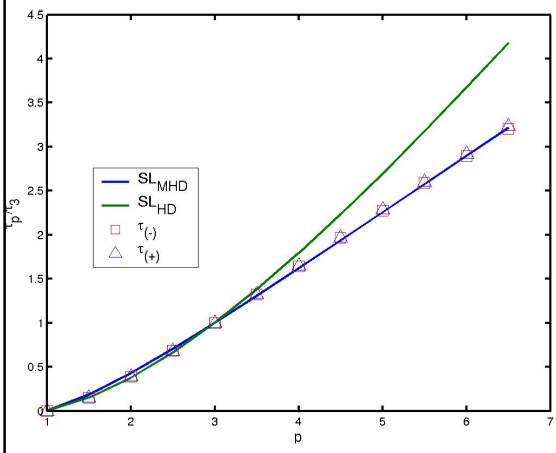
Recover Scaling Exponents



She Leveque model

•Cascade by random eddy scrambling (K41)

•Structures that dissipate most intensely are sheet-like

Scaling consistent with previous analyses

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However ….

•Refined similarity hypothesis not verified to high order

Investigate 2D DNS

- •High order finite difference code in 2D
- •Fourth order time stepping
- •Compressible (isothermal)
- •Higher Reynolds number for given computing power
- •Driven so allows statistics to be gathered over a long period
- •Driven at 4^{th} harmonic to allow inverse cascade to develop
- •Find we can investigate the refined similarity hypothesis directly

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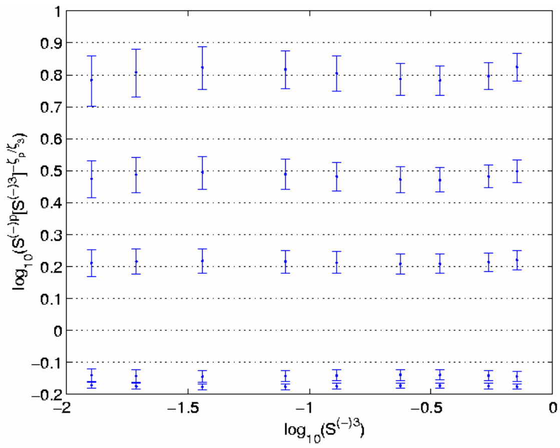
ESS recovered in *S* and χ again



- Here shown ESS in $\boldsymbol{S^{(-)}}$
- p=5(top) to p=1(bottom) p=3 is exluded
- Error bars show standard error in time average
- y axis normalised by ζ_p so ESS appears as horizontal line

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Direct test of Refined Similarity Hypothesis

–	p=1	p=2	p=3	p=4	p=5
K41 +	1.02	1.01	—	0.97	0.93
K41 -	1.02	1.02	—	0.96	0.91
IK +	1.12	1.10	1.06	—	0.94
IK -	1.14	1.11	1.07	—	0.93

Perfect agreement indicated by a value of 1

$$S_l^{(\pm)p} \sim \chi_l^{p/3}\left(S_l^{(\pm)3}\right)^{p/3} \quad K41$$

$$S_l^{(\pm)p} \sim \chi_l^{p/4}\left(S_l^{(\pm)4}\right)^{p/4} \quad IK$$

- Neither case agrees to high order
- Low order measurements have greater statistical certainty
- K41 hypothesis shows better agreement for low order

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Conclusions

- Self consistent SL theory seems to exist for 3D MHD turbulence
  - Cascade governed by random eddy scrambling
  - Intermittency determined by 2D structures
  - Couldn't test refined similarity hypothesis directly
- Can explicitly test refined similarity in 2D driven simulation
  - Preliminary investigation show neither IK or K41 theories agree perfectly though K41 seems to "perform" the best
  - Does not agree with traditional high resolution structure function and power spectrum analyses which seems to favour IK
- Is there a self consistent SL theory for 2D MHD?

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