The physics of fusion power

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PREFACE

These lecture notes give a first introduction into the physics processes of importance to fusion research.

You are looking at the first version of this document. Many things can still be improved. I would like to use this opportunity to point out that I am new in the university and this is my first course. I apologize in advance for those things that are unclear, incorrect, too difficult, too easy etc. I hope you will have some patience, and give me feedback such that I can further improve these notes in the future.

It is also important to point out that these notes contain several pictures for which no copy write has been obtained, and that the text follows sometimes the text of other lecture series, presentations, or research papers. The notes do not claim to be original and should not be treated as an individual publication. They are for internal (Warwick University) use only and should not be publicly distributed.

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Contents

Chapter 1

The basics

This chapter serves to give a basic introduction into the subject. Many of the effects are introduced in a 'loose' way. Later chapters should define the physics effects more clearly. So do not despair if you do not grasp everything in one go.

1.1 Nuclear fusion

In these lecture notes fusion will refer to the controlled process in which two light atoms are fused together, generating a heavier atom, with the aim of producing energy. Fusion is a long sought after solution for the world's energy needs, with the first research starting only shortly after the Second World War. In the early days it was thought that the solution was close at hand. Many problems have, however, since then been identified, and we are still a good distance away from a working fusion reactor. For physics students this is not all bad news. The physics of a fusion reactor is an active and attractive area of research, with many unsolved problems.

The key concept behind the release of energy in fusion (and fission) reactions is binding energy. The binding energy is the energy that is released when a nucleus is created from protons and neutrons. Fig. 1.1 shows the binding energy per nuclear particle in an atom as a function of the mass number (A). The greater the binding energy per nucleon in the atom, the greater the atom's stability. Energy can therefore be released when an atom heavier than iron (Fe) is split into two (or more) lighter atoms. This is the reason behind the release of energy in fission reactors, were very heavy atoms (Uranium) are split through the interaction with neutrons. Fusing two nuclei of very small mass, such as hydrogen, will create a more massive nucleus and releases energy. From the figure it can be seen that Helium (with $A = 4$) is particularly stable, and it is therefore of interest to pick a fusion reaction that will have this atom as product.

Fig. 1.2 gives a schematic picture of the potential energy as a function of the distance between two nuclei (in this case Deuterium (D) and Tritium (T) which are both isotopes of Hydrogen (H), i.e. they both have one proton). A fusion reaction between these two nuclei will only occur when the nuclei are very close ($\approx 10^{-15}$ m) such that the strong interaction dominates. For larger distances the electromagnetic interaction dominates and since both nuclei are positive there is a large repelling force. For a fusion reaction to occur, the products must over come the repelling electro-static force or in other words

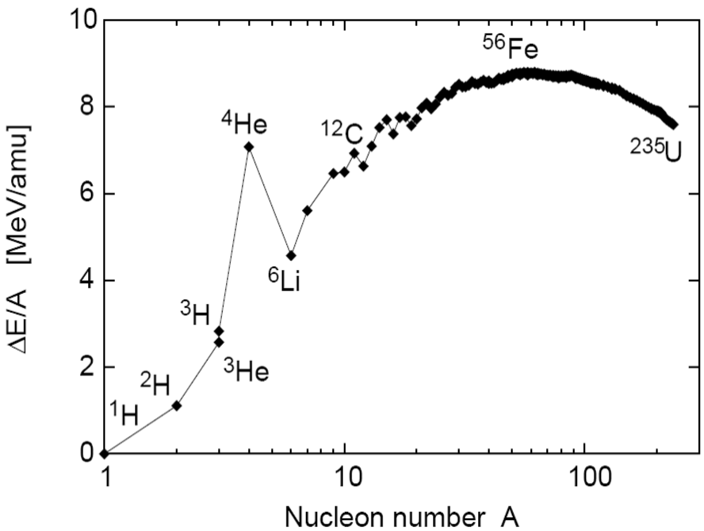

Figure 1.1: Binding energy per nucleon as a function of the mass number A

have to overcome the Coulomb barrier. This observation brings out the key problem of fusion. In order for a reaction to occur the nuclei must have a large initial energy, such that they can approach each other closely and overcome the barrier. Quantum mechanics teaches us that they do not have to have exactly the energy associated with the maximum in the potential (which is indeed very high). A reaction can occur when the particles tunnel through the barrier. But the probability for a reaction to occur decreases dramatically with lower energy. The whole problem of fusion research is the generation of the conditions under which a sufficient amount of fusion reactions occur. We will see that this is not an easy task.

Often considered fusion reactions are

$$ {}^2_1\mathrm{D} + {}^3_1\mathrm{T} \rightarrow {}^4_2\mathrm{He} + {}^1_0\mathrm{n} + 17.6 \ \mathrm{MeV} \tag{1.1} $$

$$ {}^2_1\mathrm{D} + {}^2_1\mathrm{D} \rightarrow {}^3_2\mathrm{He} + {}^1_0\mathrm{n} + 3.7 \ \mathrm{MeV} \tag{1.2} $$

$$ {}^2_1\mathrm{D} + {}^2_1\mathrm{D} \rightarrow {}^3_1\mathrm{T} + {}^1_1\mathrm{H} + 4.03 \ \mathrm{MeV} \tag{1.3} $$

$$ {}^2_1\mathrm{D} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + {}^1_1\mathrm{H} + 18.3 \ \mathrm{MeV} \tag{1.4} $$

In the reactions above the superscripts and subscripts on the left of every symbol give the total number of particles in the nucleus and the number of protons, respectively. In the reactions given above also the two isotopes of Helium (He) appear, as well as the neutron (n). Note that not every reaction must always produce the same products, since there are two possible outcomes for the fusion reaction between two Deuterium nuclei. In fact for the Deuterium-Deuterium reaction the probability of each reaction is roughly equal. On the right hand side of the reaction also the total energy release is indicated. This

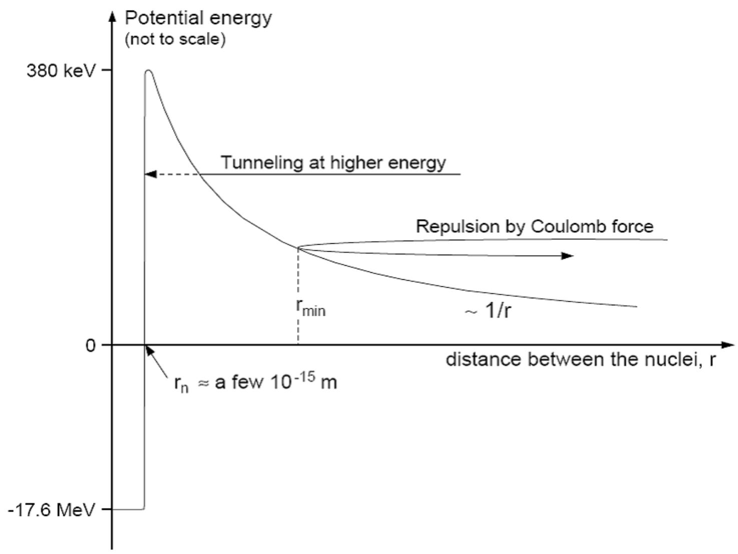

Figure 1.2: Potential energy (schematic representation) of a Deuterium-Tritium reaction as a function of their relative distance

energy release follows from the mass difference between the nuclei on the left and the right hand side of the equation. For the first reaction, for instance

$$m_D = (2 - 0.000994)m_H \qquad m_T = (3 - 0.006284)m_H$$

$$m_{He} = (4 - 0.027404)m_H \qquad m_n = (1 + 0.001378)m_H$$

Here m refers to the mass (which is obtained from high precision measurements), and the subscript indicates the species. The difference in mass (mass before (Deuterium plus Tritium) minus after (4-Helium plus the neutron)) is therefore $0.0187 m_H$. Using this in the most famous formula of physics yields an energy E

$$E = mc^2 = 0.0187 m_H c^2 = 2.8184 \cdot 10^{-12} \text{ J} = 17.56 \text{ MeV} \tag{1.5}$$

Here we have used the mass of Hydrogen $m_H = 1.6727 \cdot 10^{-27}$ kg, and the speed of light $c = 2.9979 \cdot 10^8$ m/s. The last step in the equation above gives the energy in units of electron volts, which is the energy a particle with the elementary charge e gains when it moves over a potential difference of 1 Volt

$$1 \text{ eV} = 1.6022 \cdot 10^{-19} \text{ J}, \tag{1.6}$$

one kilo-electron-Volt is 1 keV = 1000 eV, and one Mega-electron-Volt is 1 MeV = 10^6 eV. The energies of interest are often in the range of eV to MeV, the reason for which these units are popular in the field. You will find them consistently used in these lecture notes. Furthermore, we will use the same unit as a unit of temperature (T). An energy can be associated with the temperature through the use of the

Boltzmann constant k. This energy is then expressed in units of eV, i.e.

$$T = kT_k/e \text{ (eV)} = 8.617 \cdot 10^{-5} T_k \text{ (eV)} \tag{1.7}$$

where T_k is the temperature in Kelvin. In other words 1 eV corresponds to a temperature of 11605 K.

Here it is also worthwhile to note that the energy released in a fusion reaction greatly exceeds that of a typical chemical reaction (eV range). Burning 1 kg of a Deuterium / Tritium mixture would lead to a energy release of $3.4 \cdot 10^{14}$ J. This can be translated in 3.9 Giga Watt during a period of 24 hours. A large reactor would therefore burn an amount of fuel in the range of one kg per day. The energy released in the fusion reaction is released as kinetic energy of the final products. This energy is not distributed equally among the products since both energy as well as momentum need to be conserved. For the reaction that yields two products A and B, the conservation laws yield a set of two equations in the centre of mass frame

$$\frac{1}{2} m_{\rm A} v_{\rm A}^2 + \frac{1}{2} m_{\rm B} v_{\rm B}^2 = E_{fus} \tag{1.8}$$

$$m_{\rm A} v_{\rm A} + m_{\rm B} v_{\rm B} = 0 \tag{1.9}$$

where the initial kinetic energy has been neglected (it is usually much smaller, see below), v denotes the velocity of the particles, and E_{fus} is the energy released in the fusion reaction. These equations can easily be solved to obtain

$$E_A = \frac{1}{2} m_{\rm A} v_{\rm A}^2 = \frac{m_B}{m_A + m_B} E_{\rm fus} \qquad E_B = \frac{1}{2} m_{\rm B} v_{\rm B}^2 = \frac{m_A}{m_A + m_B} E_{\rm fus} \tag{1.10}$$

For the first fusion reaction involving Deuterium and Tritium this means the following. Since the Helium atom is roughly four times heavier than the neutron, it will acquire only one fifth of the energy released in the fusion reaction, i.e. 3.5 MeV, whereas the neutron will obtain 80%, i.e. 14.1 MeV.

The likelihood of a fusion reaction is expressed in terms of a cross-section. A cross section has the dimension of an area, and one can roughly think of it as the size of the particles. A snooker ball, for instance, would have a cross section πr^2 where r is the radius of the ball. For a fusion reaction, the reaction itself involves quantum mechanics, and one can only work with probabilities. The cross section therefore is an 'average' size, where the averaging is over the probability for the reaction to occur. The cross sections for fusion reactions are available from measurements, and are parameterized (i.e. interpolated) using theory based models. Fig. 1.3 shows the cross sections for three of the fusion reactions described above. It is of interest to operate a fusion reactor at the lowest possible energy, and it can be seen that there is a large difference in cross section for these reactions in the lower energy range (up to 10 keV). The present plan for the first fusion reactor is, therefore, based on the first of the fusion reactions given above, i.e. the one between Deuterium and Tritium. Current experiments aimed at the investigation of the Deuterium Tritium fusion reaction, however, often use Deuterium only. This is because Tritium is radioactive, whereas Deuterium is a stable isotope of hydrogen.Furthermore, the neutron yield of a Deuterium mixture is much smaller because far less fusion reactions occur at the same density and temperature. Both reasons allow for an experiment with only a small amount of radio-activity. Of course, many of the physics processes can be studied without the fusion reactions themselves.

In Fig. 2.1 the cross section of the fusion reactions is compared with the cross section of a Coulomb collision. We will discuss this cross section in more detail later. Here it is sufficient to understand that this is the cross section for a 90 degree angle scattering elastic collision (after the collision the trajectory of

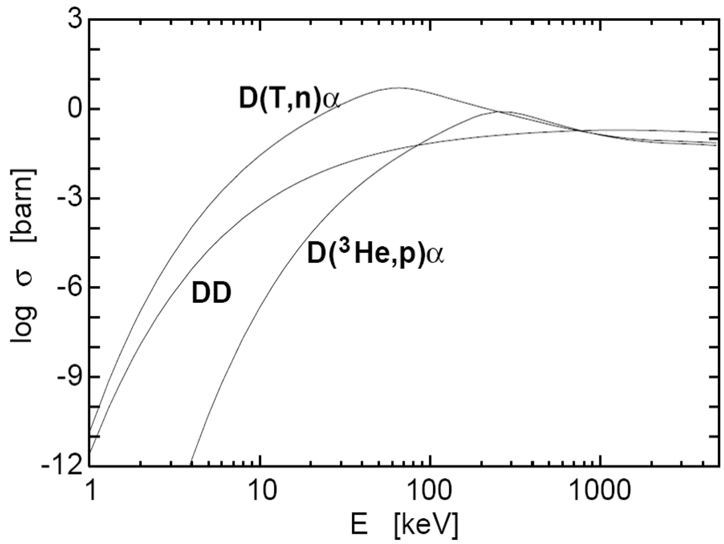

Figure 1.3: Cross sections of the various fusion reactions (1 barn = 10^{-28} m^{-3}). as a function of the centre of mass energy. The curve labeled DD represents the sum of both possible reactions

the colliding particle is changed by 90 degrees) that is generated solely by the Coulomb interaction. Due to the high repulsive force and its long range, the probability of a scattering collision is for the energies below the maximum of the potential barrier much larger than the actual fusion reaction. This puts large constraints on the efficient generation of fusion power. Fusion reactor concepts in which a particle is lost after being scattered would generate only a small amount of fusion reactions since most of the collisions would in fact be elastic. A successful concept, therefore, must allow for many elastic collisions without the particles being lost such that enough fusion reactions can occur. The many elastic collisions would then lead to a distribution not too far from thermodynamic equilibrium, i.e. the distribution of particles in velocity space can be described by the Maxwell's distribution function $F_m(v)$ with a well defined temperature (T)

$$F_M(v) = \frac{n}{(2\pi T/m)^{3/2}} \exp\left[-\frac{mv^2}{2T}\right] \quad \text{or} \quad F_m(E) = \sqrt{\frac{4E}{\pi T^3}} \exp\left[-\frac{E}{T}\right], \tag{1.11}$$

where n is the particle density, v is the particle velocity, m is the particle mass, and $E = mv^2/2$ is the particle energy. (Compare this with your textbook Maxwellian and note that the Boltzmann constant is missing. In plasma physics temperature is treated as a unit of energy, i.e. $k_B T \rightarrow T$, a convention which will be used from now on)

The cross sections given in Fig. 1.3 are for a given energy of the deuterium particle (on a non-moving target). When the particles are distributed in velocity according to the Maxwell distribution we must build a weighted average. Fig. 1.5 gives an illustration of such a calculation, which involves both the cross section, the relative velocity v of the colliding products and the distribution function which determines

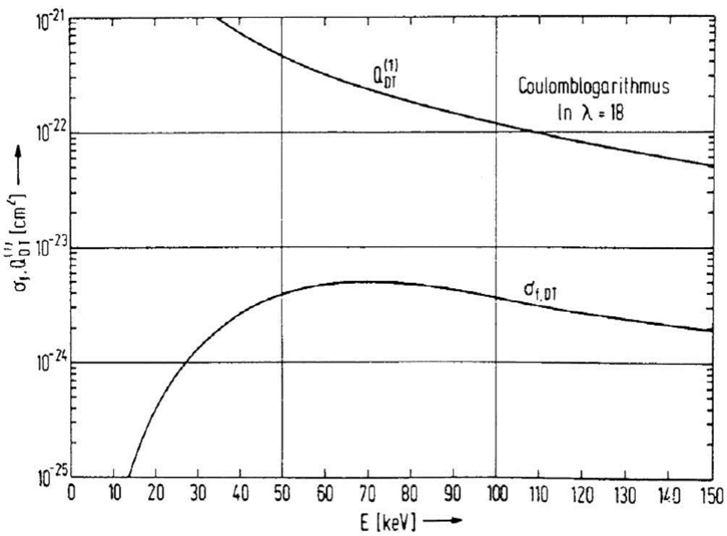

Figure 1.4: Comparison between the DT fusion cross section and the cross section for Coulomb collisions

how many particles have a certain energy E. Note that the product vF_M peaks at an energy much below that of the maximum cross section σ. The main contribution to the fusion reactions comes from an energy somewhere in between the temperature (roughly the energy with the largest particle density) and the energy for which the maximum cross section is reached. There are many more particles at low energy, but the cross section is too small to give a significant contribution. At very high energies the cross section is large, but there are simply too few particles to generate a significant amount of fusion reactions. The result of such a calculation (which we do not give here, essentially because it is boring) is the average $\langle \sigma v \rangle$, the brackets denote the integration over the Maxwell distribution and v denotes the relative velocity. This average has the dimension m^3/s, and the total amount of fusion power generated by the reaction between two species A and B can be written in the form

$$P_{\mathrm{fusion}} = n_A n_B \langle \sigma v \rangle_{AB} E_{AB} V, \tag{1.12}$$

where n is the density, E_{AB} is the energy released in one fusion reaction, and V is the volume. The quantity $\langle \sigma v \rangle$ is given for various fusion reactions in Fig, 1.6. Note that, in contrast to the figure of the cross section as a function of energy, the averaged cross section at 10 keV is within an order of magnitude of its maximum.

