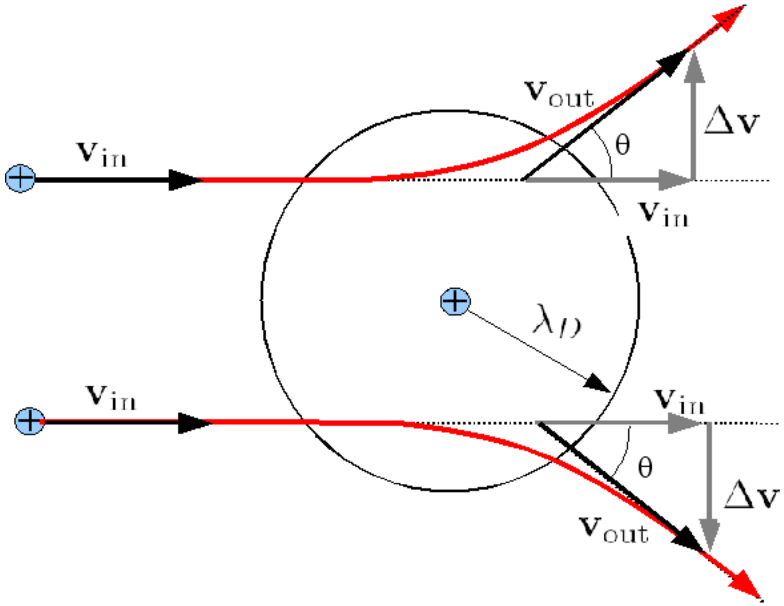
Physics of fusion power

Lecture 13 : Diffusion equation / transport

Many body problem

- The plasma has some 10^{22} particles. No description is possible that allows for the determination of position and velocity of all these particles
- Only averaged quantities can be described.
- The evolution of the averaged velocity is however influenced by a microscopic process : the collisions
- Each collision can have a different outcome depending on the unknown initial conditions



Depending on the (unknown) initial conditions the outcome of a collision can be very different

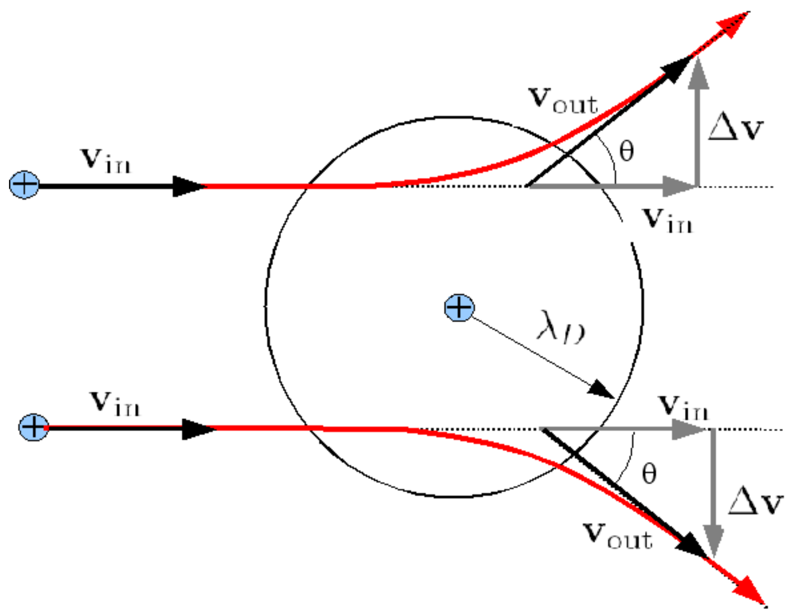
Statistical description

- For the microscopic process one can introduce a probability distribution ψ

$$\Delta P = \psi(w)\Delta(w)$$

- Ψ gives the probability that the position of the particle will change by an amount w in a given time interval τ

$$\int_{-\infty}^{\infty} \psi(w)\mathrm{d}w = \int_{-\infty}^{\infty} \mathrm{d}P = 1$$



Depending on the (unknown) initial conditions the outcome of a collision can be very different

