Characterising the Scaling Properties of Incompressible Isotropic Three-Dimensional MHD Turbulence

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1 Introduction

Turbulent fluids and plasmas display three properties that motivate development of statistical theories: (i) disorganisation in the sense that structures arise on all scales; (ii) unpredictability of detailed behaviour; and (iii) reproducibility of statistical measures. Much progress has been made by the heuristic treatment of scaling laws derived from energy cascade arguments. The basic idea is that energy carrying structures (eddies) are injected on large scales, non-linear eddy interaction causes the cascade of energy to smaller scales, and energy is finally dissipated by viscosity on small scales. A quasi-stationary state evolves where the rate of viscous dissipation matches the rate of energy injection. Heuristic arguments predict the scaling exponents ζ_p that characterise the statistical self-similarity found in structure functions S_l^p:

$$S_l^p = \langle (\mathbf{v}(\mathbf{x}+\mathbf{l},t).\mathbf{l}/l - \mathbf{v}(\mathbf{x},t).\mathbf{l}/l)^p \rangle \sim l^{\zeta_p} \tag{1}$$

Here $\mathbf{v}$ is the fluid velocity, $\mathbf{x}$ is a position vector, $\mathbf{l}$ is a differencing vector and the average is over time and space. The statistical self-similarity represented by the power-law in l is only valid within the inertial range $l_0 \ll l \ll l_d$; here l_0 is the characteristic macro-scale, and l_d is the dissipation scale at which the cascade terminates. The set of scaling exponents ζ_p is expected to be universal since it characterises the generic cascade process. Heuristic arguments can only give linear relations of ζ_p to p, for example the Kolmogorov 1941 phenomenology [3, 4] predicts $\zeta_p = p/3$. In reality ζ_p depends nonlinearly on p due to the intermittent spatial distribution of eddy activity. A commonly applied class of intermittency correction describes statistical self-similarity in the local rate of dissipation ϵ_l by means of scaling exponents τ_p:

$$\epsilon_l^p \equiv \langle \left(\frac{\nu}{4\pi l^3} \int_0^l \frac{1}{2} \left(\partial_i v_j(x+l',t) + \partial_j v_i(x+l',t) \right)^2 dl'^3 \right)^p \rangle \sim l^{\tau_p} \tag{2}$$

The scaling exponents ζ_p are then inferred by Kolmogorov's refined similarity hypothesis [4] $\zeta_p = \tau_{p/3} + p/3$. To exploit such concepts let us write the equations of incompressible MHD in Elsässer symmetric form:

$$\partial_t z^{\pm} = -z^{\mp}.\nabla z^{\pm} - \nabla\left(p + B^2/2\right) + (\nu/2 + \eta/2)\,\nabla^2 z^{\pm} + (\nu/2 - \eta/2)\,\nabla^2 z^{\mp} \quad (3)$$

Here the Elsässer field variables are $z^{\pm} = v \pm B/\left(\mu_0 \rho\right)$, p is the scalar pressure, ν is kinematic viscosity, η is magnetic diffusivity and ρ is fluid density. The symmetry of Eq.(3) suggests that statistical treatment of $z^{\pm}$ may be more fundamental than separate treatments of v and B. In light of this, structure functions are constructed in terms of Elsässer field variables hereafter

$$S_l^{p(\pm)} = \langle\left(z^{(\pm)}(x+l,t).l/\mid l \mid - z^{(\pm)}(x,t).l/\mid l \mid\right)^p\rangle \sim l^{\zeta_p^{(\pm)}} \quad (4)$$

The scaling exponents $\zeta^{(\pm)}$ were investigated by Biskamp and Müller [2] via direct numerical simulation of the incompressible MHD equations, with a spatial gid 512^3. The simulation is of decaying turbulence with initially equal magnetic and kinetic energy densities and $\nu = \eta$. Since the turbulence decays with time, structure functions are normalised by the total energy in the simulation before time averaging takes place. Direct numerical simulations must resolve the dissipation scale l_d so that energy does not accumulate at large wavenumbers, artificially stunting the cascade. Most of the numerical resolution is therefore used on the dissipation range, whereas it is only on scales much larger than l_d that dissipative effects are negligible, and scaling laws of the type discussed arise. Thus high Reynolds number simulations with an extensive inertial range are currently unavailable. However, the principle of extended self-similarity (ESS) [5] can be used to extend the inertial range scaling laws into the range of length scales that is significantly affected by dissipation but still larger than l_d:

$$S_l^{p(\pm)} \sim S_l^{q(\pm)(\zeta_p/\zeta_q)} \quad (5)$$

Biskamp and Müller extracted the ratios of scaling exponents ζ_p/ζ_3 by this method, and found them to match a variant of the She-Leveque (SL) 1994 model [1], see Fig. 1. In SL models, once the basic fluid scaling is determined, only one more parameter is required - the dimension of the non-spacefilling coherent structures that are most intensely dissipating. The SL model postulated by Biskamp and Müller [2, 6] combines the Kolmogorov fluid scaling with two dimensional most intensely dissipating structures, consistent with the tendency of MHD flows to dissipate through the formation of current sheets. For the application of this model to be theoretically consistent, ESS scaling must be present in the local rate of dissipation. This is investigated in the present work.