Although the temperatures at which the current fusion reactors are supposed to operate (5-12 keV) are low when compared with the energy at which the maximum cross section is obtained, it must be noted that energies of 10 keV corresponds to a temperature of roughly 100 million Kelvin. At these temperatures matter is fully ionized and thus the matter is in the plasma state. Because of the large energies, the charged particles would leave any vessel in a very small amount of time. An idea of the

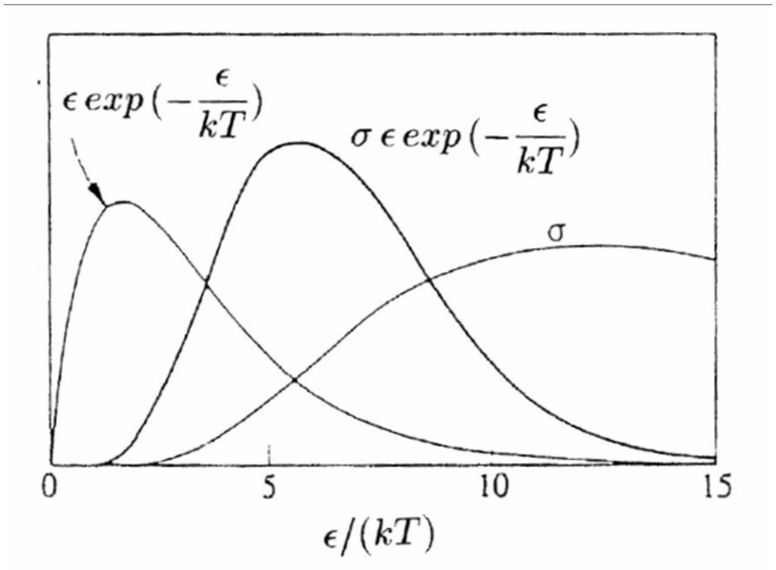

Figure 1.5: Calculation of $\langle \sigma v \rangle$ through an average of the cross section over the Maxwellian distribution

timescales involved can be obtained by considering the thermal velocity v_{th} of a 10 keV particle

$$v_{th} = \sqrt{2T/m} \tag{1.13}$$

This velocity is around 10^6 m/s for a Deuterium nuclei, and $6 \cdot 10^7$ m/s for an electron. If the reactor would have a typical size of 10 m, all the material would be lost to the wall in 10 micro-seconds. This of course demands a special scheme of operation in which one either tries to prevent the rapid loss, or finds a way to generate enough fusion reactions within this short time.

The natural abundance of Deuterium is one in 6700. There is enough water in the ocean to provide energy for $3 \cdot 10^{11}$ years at the current rate of energy consumption. This number is larger than the age of the universe !!! Apart from its availability, Deuterium is also very cheaply obtainable. Calculating the price of electricity solely on the basis of the cost of Deuterium, would lead to a drop of 10^3 in your electricity bill. For those concepts that rely on the fusion of Deuterium and Tritium, it is the Tritium that is more problematic. Tritium is unstable with a half age of 12.3 years. There is virtually no naturally resource of Tritium. It, however, can be bred from Lithium through the reactions

$${}^6_3\mathrm{Li} + {}^1_0n \rightarrow {}^4_2\mathrm{He} + {}^3_1\mathrm{T} + 4.8\ \mathrm{MeV} \tag{1.14}$$

$${}^7_3\mathrm{Li} + {}^1_0n \rightarrow {}^4_2\mathrm{He} + {}^3_1\mathrm{T} + {}^1_0n - 2.5\ \mathrm{MeV} \tag{1.15}$$

The isotope ^{6}Li has a natural abundance of 7.4%, and is the principal component for the breading of Tritium. Note, that one can use the neutron released in the fusion reactions to breed Tritium. The availability of Lithium on land is sufficient for at least 1000 if not 30000 years of energy production, and the cost per kWh would be even smaller than that of Deuterium. If the availability of Lithium in the

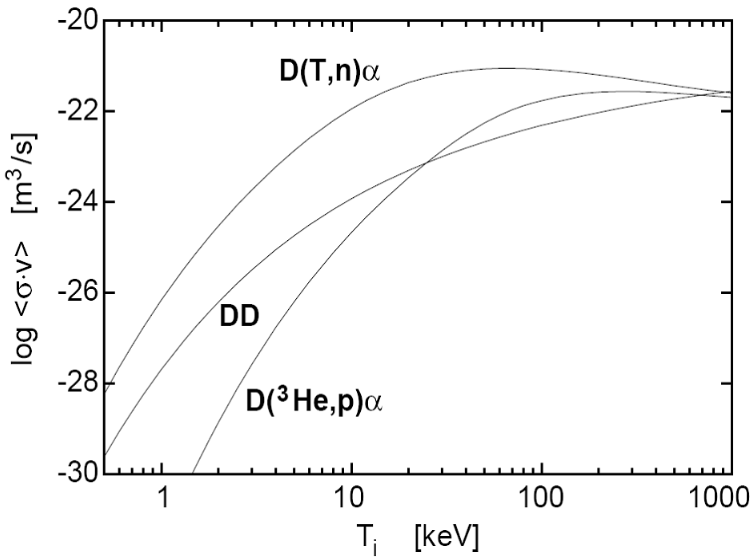

Figure 1.6: The reaction product $\langle \sigma v \rangle$ as a function of the temperature for various fusion reactions

oceans is included the estimation is that there is enough fuel for $3 \cdot 10^7$ years. In conclusion there is an amazing amount of fuel available at virtually no cost.

1.2 Why fusion?

The energy production through the use of nuclear fusion would have several advantages over current power plants.

- There is a large amount of fuel available, at a very low price.
- Fusion is CO_2 neutral.
- It would yield only a small quantity of high level radio active waste.
- There is no risk of uncontrolled energy release.
- The fuel is available in all locations of the earth. Fusion is of interest especially for those regions that do not have access to other natural resources.
- There is only a small threat to non-proliferation of weapon material

The first two points on this list are directly related to the energy problem, and the problems associated with the current use of fuel. Indeed a working fusion reactor sounds like Christmas the whole year through. The cost argument, although very popular, is only partly true. The cost of electricity generated by a fusion reactor will not be due to the cost of the fuel, but rather the cost of the reactor itself, which is predicted to have a rather limited lifetime due to the forces on the materials and the constant neutron flux on them. In fact it turns out to be rather hard to predict the costs at present, since one has to make assumptions on a solution that does not yet exist. The size of the machine strongly determines the final cost of electricity. Present studies indicate a price around the current price or somewhat above. These studies are, however, not all that reliable. Nevertheless, fusion presents a possible solution to a very important problem. Not only could it provide for a virtually unlimited source of energy, it is also CO_2 neutral, i.e. it will not contribute to the greenhouse effect.

The third and fourth point form more or less a contrast with the solution of a fission reactor. The products of a fission reaction are radio-active with rather long half-ages and, therefore, there is a natural problem with radio-active waste. The fusion reactions currently envisioned, involve only stable nuclei or nuclei with a rather short half-age. From the reactions themselves there is therefore virtually no radio-active waste. It should be noted though that the neutrons released in the fusion reactions will also interact with the material walls. The nuclear reactions resulting from the neutron bombardment, will generate a certain amount of radio-active material. There is however some freedom in choosing the materials that surround the actual core of the reactor. It is currently envisioned that the materials can be chosen such that the amount of long-lived radio active materials produced is small. Studies speak of a period of 100-200 years in which the reactor itself must be shielded. Such a period appears rather feasible. It can be seen from the fusion reactions given above that no products are generated that directly catalyze a new reaction. There is of course the breeding reaction of Tritium, but this reaction is envisioned to take place separated from the fusion reactions, i.e. no Lithium will be injected into the plasma. Therefore, there is no chain reaction that can lead to an uncontrolled release of energy. Also, it turns out that the actual fuel stored in a reactor is relatively small. Too small to lead to any serious accidents even if it is burned all in one go. A fusion reactor is therefore intrinsically safe.

Finally point five and six are more of a political nature. The localization of natural resources is a potential treat to international stability.

1.3 Two approaches to fusion

In the previous section, we saw that in order to have fusion reactions, the charged particles involved must have very large energies and velocities. This lead to the fact that these particles cannot be contained in an ordinary vessel but for a very small amount of time. There are two different lines of research that try to deal with this problem. One is based on the rapid compression, and heating of a solid fuel pellet through the use of laser or particle beams. In this approach one tries to obtain a sufficient amount of fusion reactions before the material flies apart, hence the name, inertial confinement fusion (ICF). The other approach, known as magnetic confinement, uses a magnetic field to confine the plasma. The particle trajectories will be discussed later in much detail. Here, it is sufficient to point out that the Lorentz force connected with the magnetic field will prevent charged particles from moving over large distances perpendicular to the field. Charged particles will gyrate around the magnetic field (see Fig. 1.7) with a typical size of the orbit, known as the Larmor radius (ρ) roughly equal to (see later in these lecture notes)

$$\rho = \frac{mv_{th}}{ZeB} \tag{1.16}$$

where Z is the charge number of the particle. For a magnetic field of 5 Tesla, and a temperature of 10 keV the Larmor radius would is 4 mm for the Deuterium nuclei and 0.07 mm for the electrons. Such length scales are small compared to the size of any feasible reactor, and one can conclude that the plasma can be very efficiently confined by the magnetic field. Of course, the problem is more complicated since the particles are free to move along the magnetic field lines, and the currents in the plasma can, furthermore, influence the magnetic field. But the discussion of these phenomena is postponed to later chapters.

We note here that only charged particles are confined by the magnetic field. The neutrons generated by the fusion reactions do not interact with the field. They leave the plasma on a time-scale of micro-seconds and, consequently, lead to a large flux on the material walls. The charged particles which are generated by the fusion reactions are confined by the magnetic field. Here, it must be noted that their energy will exceed the plasma temperature by a larger factor. The Larmor radius of the Helium nuclei (the alpha particle) for the parameters described above would be as large as 5.4 cm. There is a good reason to build a reactor such, that these particles are well confined by the magnetic field. They can provide heating in the plasma. One speaks of an ignited reactor, when this heating by the fusion products is sufficient to maintain the temperature at the level needed for the fusion to occur.

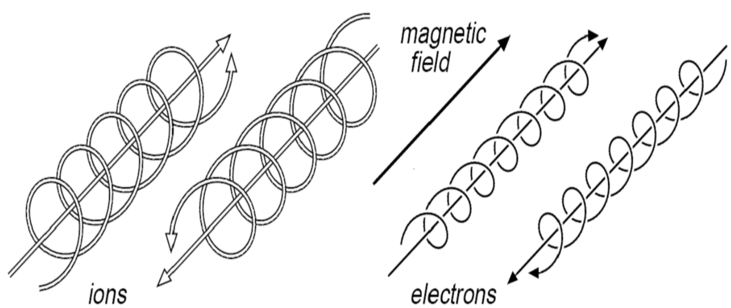

Figure 1.7: Gyro-motion of the ions and electrons in an magnetic field

Both of the lines of research have achieved considerable advances in recent years. The most developed solution, however, is the one of magnetic confinement, and it is this approach on which these lecture notes will concentrate.

1.4 Lawson / Ignition criterion

In the early days of fusion (the 50s) some physicists had the wildest ideas of generating energy through fusion reactions. Examples include two colliding beams, or beam target reactions. It was Lawson (an engineer) who pointed out that it might be a good idea if a fusion reactor would produce more fusion energy compared with the energy that one has to put in to keep it going. The criterion he derived is a limit for the product of density and what is know as the confinement time. We will extend it here to include the dependence on the temperature. The Lawson criterion is also known as the break-even condition. The latter is defined to be the condition for which the total generated fusion power is equal to the power that heats the plasma. Note that in this condition it is not specified how the plasma is heated. For some fusion reactions not all the fusion power is available for plasma heating (if a neutron is involved it will not heat the plasma). For these reactions the plasma must be heated with some additional external power even if the break-even condition is reached. In general, therefore, a more difficult to satisfy criterion is the ignition condition, in which the heating power is assumed to be that fraction of the total fusion power available for plasma heating. We will work this criterion out for the case of magnetic confinement, but equal arguments apply to the case of inertial confinement fusion.

Consider a plasma containing a mixture of Deuterium and Tritium. It will be discussed in the next chapter that a plasma must be neutral, i.e. the charge density is zero. From this it follows

$$n_D + n_T = n_e = n \tag{1.17}$$

where n_e is the electron density, which will, for simplicity, be denoted as n. If the Deuterium and Tritium density (n_D and n_T) are equal one obtains a total fusion power per unit of volume

$$P_{\rm fusion} = \frac{1}{4} n^2 \langle \sigma v \rangle_{DT} E_{\rm fus} V \tag{1.18}$$

This power is to be compared with the power that heats the plasma $P_{\rm heat}$. This comparison is somewhat complicated since the fusion power depends on plasma parameters like density and temperature. The latter enters through the cross section. Fortunately, in the temperature range in which one wants to operate a reactor (5-15 keV) the cross section can be well approximated by

$$\langle \sigma v \rangle_{DT} \approx 1.1 \cdot 10^{-24} T_k^2 \ \ {\rm m}^3/{\rm s} \tag{1.19}$$

where the temperature T_k is the temperature in keV. Substituting the energy release per fusion reaction ($E_{\rm fus} = 17.56$ MeV) yields

$$P_{\rm Fusion} = 7.7 n_{20}^2 T_k^2 V \ \ {\rm kW} \tag{1.20}$$

Where n_{20} is the density in units of 10^{20} m^{-3}. (I apologize for these engineer type formulas. The proper treatment of units is lost after the approximation of the fusion cross section, and so I decided to also express the density in units that, when used for a fusion reactor, yield typical values around 1). In the equation above $P_{\rm Fusion}$ is the total fusion power, i.e. integrated over the plasma volume V. In general the density and temperature will not be uniform over the volume, and the equation above is an approximation of the integral. The density and temperature in this equation should therefore be understood as averaged (over the volume) values. Of course the approach is not exact, but we are interested in an order of magnitude estimate.

The equation above makes clear that the fusion power scales as the plasma pressure squared. This plasma pressure depends on density and temperature. The latter will be influenced by how much power is available to heat the plasma. In order to address the pressure as a function of the heating power we introduce the typical time (τ_E) on which the plasma loses energy (W)

$$\frac{\partial W}{\partial t} = -\frac{W}{\tau_E} + P_{\rm heat} \tag{1.21}$$

The time τ_E is known as the (energy) confinement time. It describes, in a very rough way, all kinds of physics processes through which the plasma loses energy. The plasma stored energy W must be obtained through an integral over the density and temperature profiles. Since we work with an estimate here, we will not perform the integral, but rather use averaged values of density and temperature. The stored energy can then be expressed as

$$W = \left[\frac{3}{2} n_D T_D + \frac{3}{2} n_T T_T + \frac{3}{2} n_e T_e \right] V \approx 3 n T V \tag{1.22}$$

In a steady state ($\partial W/\partial t = 0$) the heating power will be balanced by the loss of energy, which allows one to express the averaged pressure as a function of the heating power

$$nT = \frac{1}{3V} P_{\text{heat}} \tau_E \tag{1.23}$$

This equation for the pressure can then be used to derive the equation for the ratio of total fusion power P_F and total heating power of the plasma

$$\frac{P_{\text{Fusion}}}{P_{\text{heat}}} = 0.16 n_{20} T_k \tau_E \tag{1.24}$$

where τ_E is in seconds.

The break even condition can now be calculated. For break even we assume $P_{\text{Fusion}}/P_{\text{heat}} = 1$. The equation above then directly gives

$$n_{20} T_k \tau_E > 6 \ \text{Break} - \text{even} \tag{1.25}$$

The condition for ignition of the plasma can be derived from the same formula. Here, it must be assumed that there is no external heating power. Since only the alpha particles heat the plasma $P_{\text{heat}} = P_{\text{Fusion}}/5$ and one directly arrives at

$$n_{20} T_k \tau_E > 30 \ \text{Ignition} \tag{1.26}$$

The most advanced concept for magnetic fusion, the tokamak, which will be discussed in detail below, can not operate at arbitrary density. There is a density limit that will be briefly touched upon later. For the time being it is sufficient to take a typical number of $n = 10^{20}$ m^{-3} ($n_{20} = 1$). Assuming, furthermore, an averaged temperature of 10 keV, one arrives at a confinement time of at least 3 s in order for the plasma to ignited. Loosely speaking one could say that each particle should remain in the plasma for at least 3 seconds before it is lost to the wall. This time might not seem extraordinarily long, but one must remember that the thermal velocity is extremely large. When the ignition criterion is satisfied the electrons move over a distance of $1.8 \cdot 10^8$ meters before it is lost from the plasma. This is roughly 5 times around the earth at the equator. The orbit of a particle in a reactor should be such that it does not touch the wall before it can move over this distance. It should be clear that one demands a nearly perfect confinement of the particles, and it might be clear that this is indeed one of the problems.

The product $nT\tau_E$ has become the measure of the progress in fusion. The projected reactor values of the individual parameter (density, temperature, and confinement time) have all been achieved, but never all at the same time. Fig. 1.8 plots the values of the product for different machines, and as a function of time. Discharges on the Joint European Torus (JET) experiment have reached values close to break even. The future experiment ITER should reach values close to ignition.

1.5 What do I have to know for the exam ??

The exam will be on physics issues. You will not get any questions related to how much Deuterium there is in the ocean. At maximum a problem will specify how much Deuterium there is and ask you to work out how much energy production it relates. Of course you are expected to be able to calculate the released energy in a fusion reaction using $E = mc^2$ (with the masses given), and you are expected to know the definitions as well as the derivations of the break-even and ignition conditions.