Distribution function

- Since one deals with the probability of a certain change in position it is useful to also describe the particle distribution as a probability

$$\mathrm{d}P = F(x)\mathrm{d}x$$

- The evolution of the distribution is due to the process described by the function ψ. The distribution at time t can be found from the distribution at t – τ, by multiplying the probability of finding a particle at the position x – w with the probability that its position changes by an amount w

$$F(x,t) = \int_{-\infty}^{\infty} \mathrm{d}w\, F(x-w, t-\tau)\psi(w)$$

Evolution equation

$$F(x,t) = \int_{-\infty}^{\infty} \mathrm{d}w\, F(x-w, t-\tau)\psi(w)$$

- Assuming a small time interval t and small step w one can use a Taylor expansion

$$F(x-w, t-\tau) = F(x,t) - w\frac{\partial F}{\partial x}(x,t) + \frac{1}{2}w^2\frac{\partial^2 F}{\partial x^2}(x,t) - \tau\frac{\partial F}{\partial t}(x,t)$$

- This yields the equation

$$F(x,t) = \int_{-\infty}^{\infty} \mathrm{d}w \left[F(x,t) - w\frac{\partial F}{\partial x} + \frac{1}{2}w^2\frac{\partial^2 F}{\partial x^2} - \tau\frac{\partial F}{\partial t} \right] \psi(w)$$

Evolution equation

$$F(x,t) = \int_{-\infty}^{\infty} \mathrm{d}w \left[F(x,t) - w\frac{\partial F}{\partial x} + \frac{1}{2}w^2\frac{\partial^2 F}{\partial x^2} - \tau\frac{\partial F}{\partial t} \right] \psi(w)$$

$$= F(x,t)\int_{-\infty}^{\infty} \mathrm{d}w\, \psi(w)$$

The integral = 1 (integration over all probabilities)

$$-\frac{\partial F}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}w\, w\psi(w)$$

$$+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\int_{-\infty}^{\infty} \mathrm{d}w\, w^2\psi(w)$$

$$-\tau\frac{\partial F}{\partial t}\int_{-\infty}^{\infty} \mathrm{d}w\, \psi(w)$$

The integral = 1 (integration over all probabilities)

Evolution equation

$$\tau \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial x} \int_{\infty}^{\infty} \mathrm{d}w \, w \psi(w) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \int_{-\infty}^{\infty} \mathrm{d}w \, w^2 \psi(w)$$

- With the definitions

$$V = \frac{1}{\tau} \int_{-\infty}^{\infty} \mathrm{d}w \, w \psi(w) \qquad D = \frac{1}{2\tau} \int_{-\infty}^{\infty} \mathrm{d}w \, w^2 \psi(w)$$

- Can be written as a convection diffusion equation

$$\boxed{\frac{\partial F}{\partial t} = -V \frac{\partial F}{\partial x} + D \frac{\partial^2 F}{\partial x^2}}$$

Simple form ….

- Previously in estimating diffusion coefficient we have used a fixed step which could be either positive or negative

$$\psi(w) = \frac{1}{2}\delta(w - \Delta x) + \frac{1}{2}\delta(w + \Delta x)$$

- This directly yields

$$V = \frac{1}{\tau}\int_{-\infty}^{\infty} w\left[\frac{1}{2}\delta(w - \Delta x) + \frac{1}{2}\delta(w + \Delta x)\right] = \frac{1}{2\tau}[\Delta x - \Delta x] \boxed{= 0}$$

$$D = \frac{1}{2\tau}\int_{-\infty}^{\infty} \mathrm{d}w\, w^2\left[\frac{1}{2}\delta(w-\Delta x)+\frac{1}{2}\delta(w+\Delta x)\right] =$$

$$= \frac{1}{2\tau}\left[\frac{1}{2}(\Delta x)^2+\frac{1}{2}(\Delta x)^2\right] \boxed{= \frac{(\Delta x)^2}{2\tau}}$$

Gaussian distribution

- If the probability is Gaussian with half-width σ one can use σ as the typical step length

$$\psi(w) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{w^2}{2\sigma^2}\right]$$

$$V = \frac{1}{\tau}\int_{-\infty}^{\infty} \mathrm{d}w\, w\psi(w) = 0$$

$$D = \frac{1}{2\tau}\int_{-\infty}^{\infty} \mathrm{d}w\, w^2\psi(w) = \frac{\sigma^2}{2\tau}$$