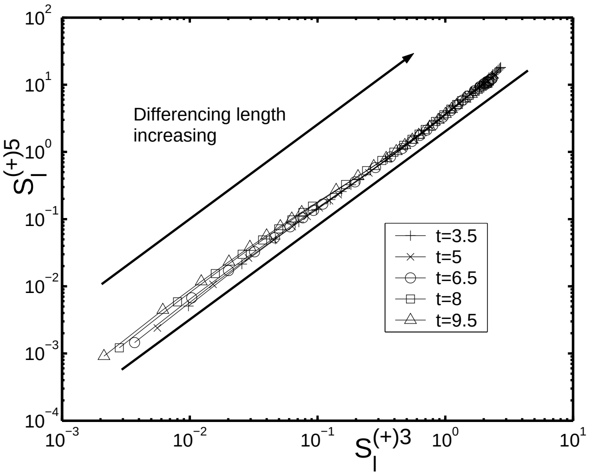

Fig. 1: Extended self-similarity for the Elsässer field variable $z^{(+)}$ (order five against order three), for decaying MHD turbulence where structure functions are normalised by the total energy before time averaging. This normalisation reveals the same underlying scaling for points from different simulation times, as shown. After Biskamp and Müller [2].

2 Results

The gradient squared measure $(\partial_i z_i^{(\pm)})^2$ is used as a proxy for the local rate of dissipation $(\partial_i B_j - \partial_j B_i)^2\,\eta/2 + (\partial_i v_j + \partial_j v_i)^2\,\nu/2$, so that statistical self-similarity implies [4]

$$\chi_l^{p(\pm)} \equiv \langle \left(\frac{1}{l} \int_0^l \left(\partial_i z_i^{(\pm)}(x+l',t) \right)^2 dl' \right)^p \rangle \sim l^{\tau_p^{(\pm)}} \tag{6}$$

and the SL model adopted by Biskamp and Müller predicts

$$\tau_p^{(\pm)} = -2p/3 + 1 - (1/3)^p \tag{7}$$

Normalisation by the spatial average of viscous plus Ohmic rates of dissipation allows time averaging to be performed. Figure 2 shows an example of the ESS and normalisation procedure for $\chi_l^{(+)}$ order five against order three. Statistical self-similarity is recovered at least for smaller values of l. The roll-off from power law behaviour at large l may be due to the finite size of the system, since a more extensive part of the simulation domain is encompassed by the spatial average as l increases in Eq. (6). In Fig. 2 points identified with this roll-off are removed, and ratio of scaling exponents (τ_p/τ_3) is calculated from the remaining points by linear regression. These ratios are shown in Fig. 3. The solid line in Fig. 3 shows the ratio predicted by Eq. (7), in contrast to the dashed line which shows the ratio predicted by the SL theory for hydrodynamic turbulence [1].

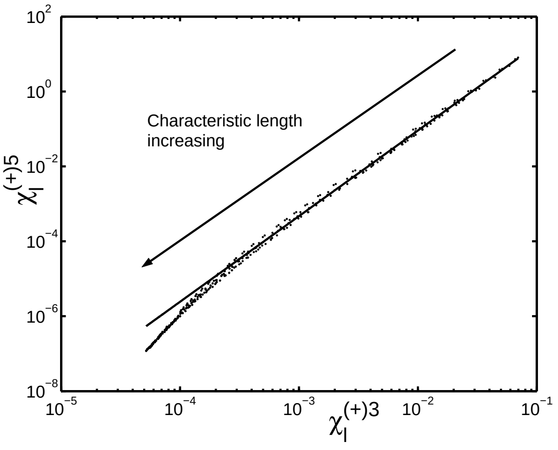

Fig. 2: Extended self-similarity in the Elsässer field variable $z^{(+)}$ gradient squared proxy for the local rate of dissipation (order five against order three). Normalisation by the space averaged local rate of viscous and Ohmic dissipation allows time averaging in spite of the decay process. Deviation from power law scaling at large l is probably a finite size effect. Solid line is the best fit in the linear region.

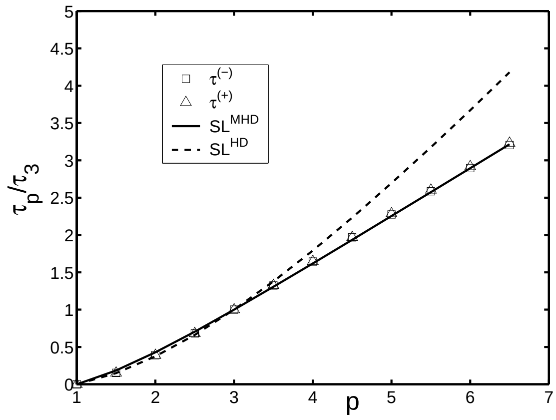

Fig. 3: Ratio of scaling exponents (order p over order three) obtained via extended self-similarity from the Elsässer field gradient squared proxy for the local rate of dissipation. Solid line shows ratios predicted by a She-Leveque theory based on Kolmogorov fluid scaling and sheet-like most intensely dissipating structures. Dashed line shows ratios predicted by hydrodynamic She-Leveque [1].

3 Conclusions

Extended self-similarity is recovered in the gradient squared proxy for the local rate of dissipation of the Elsässer field variables $z^{(\pm)}$ computed by Biskamp and Müller. We believe this is the first time this has been shown for MHD flows. This result supports the application to Elsässer field scaling exponents $\zeta_p^{(\pm)}$ of turbulence theories that require statistical self-similarity in the local rate of dissipation, even when $\zeta_p^{(\pm)}$ are extracted from relatively low Reynolds number flows via ESS. Furthermore the ratio of exponents recovered is that predicted by the SL theory proposed by Biskamp and Müller [2]. This supplies further evidence that the cascade mechanism in MHD turbulence is non-linear random eddy scrambling, with the level of intermittency determined by dissipation through the formation of current sheets.

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