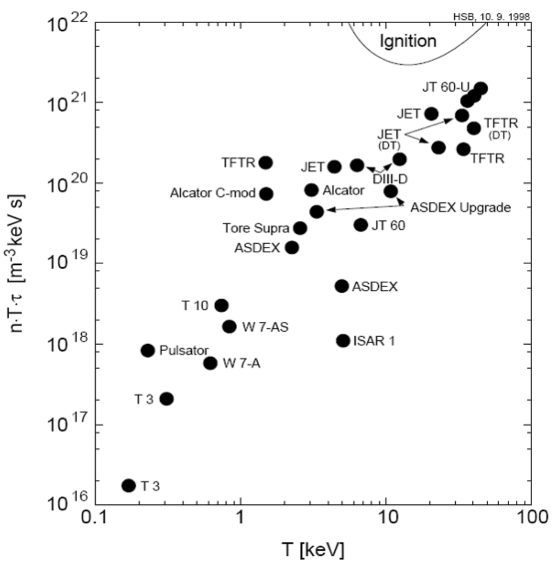

Figure 1.8: The $nT\tau_E$ product achieved in various experiments as a function of the temperature

Chapter 2

Some plasma physics

2.1 Quasi-neutrality

Due to the large temperatures connected with fusion, the hydrogen mixture will be fully ionized. This state is known as a plasma, i.e. a mixture of nuclei and free electrons (although the exact definition of a plasma is somewhat more precise). The key property of a plasma is that of quasi-neutrality. To a very high degree of accuracy the electron density is such that its charge exactly balances the charge of the ions.

$$\sum_s Z_s n_s = 0, \tag{2.1}$$

where the sum is over all species. One can make oneself clear that this property must hold by considering the opposite example: Separate 10^{20} electrons and ions by a distance of 1 m, and calculate the force between them. You will end up with an enormous force which, under any normal conditions, can not be balanced. The separation between the electrons and the ions must, therefore, be very small, and to a good approximation their densities are equal.

One can make the statements made above more precise by considering a homogeneous mixture of electrons and ions, with density n and temperature T. We place a test particle with charge Q in this homogeneous mixture, and calculate the electric potential ϕ as a function of the distance r to the charge. The electric potential follows from Poisson's equation

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = \frac{\rho}{\epsilon_0} = \frac{Q}{\epsilon_0}\delta(\mathbf{r}), \tag{2.2}$$

where δ is the delta function. Of course, this problem is solved in spherical coordinates

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \phi}{\partial r}\right]. \tag{2.3}$$

The potential can now be directly obtained by integrating over a sphere with radius r using Gauss's theorem

$$\int \mathrm{d}^3\mathbf{r}\, \nabla^2 \phi = 4\pi r^2 \frac{\partial \phi}{\partial r} = -\frac{Q}{\epsilon_0}, \tag{2.4}$$

or

$$\phi(r) = \frac{Q}{4\pi r \epsilon_0}, \tag{2.5}$$

where the constant of integration has been chosen such that ϕ goes to zero for $r \rightarrow \infty$. (The mathematically oriented student might have noticed that I did not treat the problem of the $r = 0$ limit in the integral very carefully. This student can verify that the final solution is indeed a solution of the initial equation) For the case of a plasma, however, one has to consider also the charge density of the plasma surrounding the charge Q. If the charge is positive, then ions are repelled while electrons are attracted. This gives a negative charge density around the positive charge, which shields (part of the) charge Q. The potential can be expected to fall off stronger with the distance to the charge compared with the vacuum case. To model the response of the electrons and ions to the charge we will assume a Boltzmann distribution

$$n_e = n_0 \exp\left[\frac{e\phi}{T}\right] \qquad n_i = n_0 \exp\left[-\frac{e\phi}{T}\right]. \tag{2.6}$$

where n_0 is the density far from the charge Q. We will furthermore assume that the kinetic energy is much larger than the potential energy $e\phi \ll T$. This is known as the ideal plasma approximation. It allows one to expand the exponents retaining only the first order term of the Taylor expansion. From this one obtains the charge density as a function of the potential

$$\rho = en_i - en_e = -\frac{2e^2 n_0}{T}\phi, \tag{2.7}$$

And the equation for the potential is

$$-\nabla^2 \phi = \frac{1}{\epsilon_0}\left[Q\delta(\mathbf{r}) - \frac{2e^2 n_0}{T}\phi\right]. \tag{2.8}$$

To solve this equation one can substitute a modified vacuum solution

$$\phi = g(r)\frac{Q}{4\pi r \epsilon_0} = g\phi_v, \tag{2.9}$$

where g is an unknown function and ϕ_v is the vacuum solution. Writing out the Poisson equation then yields

$$\phi_v \frac{\partial^2 g}{\partial r^2} + 2\left[\frac{\partial \phi_v}{\partial r}\frac{\partial g}{\partial r} + \frac{\phi_v}{r}\frac{\partial g}{\partial r}\right] + g\nabla^2\phi_v = -\frac{1}{\epsilon_0}\left[Q\delta(\mathbf{r}) - \frac{2e^2 n_0}{r} g\phi_v\right] \tag{2.10}$$

This equation can be simplified through the use of

$$\frac{\partial \phi_v}{\partial r} = -\frac{Q}{4\pi\epsilon_0 r^2} = -\frac{\phi_v}{r} \tag{2.11}$$

and, furthermore through the assumption that $g(r = 0) = 1$ such that

$$g\nabla^2\phi_v = -\frac{1}{\epsilon_0} Q\delta(\mathbf{r}) \tag{2.12}$$

leading to an equation for g.

$$\frac{\partial^2 g}{\partial r^2} = \frac{2e^2 n_0}{\epsilon_0 T} g. \tag{2.13}$$

With the solution

$$g = \exp\left[-\frac{r}{\lambda_D}\right] \qquad \lambda_D = \sqrt{\frac{\epsilon_0 T}{2e^2 n_0}} \rightarrow \phi = \frac{Q}{4\pi r \epsilon_0} \exp\left[-\frac{r}{\lambda_D}\right] \tag{2.14}$$

The vacuum potential is therefore screened over a distance λ_D known as the Debye length. For typical parameters this length is of the order of 10^{-5} m, i.e. much smaller than the system size. In fact, this is one of the criterion use to determine if one can speak of a plasma. The system size L has to be larger than the Debye length

$$L \gg \lambda_D \tag{2.15}$$

The other condition for a classical plasma is that, although the Debye sphere is small, there are a large number of particles in the Debye sphere. This condition can be formulated as

$$\frac{4\pi}{3} n_0 \lambda_D^3 \gg 1 \tag{2.16}$$

In fact the criterion above is necessary for our derivation of the Debye length to be valid. We have used a continuum description of the electron and ion density. Such a description is not valid if the number of particles in the sphere is smaller than or of the order of one. (Note that the latter condition is satisfied for the description of free electrons in some metals).

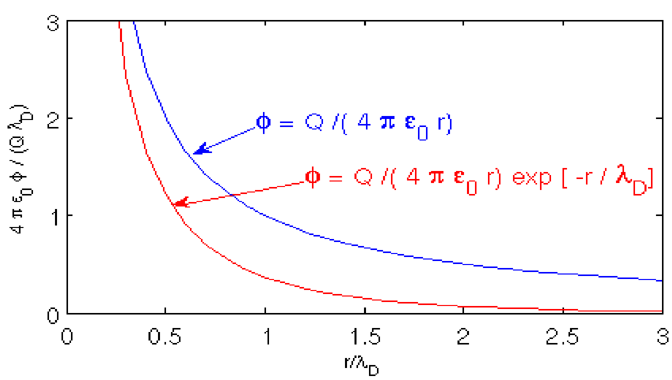

Figure 2.1: Plot of the normalized potential $(4\pi\epsilon_0\phi/(Q\lambda_D))$ as a function of the normalized distance (r/λ_D) for the vacuum as well as the plasma solution. Note fall of for the plasma case.

The screening of the charges over very small distances, should not be misunderstood by assuming that there can not be any electric field in the plasma. The problem above contains only one charge and no other external forces on the plasma. In fact if one would exert a force on all the electrons in one direction and on the ions in the opposite direction, they would move apart and an electric field would build up that balances this external force. The statement of quasi-neutrality is more the statement that a very small deviation $n_e - n_i \ll n_e$ would be sufficient to build up this electric field. To a good approximation one can still use $n_e = n_i$ for the calculation of all physical effects. It should be noted that the electric field in this approximation can no longer be calculated from the Poisson equation, since it would predict zero

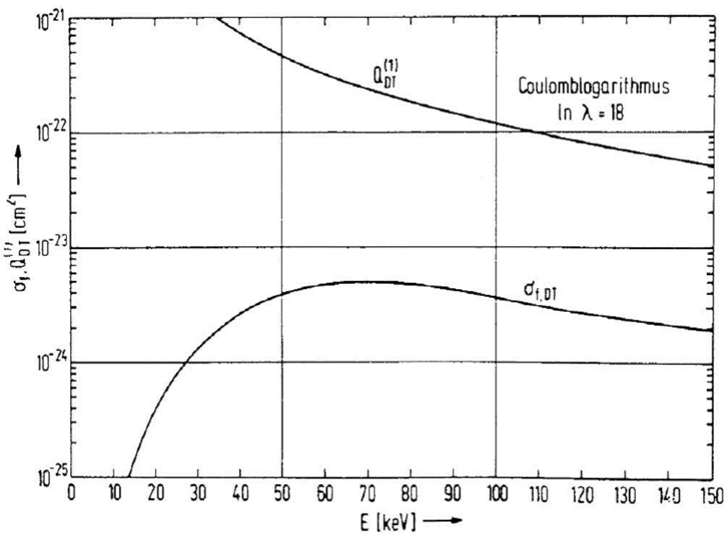

Figure 2.2: The potential of a charge in vacuum, and with a surrounding plasma.

electric field. The Poisson equation can therefore not be used, and the electric field is directly calculated from the force balance. This yields a closed set of equations, that can give a consistent description of the plasma phenomena.

Note too that the quasi-neutrality approximation can be used for most but not all phenomena. To see the limitations we investigate the continuity equation of electrons and ions

$$\frac{\partial n}{\partial t} + \nabla \cdot [n\mathbf{u}] = 0 \tag{2.17}$$

where $\mathbf{u}$ is the fluid velocity. For all of you who have not seen this equation before, it is simply the statement of particle conservation. Consider a velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x})$. The particle flux ($\Gamma$) through a small area $\mathrm{d}^2\mathbf{A}$ is

$$\Gamma = n\mathbf{u} \cdot \mathrm{d}^2\mathbf{A} \tag{2.18}$$

(note that the dimension is s^{-1}, i.e. number of particles per second). Taking a volume (V) enclosed by an area (A) the total number of particles (N) changes according to

$$\frac{\partial N}{\partial t} = -\int \mathrm{d}^2\mathbf{A} \cdot n\mathbf{u}. \tag{2.19}$$

The minus sign appears in this equation because the norm of the surface (given by the vector $\mathrm{d}^2\mathbf{A}$) points outward. i.e. a negative flux leads to an increase in the particle number. The total number of particles can be calculated by integrating the particle density ($n(\mathbf{x})$) over the volume

$$N = \int \mathrm{d}^3\mathbf{V}\, n \tag{2.20}$$

Then applying Gauss's theorem (divergence theorem) to the right hand side of the equation one obtains

$$\int \mathrm{d}^3\mathbf{V} \left[\frac{\partial n}{\partial t} + \nabla \cdot [n\mathbf{u}] \right] = 0 \tag{2.21}$$

Since this equation must be valid for any arbitrary chosen volume, it follows that equation 2.17 must be satisfied. Note too that in a fully ionized plasma the continuity equation is satisfied for each species separately. It has the same form for both ions and the electrons.

Multiplying the continuity equation with the charge e for the ions and $-e$ for the electrons yields, after summation, the equation for the continuity of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{2.22}$$

where $\mathbf{J}$ is the current

$$\mathbf{J} = e n_i \mathbf{u}_i - e n_e \mathbf{u}_e \tag{2.23}$$

with the subscripts indicating the species ions and electrons. The quasi-neutrality condition ($\rho = 0$), therefore, demands that the current in the plasma is divergence free

$$\nabla \cdot \mathbf{J} = 0. \tag{2.24}$$

This condition can easily be understood since a non-divergence free current would lead to an accumulation of charge. The divergence free nature of the current on the other hand has direct consequences for the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{2.25}$$

Taking the divergence of this equation leaves only the displacement current since the divergence of a curl is zero, and the current is divergence free

$$\frac{\partial \nabla \cdot \mathbf{E}}{\partial t} = 0. \tag{2.26}$$

For a consistent application of the quasi-neutrality condition one must therefore neglect the displacement current, and the equation that determines the magnetic field becomes

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{2.27}$$

Neglecting the displacement current is equivalent to neglecting electro-magnetic wave propagation, or in other words the magnetic field responds instantaneously to the current. High frequency phenomena can, therefore, not be treated within the quasi-neutrality approximation.

We are left with a set of equivalent statements

- quasi-neutrality applies
- Scale length of the phenomena is larger than the Debye length
- The current is divergence free
- The displacement current can be neglected

Can a high frequency electromagnetic wave propagate in a plasma? Of course it can, only the quasi-neutrality approximation does not apply in this case.

2.2 Force on the plasma

Here we will derive the equation for the force on the plasma in an heuristic way. We start by simply considering the force on a particle ($\mathbf{F}_i$, we will denote individual particles by the index i) due to the electromagnetic fields

$$\mathbf{F}_i = Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}], \tag{2.28}$$

where Z_s is the charge number (the index s refers to the species) and $\mathbf{v}_i$ is the velocity of the particle. Consider a small volume with a number of particles (electrons or ions) $N_s = n_s \mathrm{d}^3 V$. The force per unit volume on species s is

$$\mathbf{F}_s = \frac{1}{V} \sum_{i=1}^{N_s} Z_s e[\mathbf{E} + \mathbf{v}_i \times \mathbf{B}] = Z_s e n_s [\mathbf{E} + \mathbf{u}_s \times \mathbf{B}], \tag{2.29}$$

where we have introduced the mean velocity $\mathbf{u}_s$, which in some sense we used already in the continuity equation.

$$\mathbf{u}_s = \frac{1}{N_s} \sum_{i=1}^{N_s} \mathbf{v}_i. \tag{2.30}$$

Summing over all species and using quasi-neutrality will eliminate the electric field, while the velocities of the species will combine to the current ($\mathbf{J}$). The force density on the neutral plasma can be then written as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}, \tag{2.31}$$

i.e. is given by the Lorentz force. For a hot plasma one has to consider as well the thermal motion in the total force. This can be added in a lowest order approximation as the pressure gradient. We, therefore, arrive at a total force

$$\mathbf{F} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{2.32}$$

where p is the total pressure

$$p = \sum_s n_s T_s \tag{2.33}$$

The magnetic field and the current are related over the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.34}$$

We can use this to reformulate the Lorentz force into an expression that contains only the magnetic field

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left[\frac{B^2}{2\mu_0}\right] + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}, \tag{2.35}$$

where in the last step the vector identity B.9 has been used. The Lorentz force can be thought to be a combination of the magnetic field pressure ($\nabla B^2/2\mu_0$) and a magnetic field tension $(\mathbf{B} \cdot \nabla)\mathbf{B}/\mu_0$. The latter force will try to prevent the bending of the field lines. Of course both forces can act as restoring force for wave propagation. The magnetic pressure is connected with the so called compressional Alven wave, whereas the tension is connected with the so called shear Alfven wave. The total force on the plasma can then be written as

$$\mathbf{F} = -\nabla \left[p + \frac{B^2}{2\mu_0}\right] + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}. \tag{2.36}$$

There is an important parameter, simply known as 'plasma beta' (β), connected with the ratio of the plasma pressure to the magnetic field pressure

$$\beta = \frac{p}{B^2/2\mu_0} \tag{2.37}$$

This parameter can generally be seen as an efficiency parameter. The fusion power scales roughly with the pressure squared, whereas the magnetic field must be supplied from the outside and is a very large cost factor for a reactor. The quantity therefore roughly measures the ratio of the produced power and the price needed to generate that power. It should also be clear that the plasma wants to expand through the pressure gradient. It is the magnetic field that should confine the plasma, and it is easy to guess that in order to do so it must be sufficiently strong. We will see later that there is no plasma confinement if $\beta > 1$, and that several reactor concepts operate well below this boundary.

If we take one step back and consider the equilibrium condition (i.e. no net force) for a single species

$$\mathbf{F}_s = -\nabla p_s + Z_s e n_s[\mathbf{E} + \mathbf{u}_s \times \mathbf{B}] = 0 \tag{2.38}$$

We can solve this equation to obtain the velocity perpendicular to the field lines $\mathbf{u}_\perp$.

$$\mathbf{u}_{\perp s} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mathbf{B} \times \nabla p_s}{Z_s e n_s B^2} \tag{2.39}$$

The perpendicular electric field generates a motion across the field lines, known as the $E \times B$ (say E-cross-B) velocity, that is independent of the charge and mass of the species. The pressure gradient, on the other hand, leads to a motion perpendicular to the field which depends on the charge. When summed over all species this contribution generates a current rather than a collective motion of all the species.