Brief look back at the collisions

- Here the distribution was defined for the angle of the velocity

$$\mathrm{d}P = F(\theta)\mathrm{d}\theta$$

- This distribution satisfies a diffusion equation

$$\frac{\partial F}{\partial t} = D\frac{\partial^2 F}{\partial \theta^2}$$

- With the diffusion coefficient

$$D = \frac{(\Delta\theta)^2}{2\tau}$$

Typical change in the angle of the velocity due to one collision

Typical time on which the collisions take place

Many particles

- For many particles that are independent (uncorrelated) the probability for finding a particle in a certain position is the same for all particles
- For many particles the one particle probability distribution can be thought of as a distribution of density

$$n = \frac{N}{V} F(x)$$

- Consequently,

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}$$

Key things to remember

- Many body problem can only be described in a statistical sense
- Outcome of a microscopic process often depends on unknown initial conditions
- The microscopic processes can be described by a probability distribution for a certain change
- If the time interval is short and the step size is small the time evolution can be cast in a convection / diffusion equation
- The convection is zero for many phenomena, and the diffusion coefficient is proportional to the step size squared over 2 times the typical time

Transport in a homogeneous magnetic field

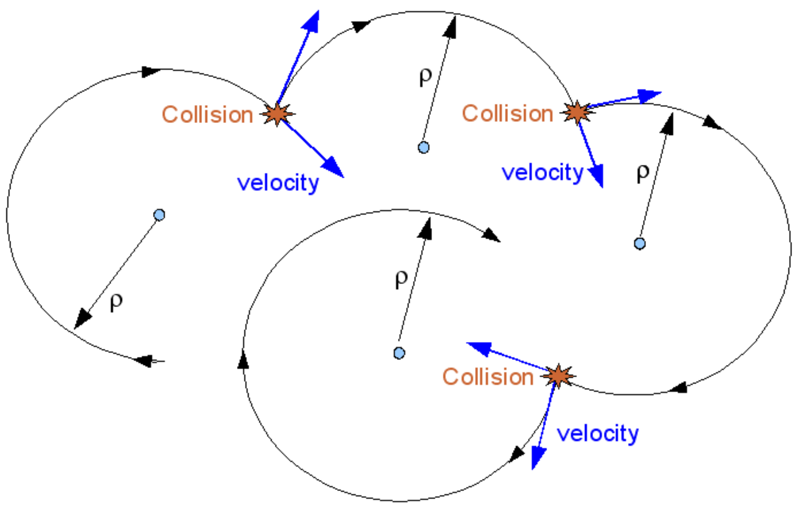
- Particles undergo scattering. The diffusion coefficient

$$D = \frac{(\Delta x)^2}{2\tau}$$

- Typical step size is the Larmor radius, typical time the collision frequency

$$\Delta x = \rho \qquad \frac{1}{\tau} = \nu$$

$$D = \frac{1}{2}\rho^2\nu$$



Collisional scattering leads to a random walk of the particle in space

Transport in a homogeneous magnetic field

- Typical values for a reactor

$$D = \frac{1}{2}\rho^2\nu \qquad \rho = 4\,\mathrm{mm} \qquad \nu = 1000\,\mathrm{s}^{-1}$$

$$D \approx 8 \cdot 10^{-3}\ \mathrm{m}^2/\mathrm{s}$$

- The particles satisfy a diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} \quad \xrightarrow{\text{Rough estimate for } r = a \text{ gives confinement time T}} \quad \frac{n}{T} = D\frac{n}{a^2}$$

$$a = \sqrt{DT} \quad \xrightarrow{\text{For T = 3 s}} \quad a = 15\ \mathrm{cm}$$

Particle orbit in a tokamak

- In a Tokamak the particles drift away form the surface
- The drift velocity of a thermal particle scales as