We are now ready to formulate the equations of Magneto-Hydro-Dynamics (MHD in short). As stated before, the material presented in this section is not a formal derivation, we merely try to motivate the set of equations as physically meaningful. For the more lazy students it is also useful to note that we will hardly ever use the full set of equations (this should indeed be considered as an excuse to skip to the next section). Whenever we deal with MHD we will mostly discuss the force on the plasma derived above. In MHD a single fluid description is assumed which is characterized by the mass density ρ (Apologies for the confusion with the charge density, which is also denoted by ρ. We will however hardly ever use the charge density, and it is custom to denote the mass density by ρ.), the velocity $\mathbf{u}$, and the pressure p. The mass density satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.40}$$

The change in the velocity is given by the force on the plasma

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = -\nabla p + \mathbf{J} \times \mathbf{B}. \tag{2.41}$$

This equation contains the pressure, and we therefore need an equation for the evolution of the pressure. In MHD it is assumed that the pressure changes according to the law of adiabatic compression (This is a little hard to see, I suggest you simply accept it)

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \cdot \left(\frac{5}{2} p \mathbf{v} \right) = \mathbf{v} \cdot \nabla p \tag{2.42}$$

The electro-magnetic fields in these equations need to be obtained through the solution of the Maxwell equations

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.43}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.44}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.45}$$

Finally, the link between the current and the electric field is given by a generalized Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} \tag{2.46}$$

where η is the resistivity. In this equation one can also recognize that a part of the electric field is connected with a global ExB motion perpendicular to the field. For zero resistivity one speaks of ideal MHD, whereas the case of finite resistivity is referred to as resistive MHD.

2.3 The virial theorem

After deriving that a plasma is essentially quasi-neutral, and the force on the plasma can be expressed as the sum of a pressure (both kinetic (p) as well as magnetic) and a magnetic field line tension, we can ask the simply question: what is a necessary condition for an equilibrium? Of course, it would be nice if the solution was cheap. Since currents inside the plasma can generate the magnetic field, one can ask oneself if it is possible to make an equilibrium with just currents inside the plasma. The answer is no. We will show this below

For the plasma to be in equilibrium the force on the plasma must vanish, and therefore

$$-\nabla \left[\frac{B^2}{2\mu_0} \right] + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = \nabla p \tag{2.47}$$

This equation was derived before as a property of the Lorentz force. Since we will deal with tensors it is more easy to adapt the writing in which we explicitly denote the components. In this notation the second term can be written in the form

$$(\mathbf{B} \cdot \nabla) \mathbf{B} = B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} [B_\alpha B_\beta] - B_\beta \frac{\partial B_\alpha}{\partial x_\alpha}, \tag{2.48}$$

where we have used the Einstein summation convention, which implies that there is a sum over every index that appears more than once

$$B_\alpha \frac{\partial B_\beta}{\partial x_\alpha} \equiv \sum_{\alpha=1}^{3} B_\alpha \frac{\partial B_\beta}{\partial x_\alpha}. \tag{2.49}$$

Since

$$\frac{\partial B_\alpha}{\partial x_\alpha} = \nabla \cdot \mathbf{B} = 0 \tag{2.50}$$

we find

$$\frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = \frac{\partial}{\partial x_\alpha} \left[\frac{B_\alpha B_\beta}{\mu_0} \right] \tag{2.51}$$

This can be understood as the divergence of a tensor. The other terms in the equation can be formulated to have the same form, i.e.

$$\nabla p = \frac{\partial p}{\partial x_\beta} = \delta_{\alpha\beta}\frac{\partial p}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left[p\delta_{\alpha\beta}\right] \tag{2.52}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, i.e. $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$. The 'zero force' equation then takes the form

$$\frac{\partial}{\partial x_\alpha}\left[\left(\frac{B^2}{2\mu_0} + p\right)\delta_{\alpha\beta} - \frac{B_\alpha B_\beta}{\mu_0}\right] = \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = 0. \tag{2.53}$$

The equation above gives the tensor relation of the force balance.

So far we have not solved anything. We have merely written the equation in a different form. The form given above though is convenient for constructing a contradiction. We will assume that this equation is satisfied for a bounded plasma which has a finite current and pressure. We will then construct an integral of the equation above and find a contradiction which in fact simply means that our original assumption of an equilibrium can not be satisfied. The equation above still depends on the index β (not on the index α because of the summation), i.e. the expression (2.53) is in fact three equations for $\beta = 1, 2, 3$. A scalar quantity can be build by multiplying with x_β (note that a summation over β is implied). The resulting equation is then integrated over the volume

$$\int d^3\mathbf{V} x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = 0. \tag{2.54}$$

Taking $\beta = 1$ one finds

$$x_1 \frac{\partial S_{\alpha 1}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[x_1 S_{\alpha 1}] - \frac{\partial x_1}{\partial x_\alpha} S_{\alpha 1} = \frac{\partial}{\partial x_\alpha}[x_1 S_{\alpha 1}] - S_{11} \tag{2.55}$$

which makes more clear the result

$$x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}[x_\beta S_{\alpha\beta}] - \sum_\alpha S_{\alpha\alpha} \tag{2.56}$$

The first term contains the divergence of a vector, and can, when substituted in the integral over the volume, be transformed to a surface term using the theorem of Gauss

$$\int d^3\mathbf{V} x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = \int \mathrm{d}^2\mathbf{S}\, x_\beta S_{\alpha\beta} n_\alpha + \sum_\alpha \int d^3\mathbf{V} S_{\alpha\alpha} = 0, \tag{2.57}$$

where $\mathbf{n}$ is the normal of the surface $\mathbf{S}$.

If we consider a plasma of finite size with the currents in the plasma generating the magnetic field then, since there are no magnetic mono-poles, the slowest decay of the magnetic field strength with the distance from the plasma is that of the dipole

$$B \propto 1/r^3 \tag{2.58}$$

And since the pressure outside the plasma is zero one readily finds that the surface term decays as $\mathrm{d}^2 SxB^2 \propto 1/r^3$ and goes to zero if we take the limit of an infinite volume. In this limit we then obtain

$$\int \mathrm{d}^3\mathbf{V}\, x_\beta \frac{\partial S_{\alpha\beta}}{\partial x_\alpha} = -\sum_\alpha \int \mathrm{d}^3\mathbf{V}\, S_{\alpha\alpha} = 0. \tag{2.59}$$

However

$$\sum_\alpha S_{\alpha\alpha} = 3\left[\frac{B^2}{2\mu_0} + p\right] - \frac{B_1^2 + B_2^2 + B_3^2}{\mu_0} = \frac{B^2}{2\mu_0} + 3p.$$

So we find

$$-\int d^3\mathbf{V}\left[3p + \frac{B^2}{8\pi}\right] < 0$$

The integrand is always positive, and the integral on the left hand side can therefore not be zero. Hence the solution that we have been looking for can not exist. This does not mean that an equilibrium magnetic field can not be generated. We have assumed only the MHD equation to apply, such that if we generate a magnetic field with a set of coils and a power supply the derivation above does not hold. It merely states that an equilibrium will not be generated by the plasma alone. Unfortunately, we will have to impose the field from the outside. In our experiments on earth plasmas are confined using magnetic field coils, for the objects in space it is the gravitational force that keeps a plasma together.

There is, or better speaking there was, a company that was going to produce fusion energy through lightning explosions generating a confined plasma. The derivation above shows that the failure of this company to make a profit is indeed in agreement with the laws of physics. If you have read this section you would (most probably) not buy any of the stocks this company issued (although they are really cheap now).

2.4 Flux conservation

Consider a metal ring. When we try to change the magnetic flux through the ring an electric field is generated in the ring (Faraday's law) which will drive a current. The current is directed such that it generates a magnetic field that will oppose the initial field, and will try to maintain the flux through the ring. If the metal is a perfect conductor the flux will be exactly conserved. So far first year electrodynamics, but let us now consider the case of a plasma.

A plasma is a bit like a metal. The electrons can, similar to a metal, move freely through a sea of ions. Collisions of course generate a friction between the electrons and ions when they have a relative velocity, but in a thin and hot plasma the collisions are not very strong, and the plasma is, to a first approximation, a perfectly conducting fluid. Every curve inside the plasma can then be thought of as a metal ring, and the flux through the area enclosed by the curve will be exactly conserved. There is one difference with respect to the metal ring though. The metal ring is rigid, whereas the fluid can move. This means that the flux is carried by the fluid. The flux through any surface is conserved, when the surface moves with the fluid.

Fig. 2.3 shows the schematic picture of flux conservation. The flux through the blue area is proportional to the number of field lines that cross this area. When the fluid moves and the area is displaced the flux must be conserved, and the same amount of field lines must cross the area. Since this applies to all the possible areas it follows that the magnetic field is dragged with the fluid. This picture is, therefore, a very powerful way to think of the dynamics of the fluid embedded in a magnetic field.

Below we will give a more formal derivation of the statement made above. Let us start by deriving how the area changes. A small area can be represented by two vectors

$$\Delta\mathbf{S} = \mathbf{e}_1 \times \mathbf{e}_2 \tag{2.60}$$

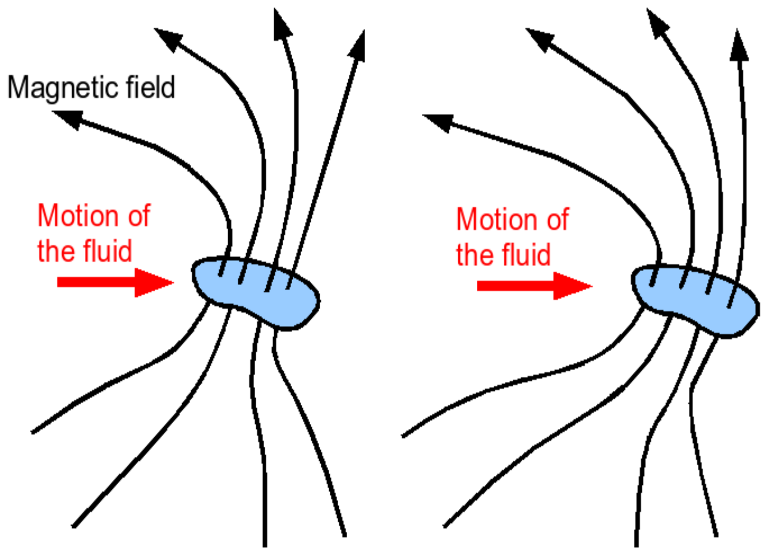

Figure 2.3: Schematic picture of flux conservation

The cross product gives both the size as well as the normal to the surface. Our problem is then reduced to deriving how a length element is changed through the motion of the fluid. Let the length element $\mathbf{l}$ be represented as

$$\mathbf{l} = \mathbf{r}_2 - \mathbf{r}_1 \tag{2.61}$$

In a small time interval δt this length element changes by

$$\delta\mathbf{l} = \delta\mathbf{r}_2 - \delta\mathbf{r}_1 = \delta t(\mathbf{v}(\mathbf{r}_2) - \mathbf{v}(\mathbf{r}_1)) = \delta t\frac{\partial\mathbf{v}}{\partial\mathbf{x}} \cdot (\mathbf{r}_2 - \mathbf{r}_1) \tag{2.62}$$

This equation can be written in the form

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = (\mathbf{l} \cdot \nabla)\mathbf{v} \tag{2.63}$$

and is obviously valid only in the limit of zero length vectors $\mathbf{l}$.

Now let us turn to the equation for the magnetic field. In ideal MHD the equation for the evolution of the magnetic field can be found by substituting $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ in Faraday's law

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{2.64}$$

Using one of the vector identities one can rewrite this equation in the form

$$\frac{\partial\mathbf{B}}{\partial t} = (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{u} \tag{2.65}$$

where we have used that the magnetic field is divergence free. Using that the time derivative in the co-moving frame (denoted by $\mathrm{d}/\mathrm{d}t$) is related to the time derivative in the Laboratory frame (denoted by $\partial/\partial t$) through

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla) \tag{2.66}$$

one can write the equation for the change of the magnetic field in the frame that moves with the fluid as

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B}\cdot\nabla)\mathbf{u} - \mathbf{B}\nabla\cdot\mathbf{u} \tag{2.67}$$

Now we take the inner product of this equation with the (small) surface area. We have to realize that this area is also a function of time, and therefore

$$\frac{\mathrm{d}(\Delta\mathbf{S}\cdot\mathbf{B})}{\mathrm{d}t} - \mathbf{B}\cdot\frac{\mathrm{d}\Delta\mathbf{S}}{\mathrm{d}t} = \Delta\mathbf{S}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u} - \Delta\mathbf{S}\cdot\mathbf{B}\nabla\cdot\mathbf{u} \tag{2.68}$$

The first term on the left hand side is obviously the magnetic flux through a small area that is conducted by the fluid. If our statement with which we have started this section is correct this term must vanish. At this point we will choose the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to be mutually orthogonal and to be perpendicular to the magnetic field. We introduce a third unit vector $\mathbf{e}_3$ parallel to the magnetic field and such that

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \tag{2.69}$$

With these assumptions we can easily work out the terms in the equations

$$-\mathbf{B}\cdot\frac{\mathrm{d}\Delta\mathbf{S}}{\mathrm{d}t} = B\mathbf{e}_3\cdot((\mathbf{e}_1\cdot\nabla)\mathbf{u}\times\mathbf{e}_2 + \mathbf{e}_1\times(\mathbf{e}_2\cdot\nabla)\mathbf{u}) = B(\mathbf{e}_1\cdot(\mathbf{e}_1\cdot\nabla)\mathbf{u} + \mathbf{e}_2\cdot(\mathbf{e}_2\cdot\nabla)\mathbf{u}) \tag{2.70}$$

where we have used the vector identity

$$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \tag{2.71}$$

$$\Delta\mathbf{S}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u} = B\mathbf{e}_3\cdot(\mathbf{e}_3\cdot\nabla)\mathbf{u} \tag{2.72}$$

$$-\Delta\mathbf{S}\cdot\mathbf{B}\nabla\cdot\mathbf{u} = -B\nabla\cdot\mathbf{u} \tag{2.73}$$

Substituting these equations in the original equation one sees that the last term is indeed canceled by the other three, leaving

$$\frac{\mathrm{d}(\Delta\mathbf{S}\cdot\mathbf{B})}{\mathrm{d}t} = 0 \tag{2.74}$$

The magnetic flux is frozen into the fluid.

This result can help us in understanding how the plasma behaves. In many applications of MHD one can picture what happens through cartoons in which one draws a number of field lines and then deforms them through a thought of motion of the fluid. The density of the field lines is then proportional to the magnetic field.

2.5 What do I have to know for the exam ??

Many of the derivations of this chapter are not part of the exam. You will not be asked to derive the charge shielding, the virial theorem, or the formal derivation of the flux conservation. Of course, you do have to understand the concept of quasi-neutrality, and apply it without thinking. The force on the plasma plays a central role in many physics phenomena, and it is suggested that you study it carefully. Finally, although you do not have to know the derivation of flux conservation, it is assumed that you understand the concept and can apply it.

Chapter 3

Cylindrical equilibriums

As pointed out before, it is impossible to confine a plasma, with a density and temperature such that it would generate a sufficient amount of fusion reactions, using only a material wall. Without a magnetic field there are three major problems: (1) the energy confinement time would be unrealistically small (2) the material walls will have to withstand the plasma pressure and (3) the wall would have to withstand a large heat flux due to the large flux of energetic particles to the wall. To confine a hot plasma in a vessel and to isolate it from the walls one, therefore, uses a magnetic field.

The magnetic field leads to a gyration of the charged particles around the field lines which prevents them from moving over large distances perpendicular to the field. This confinement of particles, reduces the heat fluxes to the walls. Furthermore, we have seen in the previous chapter that the magnetic field provides a force on the plasma that can balance the pressure gradient. The magnetic field, therefore, allows for a pressure gradient to exist within the plasma, and the plasma pressure close to the wall can be small.

The equilibrium is in general described by the force balance

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{3.1}$$

that is, the pressure gradient is balanced by the Lorentz force. The simplest equilibriums that can be considered are cylindrically symmetric, and are named after the direction of the current. They are the Θ-pinch (current wound around the cylinder). Z-pinch (current in the z-direction of the cylinder), and the screw pinch (a combination of the Θ and Z pinches). In this chapter, these three equilibriums will be described.

The study of these equilibriums is not entirely academic since they were once studied experimentally within the framework of fusion research. Although it will become clear that they do not present a reasonable concept for a fusion reactor, they allowed for the study of the plasma state, and were particularly attractive in the early days when methods of plasma heating were not as well developed as they are today. Our interest here is, however, not historically motivated. The more simple cylindrical geometry allows for the introduction of several concepts in the most easy way.

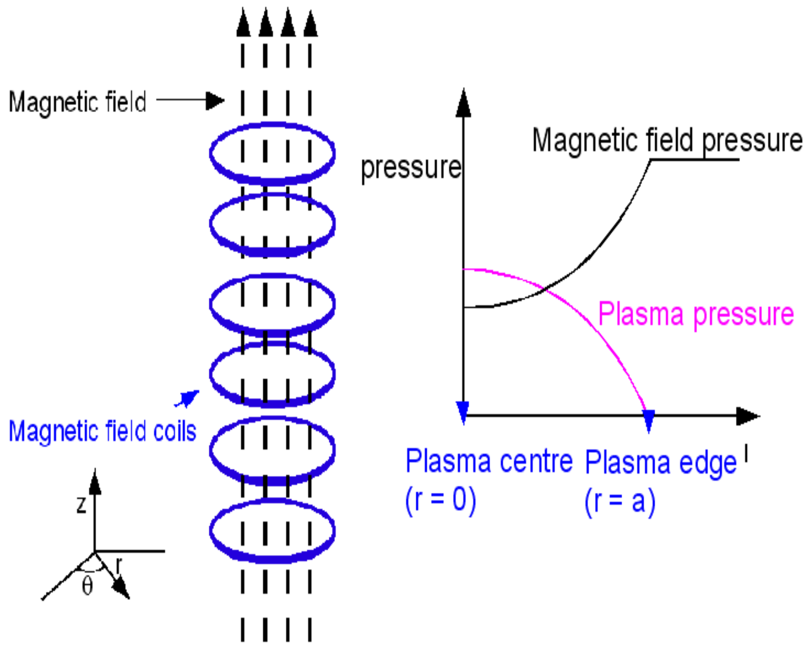

Figure 3.1: Left : Schematic picture of the Θ-pinch geometry. Right : typical pressure and magnetic field strength profiles. Cylindrical coordinates are used with the z-axis along the magnetic field and $r = 0$ corresponding to the centre of the cylindrical plasma.