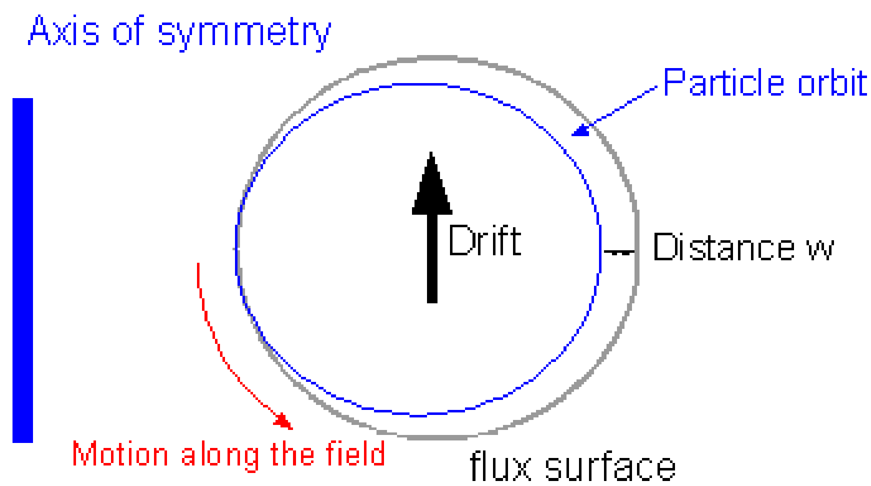
$$v_d = \frac{\rho}{R} v_{th}$$

- The drift away from the surface occurs in a time

$$\tau = \frac{l_t}{v_{th}} = \frac{\pi q R}{v_{th}}$$

- This gives a step size

$$w = v_d \tau = \pi q \rho$$



Due to the combined effect of the motion along the field line and the drift, the particle moves a finite distance (w) from the surface

Particle orbit in a tokamak

- Step size

$$w = v_d \tau = \pi q \rho$$

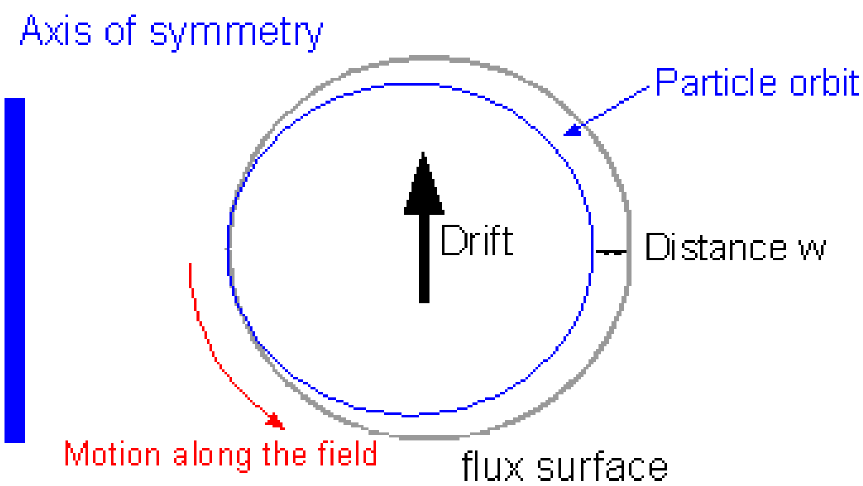
- Yields a diffusion coefficient

$$D = \frac{1}{2}(\pi q)^2 \rho^2 \nu = (\pi q)^2 D_{\mathrm{classical}}$$

$$D < 100 D_{\mathrm{classical}} = 0.8 \ \mathrm{m}^2/\mathrm{s}$$

- Which is still much smaller than the observed

$$D_{\mathrm{experimental}} = 1 - 3 \ \mathrm{m}^2/\mathrm{s}$$



Due to the combined effect of the motion along the field line and the drift, the particle moves a finite distance (w) from the surface

Key things to remember

- Collisional scattering leads to a random walk of the particles in space with a step size of the order of the Larmor radius and a typical time determined by the collision frequency
- The diffusion coefficient of this process is very small
- In a tokamak the drift leads to a large deviation from the flux surface and therefore a larger effective step size
- The diffusion coefficient in a tokamak is much larger than in an homogeneous magnetic field, but still at least a factor 4-5 times smaller than observed