3.1 The Θ-pinch

The first equilibrium discussed is the Θ-pinch. It is cylindrically symmetric, and the equilibrium is provided by an external magnetic field in the z-direction.

$$\mathbf{B} = Be_z, \tag{3.2}$$

as shown in Fig. 3.1 The equilibrium can most easily be discussed eliminating the current from the force balance using Ampere's law. As shown in the previous chapter

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) - \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} = 0. \tag{3.3}$$

The magnetic field lies in the z direction. Because of the symmetry no quantity depends on the $z-$coordinate, and the field tension term vanishes

$$-\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0} = -\frac{1}{\mu_0} B_z \frac{\partial B_z}{\partial z} = 0 \tag{3.4}$$

and the equilibrium equation is

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=0 \tag{3.5}$$

or

$$p(r)+\frac{B(r)^2}{2\mu_0}=\text{constant}. \tag{3.6}$$

Assume the plasma to be cylindrical with a radius $r=a$. In general the pressure can be assumed maximum in the centre ($r=0$) and it decreases toward the edge. Outside the plasma $r>a$ the pressure is assumed zero. An applied magnetic field through the use of the circular field coils is constant in magnitude outside the plasma. But the equation above shows that if the kinetic pressure p is rising toward the centre of the plasma the magnetic field strength must decrease, i.e. the plasma is diamagnetic. Furthermore, since the equation above can be satisfied for any pressure profile shape it follows that an equilibrium exists for any profile. Typical profiles are shown in Fig. 3.1.

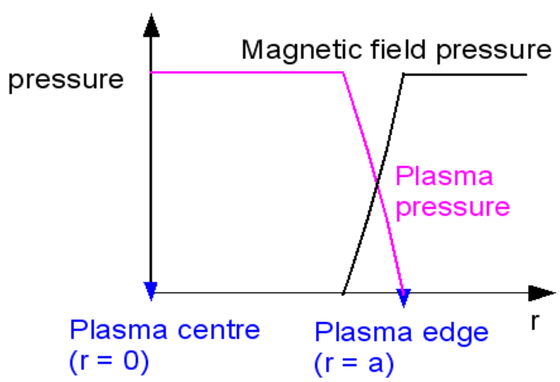

Figure 3.2: Strongly heated Θ-pinch geometry.

There is a maximum allowed central pressure which can directly be found from the equation above since this equation merely states that the sum of plasma and magnetic pressure is constant

$$p(0)+\frac{B(0)^2}{2\mu_0}=\frac{B(a)^2}{2\mu_0}, \tag{3.7}$$

It follows that

$$p(0)=\frac{B(a)^2-B(0)^2}{2\mu_0}\leq\frac{B(a)^2}{2\mu_0}, \tag{3.8}$$

The kinetic pressure has a maximum determined by the magnetic field imposed by the field coils. One could argue that for the maximum pressure the plasma beta in the centre is infinite since the pressure is finite and the magnetic field is zero. However, the measure of the plasma beta as efficiency parameter is useful only if it measures the maximum achievable pressure for an applied magnetic field. One should therefore calculate the plasma beta using the central pressure and the magnetic field of the edge. This yields

$$\beta = \frac{p(0)}{B(a)^2/2\mu_0} \leq 1 \tag{3.9}$$

The maximum plasma beta is one, as one might have expected because this corresponds to the maximum kinetic pressure being equal to the applied magnetic field pressure. Note that a strongly heated plasma which develops a large pressure gradient in the edge, will simply dig a hole in the magnetic field resulting in a large central region with zero magnetic field strength. In this region there is no magnetic field pressure and, consequently, no pressure gradient. This is shown in Fig. 3.2

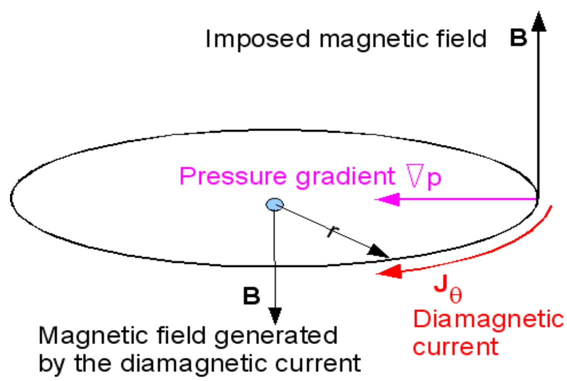

Figure 3.3: Picture of the diamagnetic current

Since the magnetic field changes in magnitude there must be a current inside the plasma. This current, of course, follows directly from the force balance equation (3.1). Taking the cross product with the magnetic field

$$\mathbf{B} \times (\mathbf{J} \times \mathbf{B}) = B^2\mathbf{J} - (\mathbf{B} \cdot \mathbf{J})\mathbf{B} = B^2\mathbf{J}_\perp = \mathbf{B} \times \nabla p \tag{3.10}$$

where the index $\perp$ refers to the component of the current perpendicular to the magnetic field. Since the pressure is only a function of the radius r it follows

$$\mathbf{J}_\perp = \frac{\mathbf{B} \times \nabla p}{B^2} \quad \rightarrow \quad J_\theta = \frac{1}{B}\frac{dp}{dr} \tag{3.11}$$

This current is sketched in Fig. 3.3 and it can be seen that its direction is such that the magnetic field it generates is in the opposite direction compared with the magnetic field applied. This current is the reason for the decrease of the magnetic field strength inside the plasma and is therefore referred to as diamagnetic current.

Although it is clear that a current must exist, the nature of this current might be somewhat puzzling. Using the simple picture of the particles gyrating around the magnetic field and, therefore, being automatically confined might seem in contradiction with the existence of a current, since one might be lead to think that something must drive the current. It turns out that the gyro-motion of the particles automatically generates the diamagnetic current. The basic physical picture is shown in Fig. 3.4. In a chosen point X, the particles that move downward are the gyrating particles for which the centre of

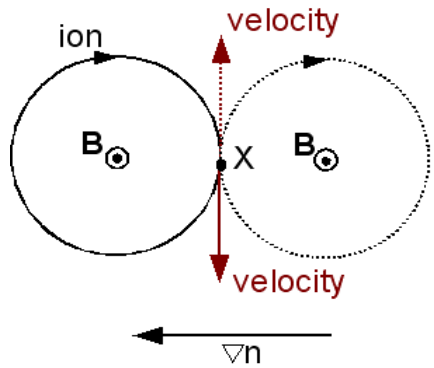

Figure 3.4: Schematic picture of how the diamagnetic current is generated

the orbit is displaced to the left of X, whereas the particles that move upward have their centre more to the right. If the plasma is homogeneous the density of particles is independent of the position, and in X there as many particles moving downward as there are moving upward. In this case there is no net ion flow at position X. However if there is a density gradient (pointing to the left) there are more particle on the left orbit compared with the right orbit which results in a net downward motion in the point X. Here, it should be noted that the picture is drawn for the ions. A similar picture applies for the electrons with the exception that they rotate in the opposite direction due to their negative charge. The mean electron velocity will therefore be upward. The relative motion of the ions and electrons is the diamagnetic current, and from the description it is clear that this current is a direct consequence of the gyro-motion in the presence of a density (pressure) gradient.

One can make a crude estimate of the current from the picture above. The current density can be estimated using a two point estimate

$$J = -en(X - \rho)v + en(X + \rho)v \tag{3.12}$$

where v is the velocity of the particles and n is the density. Assuming that the Larmor radius is small one can make a Taylor expansion

$$n(X - \rho) = n(X) - \rho\frac{\partial n}{\partial x} + \dots \qquad n(X + \rho) = n(X) + \rho\frac{\partial n}{\partial x} + \dots, \tag{3.13}$$

where terms of the order ρ^2 have been neglected. The current then is

$$J = 2e\rho v\frac{\partial n}{\partial x}, \tag{3.14}$$

and using the equation for the Larmor radius $\rho = mv/eB$ one obtains

$$J = 2\frac{mv^2}{B}\frac{\partial n}{\partial x} \tag{3.15}$$

Finally the velocity will be chosen to be the averaged velocity of a plasma with a temperature T

$$mv^2 = 2T \tag{3.16}$$

$$J = 4\frac{T}{B}\frac{\partial n}{\partial x} \tag{3.17}$$

Since the example considers a density gradient but no temperature gradient the pressure is

$$\nabla p = T\nabla n \tag{3.18}$$

and the estimate more or less recovers the expression of the diamagnetic current (3.11). The numerical factor is wrong, but this is simply related to the crudeness of the two point estimate. A proper average can be shown to lead to the exact recovery of the diamagnetic current. For the temperature gradient one has then to consider that the particles on the two orbits do not have the same averaged speed, nor the same Larmor radius.

In the early days plasma heating was not as well developed as it is today. The Θ-pinch can, however, be heated by ramping the magnetic field. The principle effect is shown in Fig. 3.5. One simply ramps the current in the magnetic field coils. This increases the magnetic field strength at the plasma boundary and must therefore lead to a larger gradient in the magnetic field pressure. When increasing this gradient it exceeds the kinetic pressure gradient, and the plasma is no longer in equilibrium. The net force is directed inward and will compress the plasma. The work done against the pressure gradient will then lead to an increase in the kinetic pressure of the plasma which will finally reach a value such that it can again balance the magnetic pressure. This heating effect is of course similar to the heating obtained when compressing a ordinary gas.

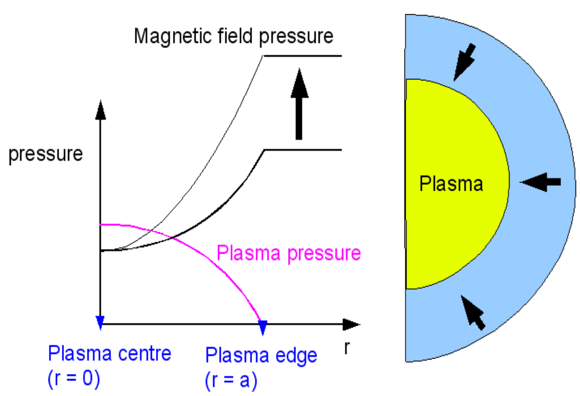

Figure 3.5: Heating of the theta pinch through compression

Finally, the stability of the Θ-pinch is discussed. This can be done on the basis of flux conservation and the force on the plasma. Simple cartoons allow us to asses the stability of the equilibrium. In these cartoons the plasma and a set of magnetic field lines is drawn. The magnetic field strength can be thought of as the density of the magnetic field lines. Furthermore, when perturbing the plasma position the magnetic field lines move with the plasma due to the conservation of magnetic flux. The resulting magnetic field structure can be analyzed using the expression for the force on the plasma derived in the previous chapter. If the force is in the direction of the original perturbation then the resulting acceleration will further enhance this perturbation and the plasma will be unstable. Fig. 3.6 shows the magnetic field structure. When the plasma is bend, so is the magnetic field. The tension that results from this bending is such that the Lorentz force is directed in the direction opposite to the original motion. The plasma is therefore stable to any perturbation that bends the plasma column. Similarly, when one squeezes the plasma the density of magnetic field lines is increased. This leads to a larger magnetic field pressure, and the force due to the gradient of this field pressure will be outward. Consequently, the Θ-pinch is also stable to any perturbation that leads to a modulation of its radial width. In conclusion, a Θ-pinch is a nicely stable configuration.

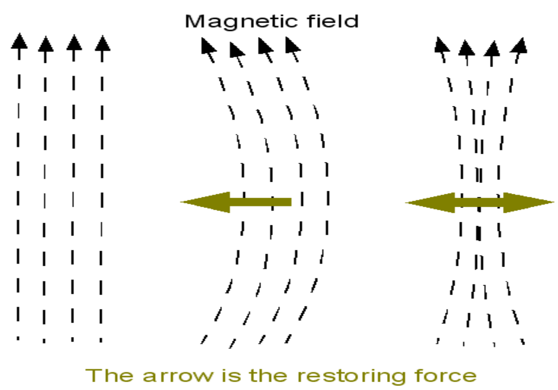

Figure 3.6: The stability of the Θ-pinch

The theta pinch is an easy to realize geometry. It is stable, and can reach β values of around 1. So what is wrong with it? The problem is the end losses. Although the particles are well confined perpendicular to the magnetic field, they are free to move along the field lines. In practice the finite length of the device will then lead to large losses of particles and energies at both ends of the device where the magnetic field must touch the material plates. In many ways the confinement is this device is only marginally better compared with the case without magnetic field.

3.2 The Z pinch

The second equilibrium considered is the Z-pinch. In this case a electric current in the z-direction is imposed from the outside.

$$\mathbf{J} = J\mathbf{e}_z. \tag{3.19}$$

The geometry of this equilibrium is shown in Fig. 3.7 For simplicity it will be assumed that the current density (J) is constant inside a cylindrical plasma with radius $r = a$. Since it is the current that is imposed from outside all quantities will be expressed in the current density (J). The magnetic field is solely generated by the current and lies in the (θ) direction. It can be easily solved for by integrating the

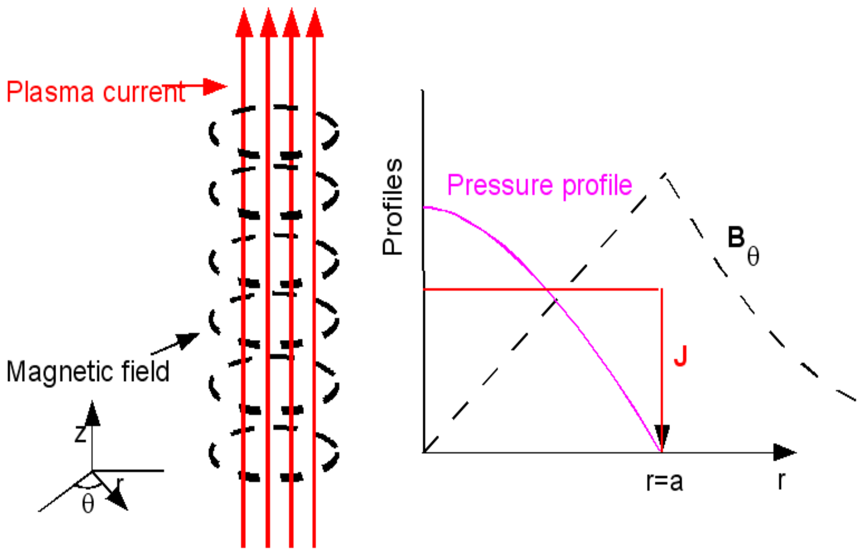

Figure 3.7: The z-pinch configuration

Maxwell equation

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} \tag{3.20}$$

over a circular surface with radius r lying in the z =constant plane

$$2\pi r B_\theta = \int \mathrm{d}\mathbf{S} \cdot \nabla \times \mathbf{B} = \mu_0 \int \mathrm{d}\mathbf{S} \cdot \mathbf{J} = \mu_0 \pi r^2 J, \tag{3.21}$$

where it is assumed that $r < a$. In the equation above B_θ refers to the component of the magnetic field in the θ-direction. For $r > a$ one finds

$$2\pi r B_\theta = \mu_0 \pi a^2 J. \tag{3.22}$$

Therefore

$$B_\theta = \frac{\mu_0 J \min(r^2, a^2)}{2\, r} \tag{3.23}$$

This is the equation for the magnetic field generated by the current. One point can immediately be noted. For the Z-pinch the magnetic field lines are circular and close upon themselves. The free parallel motion along the field lines therefore does not lead to any particle or energy losses.

The force balance equation inside the plasma yields

$$\mathbf{J} \times \mathbf{B} = -J B_\theta \mathbf{e}_r \tag{3.24}$$

The pressure gradient inside the plasma, therefore, is

$$\frac{\mathrm{d}p}{\mathrm{d}r}\mathbf{e}_r = \mathbf{J} \times \mathbf{B} = -\frac{\mu_0 J^2 r}{2}\mathbf{e}_r, \tag{3.25}$$

while outside the plasma the equation is trivially satisfied since both the current and the pressure are zero. The equation above can be easily integrated to obtain

$$p(r) = -\frac{\mu_0 J^2 r^2}{4} + C \tag{3.26}$$

The integration constant C can then be found through the boundary condition that the pressure at the edge of the plasma is zero

$$p(r) = \frac{\mu_0 J^2}{4}(a^2 - r^2) \tag{3.27}$$

For the Z-pinch the pressure profile that can be sustained is parabolic. Different profile forms can only be obtained when the current is non-uniform. Furthermore, it follows that the maximum plasma pressure is proportional to the current squared. This is because the current appears in the force balance but is also the source of the magnetic field. The typical profiles are shown in Fig. 3.7.

One can see from the figure containing the profiles that inside the plasma the magnetic field strength increases with radius. Consequently, there is a magnetic field pressure that increases with radius such that it confines the plasma pressure. The magnetic field pressure is, however, not the only force that contributes to the plasma confinement. To see this one needs to go back to the force balance equation

$$-\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \nabla p \tag{3.28}$$

One can investigate how much of the Lorentz force is due to the gradient of the magnetic pressure and how much is due to field bending The second term on the left hand side yields

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = B_\theta(\mathbf{e}_\theta\cdot\nabla)(B_\theta\mathbf{e}_\theta) = B_\theta^2(\mathbf{e}_\theta\cdot\nabla)\mathbf{e}_\theta. \tag{3.29}$$

Although $\mathbf{e}_\theta$ is a the unit vector in the θ direction, the last term on the right hand side is not zero. This is because the unit vector changes direction with θ. One can derive

$$(\mathbf{e}_\theta\cdot\nabla)\mathbf{e}_\theta = -\frac{1}{r}\mathbf{e}_r \tag{3.30}$$

Therefore

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = -\frac{B_\theta^2}{r}\mathbf{e}_r \tag{3.31}$$

Substituting the equation for the poloidal field

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = -\frac{\mu_0 r J^2}{4}\mathbf{e}_r \tag{3.32}$$

Therefore, half of the pressure gradient is balanced by the magnetic field tension and the other half by the gradient of the magnetic field strength. The combination of the two forces can also be seen to lead to the decay of the magnetic field beyond the plasma radius $r = a$. Of course, you are quite familiar with the concept that the magnetic field decays as $1/r$ away from a current carrying wire. In the language of the pressure and tension it means that the inward directed force due to the tension is compensated by the force due to the magnetic field pressure.

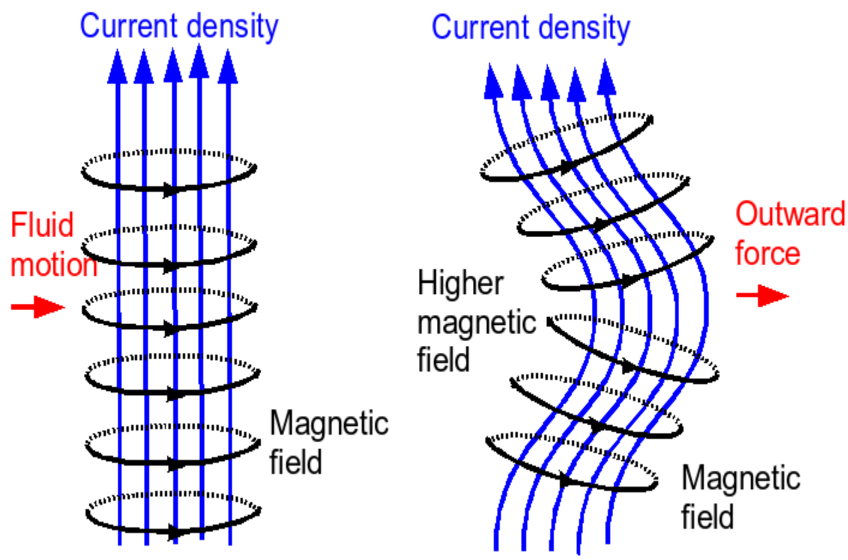

Figure 3.8: The kink instability

The Z-pinch can be heated in much the same way as the Θ-pinch. In this case the current is ramped, leading to a larger magnetic field which again compresses the plasma through the Lorentz force. The compression leads to an increase in pressure. The Z-pinch is, however, also heated by the current. The plasma will have a finite resistivity, and therefore an Ohmic heating associated with the current. This makes that the z-pinch can be more efficiently heated compared with the θ pinch. Furthermore, the z-pinch reaches a high beta value. Taking again the kinetic pressure in the centre and the magnetic field at the edge

$$p(0) = \frac{\mu_0 J^2}{4} a^2 \qquad B_\theta(a) = \frac{\mu_0 J a}{2} \tag{3.33}$$

One obtains

$$\beta = 2 \tag{3.34}$$

The plasma beta is double that of the Θ pinch, loosely speaking because both magnetic field pressure as well as tension contribute to the confinement.

The Z-pinch is an easy to generate equilibrium, which does confine the particles, can be relatively easy heated, and reaches a high beta value. It is nevertheless not of interest to magnetic confinement fusion research because it isn't stable. The most important instability is shown in Fig. 3.8. A displacement that bends the plasma will lead to a larger density of field lines on the inside whereas the density of field lines on the outside is smaller. The magnetic field strength is therefore higher on the inside compared with the outside, and the resulting force due to the magnetic field pressure is in the direction of the original perturbation. It will therefore accelerate the plasma and lead to an instability. This instability is known as the kink, and it plays an important role in the tokamak, the most developed reactor concept, as well.

3.3 General cylindric equilibriums - screw pinch

The third equilibrium known as the screw pinch is simply a combination of the other two. Now both the current as well as the magnetic field have a z as well as a θ component. The z-component again does not contribute to the field line bending due to the symmetry, and the force balance can be written in the form

$$-\nabla\left(\frac{B_z^2+B_\theta^2}{2\mu_0}\right)-\frac{B_\theta^2}{\mu_0 r}\mathbf{e}_r=\nabla p$$

Also the equation for B_θ that was derived in the case of the Z pinch is still valid

$$B_\theta=\mu_0 r J_z/2 \tag{3.35}$$

Therefore

$$p_0=\frac{\mu_0 J_z^2 a^2}{4}+\frac{B_{za}^2-B_{z0}^2}{2\mu_0} \tag{3.36}$$

The first equation of this section can also be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[p+\frac{B_z^2}{2\mu_0}\right]+\frac{B_\theta}{\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}[rB_\theta]=0 \tag{3.37}$$

where it has been used that all quantities are only a function of the radius r. Since the poloidal field is related to the toroidal current

$$rB_\theta=\int_0^r xJ(x)\mathrm{d}x \tag{3.38}$$

One directly observe that as long as the current has the same sign over the entire radius, the last term in the equation above is always positive. For zero pressure this means that the first term is negative and the plasma is paramagnetic. In the case of zero pressure

$$\mathbf{J}\times\mathbf{B}=0 \quad\rightarrow\quad \mathbf{J}\ \|\ \mathbf{B}, \tag{3.39}$$

the current is parallel to the magnetic field. The current in the $z-$ direction generates a magnetic field in the θ direction and, therefore, there must also exist a current in the θ direction. One can make oneself clear that this poloidal current is always directed such that it enhances the magnetic field in the z-direction. The paramagnetic solution is obtained for a small kinetic pressure (at higher pressure the diamagnetic effect dominates). For a small kinetic pressure the magnetic field pressure due to B_θ is too large to be balanced by the pressure gradient. The plasma then compresses the magnetic field in the z-direction until the forces balance. This compression of the magnetic field leads to the paramagnetic effect.

The most important reason for discussing this equilibrium is the stabilizing effect of the z-component of the magnetic field shown in Fig. 3.9. It is clear from the discussion of the Z-pinch and Θ-pinch that the bending of the plasma will lead to a destabilizing as well as a stabilizing effect. The former is related with the magnetic field generated by the $z-$component of the current, whereas the latter is generated by the bending of the externally imposed magnetic field. One can easily imagine that stability depends on the relative magnitude of the $z-$ and Θ components of the magnetic field. One needs to supply a sufficiently strong magnetic field in the z direction to obtain a stable screw pinch.

Although the stabilization of the Z-pinch with an external field leading to the screw pinch is nice, it also brings back an old problem. The combined magnetic field now winds helically upward and the end losses reappear. Also the screw pinch is not a solution. But it will be seen later that one needs to add one more ingredient to obtain a suitable magnetic confinement concept.

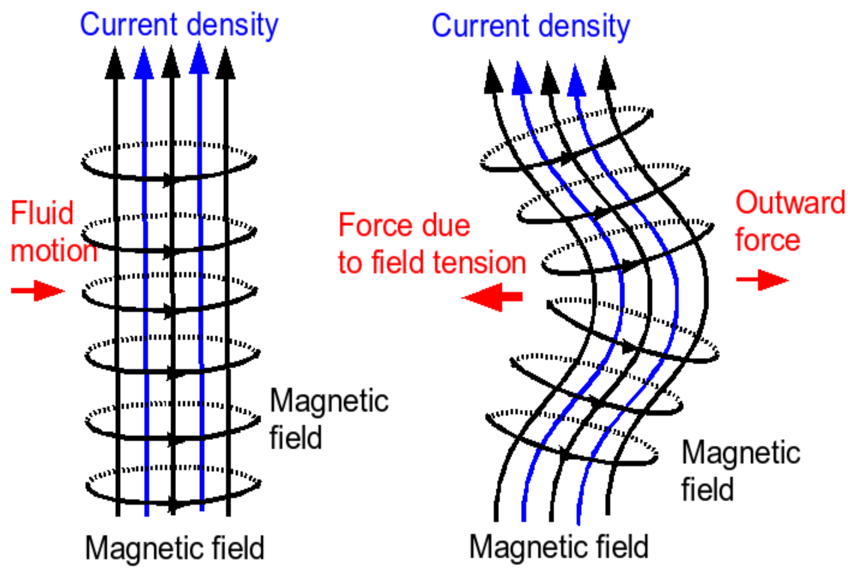

Figure 3.9: The kink for a screw pinch. The externally applied magnetic field in the z-direction stabilizes the kink is sufficient strong

3.4 What do I have to know for the exam??

Unfortunately for you almost the whole chapter is part of the exam. The only bit that you can leave out is the discussion on the equilibrium of the screw pinch and the paramagnetic effect. Be sure though that you understand the stabilizing effect of the magnetic field in the z-direction on the kink instability.

Chapter 4

Particle orbits in a magnetic field

In this chapter the particle motion in an magnetic field will be described. The derivation of the different motions is heuristic and for a more complete description one is referred to the lecture on electro-dynamics. The text is intended to give the reader an idea of what the main physics ingredients are without any claim of a rigorous derivation. If it is the night before the exam, you might also want to read the section: 'What do I have to know for the exam' at the end of the chapter before loosing too much time.

The basic equation that describes the particle motion is

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B} + \mathbf{F} \tag{4.1}$$

where q (m) is the particle charge (mass), and $\mathbf{F}$ is an arbitrary force on the particle, which will be specified later. This force is assumed to be much smaller than the Lorentz force represented by the first term on the right hand side.

4.1 Uniform electric and magnetic fields

4.1.1 Zero electric field

First we investigate the case in which there is no other force on the particle other than the Lorentz force (i.e. $\mathbf{F} = 0$). The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B} \tag{4.2}$$

The Lorentz force does not influence the parallel velocity ($v_\parallel$), i.e.

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = 0 \tag{4.3}$$

For the velocity component perpendicular to the field ($\mathbf{v}_\perp$) the Lorentz force accelerates the particle, but the acceleration is always perpendicular to the velocity (as well as to the magnetic field), and so the energy of the particle is constant

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv_\perp^2\right] = m\mathbf{v}_\perp\cdot\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = q\mathbf{v}_\perp\cdot(\mathbf{v}_\perp\times\mathbf{B}) = 0 \tag{4.4}$$

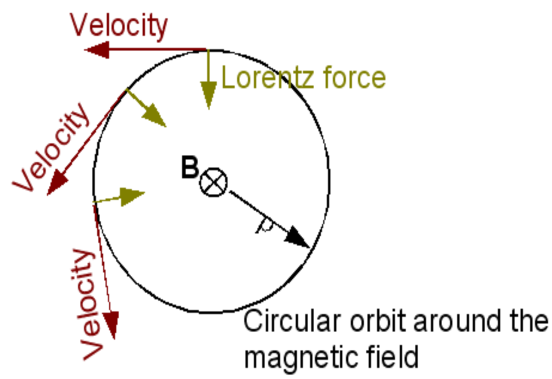

Figure 4.1: The gyro-orbit.

The acceleration due to the Lorentz force will change the direction of the velocity perpendicular to the magnetic field without changing its magnitude. The result is a circular orbit known as the gyro-orbit, which is sketched in Fig. 4.1 The radius (ρ) of this circular motion can be easily found if one balances the magnitude of the Lorentz force with a centrifugal force

$$q v_\perp B = m v_\perp^2 / \rho \qquad \rightarrow \qquad \rho = \frac{m v_\perp}{qB} \tag{4.5}$$

which is the equation for the Larmor radius that we have used before. The frequency can be calculated from the time τ a particle needs for one turn $\tau = 2\pi\rho/v_\perp$

$$\omega_c = \frac{2\pi}{\tau} = \frac{v_\perp}{\rho} = \frac{qB}{m} \tag{4.6}$$

This frequency (ω_c) is known as the cyclotron frequency. For a fusion plasma this frequency is very high indeed (80 MHz for Hydrogen ions and 130 GHz for the electrons). Many phenomena occur on a longer time scale and for a physical description of such phenomena one can assume the gyro motion to be infinitely fast. All quantities can then be averaged over the gyro-motion, removing the high frequency from the system of equations. Since also the Larmor radius is small compared to the system size one might be tempted to neglect the variations in the physical parameters over the size of the Larmor radius all together. We will, however, see below that such variations have an important effect. They lead to small drift velocities of the ring which play an important role in many phenomena.

A more formal derivation is as follows: Take the magnetic field to be in the z direction

$$\mathbf{B} = B\mathbf{z} \tag{4.7}$$

the equation of motion can be analyzed from the three differential equations

$$m\dot{v}_x = qBv_y \tag{4.8}$$

$$m\dot{v}_y = -qBv_x \tag{4.9}$$

$$m\dot{v}_z = 0 \tag{4.10}$$

where the dots indicate the differentiation toward time. Taking again the differentiation of the first and second equation toward time yields

$$\ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x \tag{4.11}$$

$$\ddot{v}_y = -\left(\frac{qB}{m}\right)^2 v_y \tag{4.12}$$

The two equations above describe a simple harmonic oscillation at the cyclotron frequency. Note that the cyclotron frequency can be negative due to its dependence on the charge.

The solution of the equations of motion is

$$v_{x,y} = v_\perp exp(i\omega_c t + i\delta_{x,y}) \tag{4.13}$$

The phase δ is chosen such that

$$v_x = v_\perp exp(i\omega_c t) = \dot{x} \tag{4.14}$$

where $v_\perp$ is a positive constant denoting the speed in the plane perpendicular to B. Then

$$v_y = \frac{m}{qB}\dot{v}_x = iv_\perp exp(i\omega_c t) = \dot{y} \tag{4.15}$$

Integrating once again one obtains

$$x - x_0 = -i\frac{v_\perp}{\omega_c} exp(i\omega_c t) \tag{4.16}$$

$$y - y_0 = \frac{v_\perp}{\omega_c} exp(i\omega_c t) \tag{4.17}$$

Using the definition of the Larmor radius

$$\rho = \frac{v_\perp}{\omega_c} \qquad \rightarrow \qquad \rho = \frac{mv_\perp}{|q|B}, \tag{4.18}$$

and taking the real part of the equations for the trajectories, one finally obtains

$$x - x_0 = \rho sin\omega_c t \tag{4.19}$$

$$y - y_0 = \rho cos\omega_c t. \tag{4.20}$$

This solution describes a circular orbit around a guiding center (x_0, y_0) which is fixed. The direction of the gyration is always such that the magnetic field generated by the charged particle is opposite to the externally imposed field. Plasma particles, therefore, tend to reduce the magnetic field, and plasmas are diamagnetic. In addition to this motion, there is an arbitrary velocity v_z along B which is not affected by the Lorentz force. The trajectory of a charged particle in space is, in general, a helix. The particles gyrate around the field lines and at the same time propagate along the field. We will formally write the solution of this section as

$$\mathbf{v} = v_\| \mathbf{b} + \mathbf{v}_g \tag{4.21}$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the magnetic field (the z-direction here) and $\mathbf{v}_g$ represents the rapidly rotating perpendicular velocity

4.1.2 General influence of an additional force

In this section the general force F, that was so far assumed zero, is considered. The equation of motion has the form

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} + \mathbf{F} \tag{4.22}$$

For the parallel motion this leads to a trivial acceleration

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = F_\parallel \tag{4.23}$$

More interesting is the perpendicular motion. The equation above is a linear ordinary differential equation for the velocity. The gyro-motion ($\mathbf{v}_g$) that was derived in the previous section is the homogeneous solution of this equation. An inhomogeneous solution can be found by putting the time derivative to zero and solving for the velocity ($\mathbf{v}$)

$$q\mathbf{v} \times \mathbf{B} + \mathbf{F} = 0$$

This equation determines only the perpendicular component of the velocity (which one can easily check by splitting the velocity in a parallel and perpendicular component). The parallel velocity is determined by Equation (4.23). Taking the cross product with the magnetic field then yields

$$\mathbf{v}_\perp = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{4.24}$$

and the velocity is the sum of the various contributions

$$\mathbf{v} = v_\parallel \mathbf{b} + \mathbf{v}_g + \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{4.25}$$

The perpendicular component of the force does not lead to an acceleration in the direction of the force. Instead it leads to a constant velocity perpendicular to both the force as well as the magnetic field. This velocity is called the drift velocity. The center of the gyration of the particle is called the gyro-center or guiding centre, and it moves with this drift velocity. Note that the drift velocity depends on the charge of the particle.

In subsequent sections the effect of various forces will be discussed using the master equation (4.25). Before doing so the physical effect that leads to the drift will be discussed. This physics picture is outlined in Fig. 4.2. A perpendicular force will accelerate a particle when it moves in its circular orbit. The velocity on the top will therefore be somewhat larger compared with the bottom. This, however, also means that the Larmor radius and, consequently, the radius of curvature is larger on the top compared with the bottom of the orbit. This difference in radius of curvature means that the orbit will not close upon itself exactly but will after one turn of the particle be slightly shifted. The result is a drift which is perpendicular to both the force as well as the magnetic field.

4.1.3 Finite electric field

To find the motion of the particle under the influence of both a magnetic as well as an electric field ($\mathbf{E}$), one can simply put the general force F of the previous section equal to the electric field force

$$\mathbf{F} = q\mathbf{E} \tag{4.26}$$

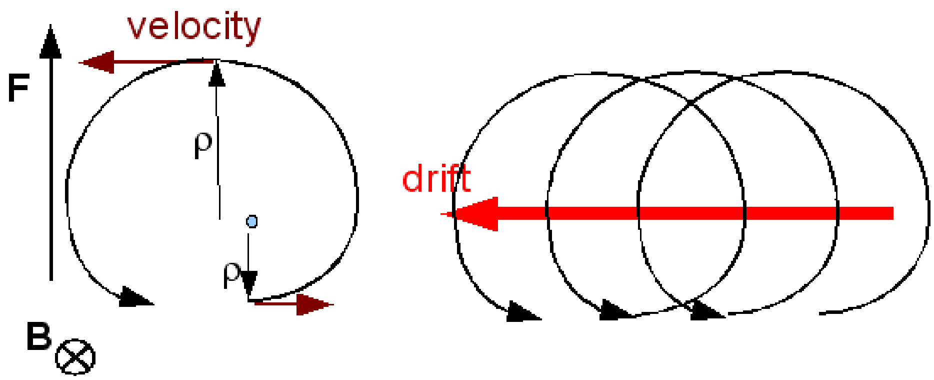

Figure 4.2: The physical picture of the drift

Substituting this in the equation for the general drift velocity, one obtains the drift of the particle due to the electric field

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{4.27}$$

This drift is known as the ExB (read E cross B) drift. Notice here that the ExB drift does not depend on the charge or mass of the particles. Indeed this drift does not lead to charge separation, since both the electrons as well as the ions are moving with the same speed, and thus no electric current is generated by the drift.

In the paragraph above we have silently assumed that the force is not a function of time. In general the derivation is approximately correct if the force does not vary on the timescale of the cyclotron motion, which is almost always the case. There is, however, an important small correction due to an electric field that is a function of time. This case is considered below. For a time dependent electric field one must solve the full equation

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} + q\mathbf{E}. \tag{4.28}$$

When the electric field varies only slowly in time one can guess that the solution at every time point is the ExB motion plus a small correction

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \mathbf{v}_1 \tag{4.29}$$

Substituting this velocity in the equation of motion one obtains

$$m\frac{\mathrm{d}(\mathbf{v}_E + \mathbf{v}_1)}{\mathrm{d}t} = q\mathbf{v}_1 \times \mathbf{B}. \tag{4.30}$$

Note that the Lorentz force due to the ExB motion has canceled the electric field on the right hand side, as it should since this is the way the ExB velocity was derived. Then assuming that the velocity $\mathbf{v}_1$ is only a small correction compared with the ExB velocity, it can be neglected on the left hand side of the equation yielding

$$q\mathbf{v}_1 \times \mathbf{B} = \frac{m}{B^2}\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} \times \mathbf{B} \tag{4.31}$$

This equation can again be solved by taking the cross product with **B** on both sides yielding

$$\mathbf{v}_1 = \frac{m}{qB^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t} = \mathbf{v}_p \tag{4.32}$$

This drift velocity is known as the polarization drift ($\mathbf{v}_p$). Note that unlike the ExB velocity it is dependent on both the mass m as well as the charge q of the particle. It is therefore in opposite directions for electrons and ions, and much larger for the ions when compared with the electrons.

4.2 Non homogeneous magnetic fields

In most magnetic confinement concepts the magnetic field is not homogeneous. The inhomogeneous magnetic field leads to a drift velocity through two effects. First is the centrifugal force a particle experiences when it moves along a curved magnetic field. The drift associated with this force is therefore called the curvature drift. The second effect is due to the variation of the magnetic field strength over the Larmor orbit which leads to a force opposite to the gradient of the magnetic field strength. The drift associated with the latter force is therefore called the grad-B drift. Below these two drifts will be discussed

4.2.1 Curvature drift

If a particle has a velocity along the magnetic field, the centrifugal force is:

$$\mathbf{F}_c = \frac{mv_\parallel^2}{|R|^2}\mathbf{R}, \tag{4.33}$$

where **R** is the radius of curvature. Using the general equation for the drift (4.24) one directly obtains

$$\mathbf{v}_{dc} = \frac{\mathbf{F}_c \times \mathbf{B}}{qB^2} = \frac{m}{q}\left(\frac{v_\parallel}{R}\right)^2 \frac{\mathbf{R}\times\mathbf{B}}{B^2} \tag{4.34}$$

There exists a general relation between the gradient of the magnetic field strength and the radius of curvature

$$\nabla B = -\frac{\mathbf{R}}{R^2}B. \tag{4.35}$$

This relation holds as long as the plasma beta is much smaller than one. Using this relation in the equation of the drift

$$\mathbf{v}_{dc} = \frac{mv_\parallel^2}{qB}\frac{\mathbf{B}\times\nabla B}{B^2}. \tag{4.36}$$

The curvature drift ($\mathbf{v}_{dc}$) can therefore be expressed directly in the magnetic field.

4.2.2 Grad-B drift

When the magnetic field strength is not constant the magnitude of the Lorentz force varies over the gyro orbit. For simplicity it is assumed here that the magnetic field points in the z-direction, and varies in strength in the x-direction. Due to the dependence of the magnetic field strength on the x-coordinate the force in the x-direction is not the same when comparing the position $\mathbf{x}_{gc} - \rho\mathbf{e}_x$ and $\mathbf{x}_{gc} + \rho\mathbf{e}_x$, where $\mathbf{x}_{gc}$ is the position of the centre of the ring. Assuming the magnetic field to vary on length scales much longer than the Larmor radius one can can make a Taylor expansion of the magnetic field strength

$$B = B(\mathbf{x}_{gc}) + \overline{\rho} \cdot \nabla B \tag{4.37}$$

Here $\overline{\rho}$ is the vector from the gyro-centre to the position of the particle and is given by the right hand side of the equations (4.16-4.17). Using the Taylor expansion the Lorentz force is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}(\mathbf{x}_{gc}) + q\mathbf{v} \times \overline{\rho} \cdot \nabla \mathbf{B} \tag{4.38}$$

The first term leads to the normal gyro-motion with a frequency determined by the magnetic field at the centre of the ring. Using the equations for the gyro-motion and choosing $\mathbf{e}_1$ in the x-direction and $\mathbf{e}_2$ in the y-direction, the second term in the equation above can be written as

$$q\mathbf{v} \times \overline{\rho} \cdot \nabla \mathbf{B} = q[-v_\perp cos(\omega_c t + \phi)\mathbf{e}_2] \times [\rho cos(\omega_c t + \phi)\mathbf{e}_1 \cdot \nabla]\mathbf{e}_z \frac{\partial B}{\partial x}\mathbf{e}_x = -\frac{mv_\perp^2}{B}\frac{\partial B}{\partial x}\cos^2(\omega_c t + \phi)\mathbf{e}_x \tag{4.39}$$

Averaging over the orbit (indicated by the angle brackets) then yields

$$\langle q\mathbf{v} \times \overline{\rho} \cdot \nabla \mathbf{B} \rangle = -\frac{mv_\perp^2}{2B}\frac{\partial B}{\partial x}\mathbf{e}_x = -\mu \frac{\partial B}{\partial x}\mathbf{e}_x, \tag{4.40}$$

where $\mu = mv_\perp^2/2B$ is the magnetic moment. The equation above is derived for a variation of the magnetic field in the x-direction, but in fact it is generally applicable. A particle in a non-uniform magnetic field will experience a force

$$\mathbf{F} = -\mu \nabla B \tag{4.41}$$

This force is directed opposite to the gradient of the magnetic field, i.e. if the particle was accelerated by this force it would move toward lower magnetic field strengths. Of course, the force will not directly lead to this motion but, rather, will create a drift. Nevertheless, the force given above expresses that the particles would like to escape from the magnetic field. It will be shown that in a fusion device this can indeed lead to problems. After having calculated the force, the evaluation of the drift is straight forward

$$\mathbf{v}_{\nabla B} = \mu \frac{\mathbf{B} \times \nabla B}{B^2} \tag{4.42}$$

4.3 Conserved quantities

Above various drift velocities have been derived. These, however, still depend on the parallel and perpendicular components of the velocity, which are generally not constant. A powerful approach to the determination of the velocity components is the use of conserved quantities. These are essentially two for a general magnetic field configuration, the magnetic moment and the kinetic energy.

The easiest to derive is the conservation of energy, which holds only in the absence of an electric field. With only the magnetic field the force on the particle is the Lorentz force, which is always perpendicular to the velocity and, therefore, never does any work. From this observation it follows directly that the kinetic energy (E) is a conserved quantity

$$E = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 \tag{4.43}$$

To derive the second conserved quantity one has to consider a particle moving in an inhomogeneous magnetic field. For simplicity, we will, however, consider a stationary particle under the influence of a changing magnetic field. The result is of course the same. Consider a particle with zero parallel velocity, gyrating in a magnetic field. The latter is assumed to increase slowly in strength ($\partial B/\partial t > 0$). Of course a stronger magnetic field implies a higher cyclotron frequency and a smaller Larmor radius. The physics effect that must be kept in describing the change in the particle motion, however, is that a changing magnetic field generates an electric field

$$-\frac{\partial B}{\partial t} = \nabla \times \mathbf{E}. \tag{4.44}$$

Consider one turn of the gyrating particle. Since the field varies slowly the orbit will not have changed much during this turn. In lowest order one can assume that the orbit is circular. Then integrating over the area enclosed by the orbit and applying the theorem of Stokes

$$-\pi\rho^2 \frac{\partial B}{\partial t} = \oint_A \mathrm{d}\mathbf{s} \cdot \mathbf{E}. \tag{4.45}$$

The norm of the surface over which was integrated determines the orientation of the integral on the right hand side. It is anti-clockwise when viewed from above, which is indicated by the index A on the integral.

The right hand side of the equation above is closely related to the acceleration a particle would undergo in the perpendicular direction

$$m\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = q\mathbf{E}_\perp. \tag{4.46}$$

This equation can most easily be solved by taking the inner product with the perpendicular velocity

$$m\mathbf{v}_\perp \cdot \frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv_\perp^2\right] = q\mathbf{v}_\perp \cdot \mathbf{E}_\perp = q\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t} \cdot \mathbf{E}_\perp, \tag{4.47}$$

where in the last step the perpendicular velocity has been written as the derivative toward time of the position vector $\mathbf{s}$. The equation above can be integrated toward time over the time interval the particle needs to go once around the magnetic field

$$\Delta\left[\frac{1}{2}mv_\perp^2\right] = q\oint_C \mathrm{d}\mathbf{s} \cdot \mathbf{E}_\perp \tag{4.48}$$

Here the direction of the integration is determined by the direction in which the particle rotates. It moves clockwise when viewed from above as indicated by the index C. If the electric field varies only slowly in time the Left hand side of this equation can be approximated as

$$\frac{\partial}{\partial t}\left[\frac{1}{2}mv_\perp^2\right]\tau = \frac{2\pi\rho}{v_\perp}\frac{\partial}{\partial t}\left[\frac{1}{2}mv_\perp^2\right], \tag{4.49}$$

where τ is the time the particle needs to go ones around. Putting the clockwise integration of the electric field equal to minus the anti-clockwise integration yields

$$\frac{2\pi\rho}{qv_\perp}\frac{\partial}{\partial t}\left[\frac{1}{2}mv_\perp^2\right] = \pi\rho^2\frac{\partial B}{\partial t}, \tag{4.50}$$

and substituting the expression for the Larmor radius ρ one obtains

$$\frac{1}{B}\frac{\partial}{\partial t}\left[\frac{1}{2}mv_\perp^2\right] - \frac{mv_\perp^2}{2B^2}\frac{\partial B}{\partial t} = 0. \tag{4.51}$$

From which it can easily be seen that

$$\frac{\partial \mu}{\partial t} = 0 \quad \text{with} \quad \mu = \frac{mv_\perp^2}{2B} \tag{4.52}$$

For an inhomogeneous magnetic field it is not the perpendicular energy that is conserved, but rather the magnetic moment μ which is the perpendicular energy divided by the magnetic field strength. One can think of this property as the conservation of the flux through the gyro-ring. This flux is

$$\psi = \pi\rho^2 B = \frac{4\pi m}{q^2}\frac{mv_\perp^2}{2B} \tag{4.53}$$

and is therefore proportional to the magnetic moment. The gyrating particle can be thought of as a ring current that does not undergo any resistive friction. The flux through the ring and, therefore, the magnetic moment are conserved.

4.4 Complete picture / meaning of the drifts

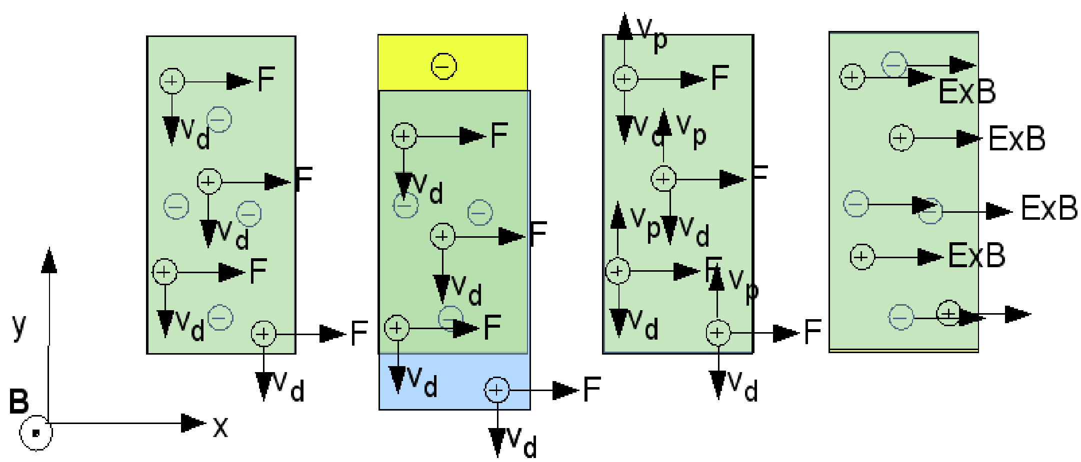

Figure 4.3: Example that should clarify the different roles played by the different drifts. From left to right 1. A force on the ions will lead to a downward ion drift, 2 This drift would lead to charge separation and the build up of an electric field. 3 Under the quasi-neutrality condition the polarization drift associated with the build up of the electric field will exactly cancel the drift due to the force. 4 The resulting ExB drift is in the direction of the force

In this section the results of the previous sections are gathered. The velocity ($\mathbf{v}$) of a particle in an inhomogeneous magnetic field ($\mathbf{B}$) including the effects of the electric field ($\mathbf{E}$) can be written as

$$\mathbf{v} = v_\parallel \mathbf{b} + \mathbf{v}_g + \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{m}{qB^2}\frac{\partial \mathbf{E}_\perp}{\partial t} + \frac{mv_\parallel^2 + mv_\perp^2/2}{qB}\frac{\mathbf{B}\times\nabla B}{B^2} \tag{4.54}$$

where the index $\parallel$ ($\perp$) refer to the parallel (perpendicular) component with respect to the magnetic field, $\mathbf{b}$ is the unit vector along the magnetic field, m (q) is the particle mass (charge), and B is the magnetic field strength. In the equation above the first term on the right hand side is the parallel motion along the field, and the second term $\mathbf{v}_g$ represents formally the rapid rotation of the velocity connected with the gyration of the particle around the magnetic field. This gyration has a frequency known as the cyclotron frequency (ω_c)

$$\omega_c = \frac{qB}{m} \tag{4.55}$$

and a radius known as the Larmor radius (ρ)

$$\rho = \frac{mv_\perp}{|q|B} \tag{4.56}$$

The third term of equation (4.54) represents the ExB (E cross B) drift, and the forth term the polarization drift. The last term, finally is the combination of the curvature and grad-B drift. To determine the motion of a particle one needs also to know how the parallel and perpendicular components of the velocity change when the particle moves in the electro-magnetic field. This can be derived using the conserved quantities, the magnetic moment μ and energy E

$$\mu = \frac{mv_\perp^2}{2B} \qquad E = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 \tag{4.57}$$

where the energy is only conserved in the absence of the electric field.

The drifts velocities are generally smaller than the free motion along the magnetic field. Assuming a typical scale L of the magnetic field, the magnitude of the curvature and grad-B drift can be estimated to be

$$\frac{mv_\parallel^2 + mv_\perp^2/2}{qB}\frac{\mathbf{B}\times\nabla B}{B^2} \approx \frac{mv^2}{qBL} \approx \frac{\rho}{L}v \tag{4.58}$$

The scale length of the magnetic field can be estimated to be the size of the machine (several meters) whereas the Larmor radius has a typical value of several mm. It, therefore, follows that the drift velocity is much smaller than the averaged velocity of the particles. A rough estimate of the drift velocity for 10 keV Deuterium ions would, however, still give a value of 1000 m/s (compared with 10^6 m/s thermal velocity). If particles are directly lost through the drift, it would still represent an unacceptable energy loss channel since the particles would leave the device on a typical timescale of milliseconds (assuming the device is several meters). One therefore has to consider the drifts when discussing the confinement of the magnetic field. It turns out that the ExB velocity is for many phenomena of interest comparable to the drift connected with the inhomogeneous magnetic field.

In the dynamics of the plasma the different drifts often play a specific role. One can get some insight into the physics of the process by considering the following example. Assume a magnetic field in the positive $z-$direction

$$\mathbf{B} = B\mathbf{e}_z \tag{4.59}$$

The plasma is assumed to have a finite size in the $y-$direction and a (non specified) force $\mathbf{F}$ works on each of the ions in the $x-$direction. This force will lead to a drift

$$\mathbf{v}_d = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} = -\frac{F}{eB}\mathbf{e}_y. \tag{4.60}$$

The ions will drift downward, whereas the electrons are stationary (See Fig. 4.3). This motion of the ions would then separate the charges of electrons and ions. Of course, a large separation of charge isn't possible due to the quasi-neutrality condition. For the sequence of the events it is nevertheless helpful to think in terms of charge separation. Charge separation will build up an electric field in the positive y-direction. A time dependent electric field leads to a polarization drift which will consequently also lie in the y-direction. One can now apply the strict neutrality condition by simply demanding that the polarization drift compensates the drift due to the force $\mathbf{F}$. This leads to an equation for the electric field

$$\frac{m}{eB^2}\frac{\partial E_y}{\partial t} = v_{py} = -v_{dy} = \frac{F}{eB} \tag{4.61}$$

And integrating toward time

$$E_y = \frac{FB}{m}t \tag{4.62}$$

Besides the polarization drift, which has been invoked to satisfy quasi-neutrality, the electric field also leads to an ExB motion

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \frac{F}{m}t \tag{4.63}$$

This velocity is in the direction of the force. One can furthermore observe that the ExB velocity satisfies the equation of motion

$$m\frac{\mathrm{d}\mathbf{v}_E}{\mathrm{d}t} = \mathbf{F} \tag{4.64}$$

i.e. the equation one would expect to be satisfied if there is no magnetic field present. The dynamics of the plasma under the action of the force therefore follows two steps. The force leads to a drift perpendicular to both the force as well as the magnetic field. For the plasma to remain neutral this drift must be balanced by the polarization drift connected with a time varying electric field. The latter electric field leads to an ExB motion in the direction of the force, with the velocity satisfying momentum conservation. Note that this result can be obtained only if the polarization drift is retained. This drift, although a small correction, is therefore of vital importance in the correct description of the dynamics.

Another instructive example that highlights the use of the conserved quantities is the mirror device. This device, shown in Fig. 4.4, was in fact extensively studied in the US in the 60s / 70s. The mirror is essentially a theta pinch with a magnetic field that is a function of the $z-$coordinate. The $z-$component of the current is generated by two magnetic field coils which are placed some distance apart. In between the field coils the magnetic field is weaker compared with the field at the coils. The idea is that the mirror force

$$\mathbf{F} = -\mu\nabla B \tag{4.65}$$

can prevent the particles from escaping. It can, however, be seen from the equation above that the mirror force is proportional to the magnetic moment μ. Not all particles will have the same perpendicular energy, and it is therefore in general not possible to confine all the particles.

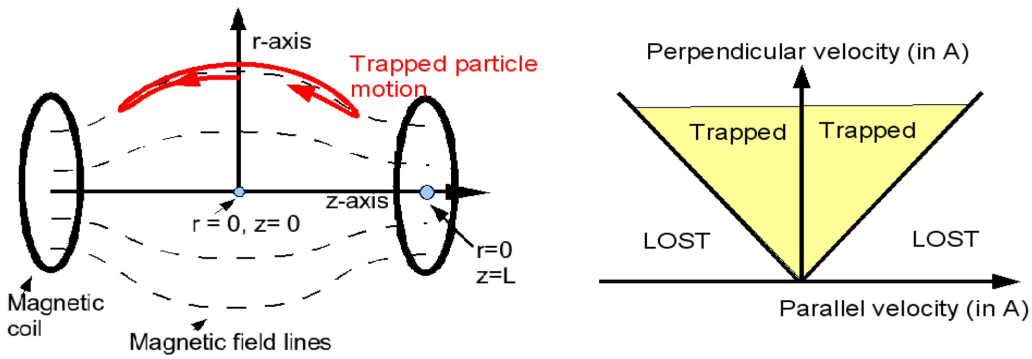

Figure 4.4: The mirror configuration. Left: the magnetic field structure and the orbit of a trapped particle. Right: the region in velocity space (schematic) for which the particles are trapped.

To work out which particles are confined, consider the case without any electric fields such that both the magnetic moment as well as the kinetic energy are conserved.

$$mv_\perp^2/2B = \text{constant} \qquad mv_\parallel^2 + mv_\perp^2 = \text{constant} \tag{4.66}$$

Denoting the quantities at the centre of the device (where the magnetic field is minimum) by the index 0, and the quantities at the end of the device (where the magnetic field is maximum) with the index m, one can derive from the conservation of the magnetic moment that the perpendicular velocity at the coil must be larger than the perpendicular velocity in the magnetic field minimum

$$mv_{\perp m}^2 = \frac{B_m}{B_0} mv_{\perp 0}^2 \tag{4.67}$$

(this is of course under the assumption that the particle will reach the coil, see below) From the conservation of energy it then follows that the parallel energy must decrease when the particle moves form the minimum magnetic field position toward the coil

$$v_{\parallel m}^2 = v_{\parallel 0}^2 + v_{\perp 0}^2 \left[1 - \frac{B_m}{B_0}\right]. \tag{4.68}$$

The mirror force accelerates the particles inward. In fact, the quadratic velocity $v_{\parallel m}^2$ from the equation above is not necessarily positive. Of course, an imaginary velocity does not exists. The particles with a negative $v_{\parallel m}^2$, simply never make it to the end of the device. The mirror force reduces the parallel velocity to zero after which the particles 'bounce' and return back the the middle of the device. After passing the minimum magnetic field position they move again into a region with a higher magnetic field, and they will again bounce back (see Fig. 4.4). Such particles are known as trapped particles. Of course particle trapping was the goal of the mirror device, since these particles are confined.

Not all particles however are trapped. It can be easily seen that the particles with zero magnetic moment, have zero mirror force and will move out of the device on a very short timescale. The boundary between trapped and passing particles can easily be determined by setting the parallel velocity at the maximum magnetic field position to zero. This yields

$$\frac{v_{\parallel 0}}{v_{\perp 0}} = \sqrt{\frac{B_m}{B_0} - 1} \tag{4.69}$$

From which it follows that it is the ratio of the parallel to the perpendicular velocity that determines whether the particles are trapped. Only those with a sufficient small parallel velocity bounce back and forth in the magnetic well, while the others are lost on the timescale L/v_{th}, where L is the length of the device and v_{th} is the thermal velocity. The region in velocity space in which the particles are trapped is shown in Fig. 4.4.

Of course, a larger magnetic field ratio between the end and center of the device will lead to a larger amount of particles being trapped. The ratio, however, can not be infinite, and one will always loose a fraction of the particles. This might be acceptable if sufficient particles remain confined. Collisions, however, make that particles are scattered into the loss region after which they are lost. Roughly speaking the confinement time is determined by the collision frequency. It is clear from the introduction that such a confinement time isn't sufficient to build a working fusion reactor. Several attempts have been undertaken to try to reduce the end losses (special mirror configurations), with rather limited success, since the fundamental problem is hard to overcome. The mirror as a concept for a reactor is made less attractive also through the numerous kinetic instabilities. The loss of particles makes that the distribution of particles in velocity space is far from that obtained in thermodynamic equilibrium (the Maxwell distribution). For a certain parallel velocity, the fast loss makes that the number of particles is strongly depleted for small values of the perpendicular velocity, whereas particles with higher perpendicular velocities are confined. One can think of this situation as the inverse population that exists in a LASER. Indeed there are waves that can tap energy from the particles and that are driven unstable by the 'excess' in perpendicular energy. For the reasons given above the mirror concept was abandoned.

4.5 What do I have to know for the exam

As already indicated in the introduction this is a rather lucky chapter. In essence you will not be asked to re-derive any of the drifts. That is good since it is most of the chapter. You are however expected to know the drifts (yes the formula of the drifts will not necessarily be given in the exam, you have to know it), and there role in the dynamics of the plasma. You are also expected to know which quantities are conserved. In essence, if you study the last section very well, you should be fine. Of course, studying the earlier sections gives you much more understanding of the physics processes, and this can be very helpful in the exam.

Chapter 5

The tokamak

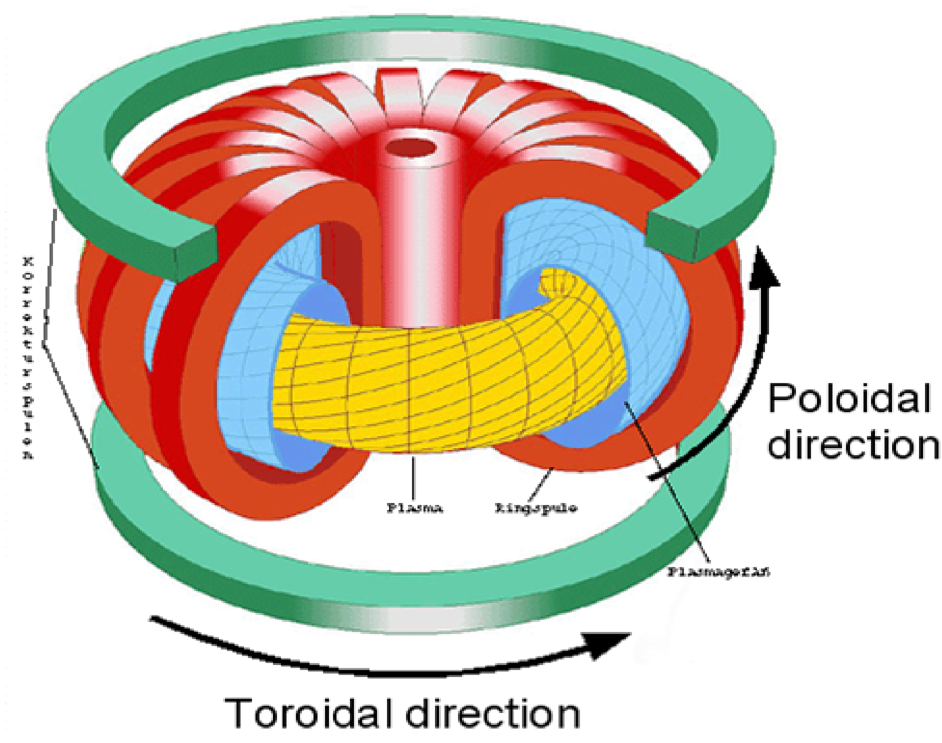

Figure 5.1: 3D representation of a tokamak

Fig. 5.1 shows the schematic picture of the tokamak configuration. In this configuration the plasma has the shape of a donut. It can be seen from the picture that the device has a rotational symmetry around the vertical axis (which will be choosen to be the z-axis). The direction the long way around the donut is refered to as the toroidal direction, whereas the plane perpendicular to this direction is known as the poloidal plane (see Fig. 5.1). A set of field coils wound around the donut generates a strong magnetic field in the toroidal direction.

With the help of the particle orbits one can easily understand why one is led to build this magnetic field configuration. To lowest order the particles gyrate around the field lines while moving freely along the field. For the time being the (small) drift motion due to the inhomogeneous magnetic field will be ignored. This is not entirely justified as will be seen later, but any device that has a rapid loss of particles through the lowest order motion isn't worth considering. As already discussed the Z-pinch is unstable and the Theta-pinch, though perfectly stable, suffers from end losses due to the rapid motion of particles along the field. A remedy to remove any end losses while staying as close as possible to the theta-pinch, can easily be thought of: Bend the theta pinch into a donut shape in which the field lines go around toroidally without ever leaving the toroidal volume. Because of the divergence free nature of the magnetic field it can be proven that such a configuration is the simplest possible topology in which a field line can remain in a finite volume of space. This concept is, of course, the previously mentioned tokamak, and was developed by the Russians in the early 60s. The tokamak rapidly overtook the mirror machines in in confinement properties, and is up to date the most successful concept in magnetic confined fusion.

The toroidal magnetic field is generated by the toroidal field coils shown in the picture. Usually 16 to 32 coils are employed. The finite number of coils makes that toroidal symmetry is not perfectly satisfied, but the ripple in the magnetic field is generally below 1% on the outboard side of the plasma, and much smaller further in. Here, unless explicitly stated otherwise, it will be assumed that the toroidal symmetry is perfectly satisfied. Due to the symmetry a natural choice for the coordinates are the cylindrical coordinates (R, ϕ, z), where R is the major radius (the distance of a point to the axis of symmetry), ϕ is the toroidal angle and z is the vertical coordinate with the z-axis coninciding with the axis of symmetry. These coordinates are sketched in Fig. 5.2. When convenient we will also use the minor radius r wich is defined to be the distance to the plasma centre, rather than the distance to the axis of symmetry. The symmetry of the tokamak is such that no quantity depends on ϕ.

5.1 Toroidal curvature has its price

An immediate consequence of the toroidally curved magnetic field is that the magnetic field strength can no longer be homogeneous. Taking a circular curve in the $z = 0$ plane with the center of the circle at $R = 0$, and integrating the maxwell equation over the enclosed surface (A) yields

$$\int \mathrm{d}^2 A\, \mathbf{n} \cdot \nabla \times \mathbf{B} = \int \mathrm{d}\mathbf{s} \cdot \mathbf{B} = 2\pi R B_t = \mu_0 \int \mathrm{d}^2 A\, \mathbf{n} \cdot \mathbf{J} = \mu_0 I \qquad (5.1)$$

where $\mathbf{n}$ is the unit vector perpendicular to the surface, the Stokes theorem has been applied to transform the surface integral into an integral along the curve, B_t is the toroidal component of the magnetic field (i.e. the component in the direction of the unit vector $\mathbf{e}_\phi$, and I is the total current flowing through the surface. This current is due to the current in the toroidal field coils which cross the surface. For a curve that has a radius large enough for it to enclose the inner legs of the coils, but small enough such that it does not enclose the whole coil, i.e. for a curve that lies inside the plasma chamber, the total current

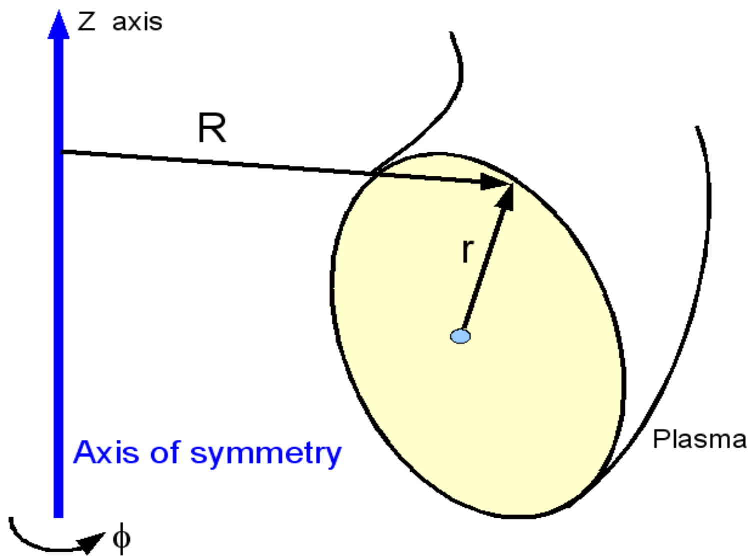

Figure 5.2: Representation of the coordinates

that crosses the surface is constant and the magnetic field varies as $1/R$

$$B_t = \frac{R_0 B_{t0}}{R} \tag{5.2}$$

Here the index 0 refers to a reference point $R = R_0$ where the magnetic field is given $B = B_{t0}$. Closer to the axis of symmetry the magnetic field becomes larger, and the part of the plasma close to the magnetic axis is sometimes referred to as high field side, in contrast with the part at large major radius which is then called the low field side. The inhomogeneity of the magnetic field leads to important physics effect as we will see below. It also has some technological implications. The magnetic field of a reactor which we have quoted before to be around 5T, in fact is set by technological constrains. If one uses superconducting coils there is a maximum magnetic field strength above which the superconductivity can not be maintained. This limits the magnetic field strengths at the coils to 11T, but due to the $1/R$ dependence of the magnetic field the averaged strength in the plasma is then only 5T (The exact numbers depend of course on the minimum major radius of the coils which is set by several technological constraints)

The immediate consequence of the inhomogeneous magnetic field is that a tokamak with only a toroidal magnetic field does not exhibit a proper equilibrium. A plasma generated in such a device would be continuously accelerated out-wards (toward larger R) until it hits the wall. In other words an equilibrium between the plasma pressure and the magnetic pressure / tension, such as has been discussed for the Z-pinch, does not exist for the tokamak if the field is purely in the toroidal direction. This can also be understood from the particle picture. The inhomogeneous magnetic field makes that

$$\mathbf{B} = B_t \mathbf{e}_\phi \qquad \nabla B = -\frac{B_t}{R}\mathbf{e}_R \tag{5.3}$$

where $\mathbf{e}_R$ is the unit vector in the direction of the major radius. The grad B drift then is

$$\mathbf{v}_d = \frac{mv_\perp^2}{2qBR}\mathbf{e}_\phi \times (-\mathbf{e}_R) = \frac{mv_\perp^2}{2qBR}\mathbf{e}_z \tag{5.4}$$

i.e. the drift velocity is in the vertical direction (see Fig. 5.3). For positive magnetic field it is upward for the ions but downward for the electrons. The opposite motion of the charged species represents a charge separation which will generate a vertical electric field (in the negative z-direction). As explained in the chapter on particle motion, the electric field can be calculated from the polarization drift. The plasma is to remain charge neutral and, therefore, the electric field will increase in time such that the polarization drift generates a current equal but opposite to the current generated by the drift. Neglecting the polarization drift of the electrons (due to the much smaller mass) compared with the ions, and replacing the perpendicular velocity of the particles with the averaged thermal velocity ($mv_\perp/2 = T$ where T is the temperature) one can derive the equation for the current in the z-direction (J_z)

$$J_z = env_{pi} + env_{di} - env_{de} = 0 \qquad \frac{m}{eB^2}\frac{\partial E}{\partial t} + \frac{2T}{eBR} = 0 \tag{5.5}$$

In the equation above single charged ions have been assumed ($q = e$). The electric field thus generated in turn leads to an ExB velocity

$$\mathbf{E} = -E\mathbf{e}_z \qquad \rightarrow \qquad \mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = -\frac{E}{B}\mathbf{e}_Z \times \mathbf{e}_\phi = \frac{E}{B}\mathbf{e}_R \tag{5.6}$$

The above two equations can then be combined to obtain

$$\frac{\partial v_{Er}}{\partial t} = \frac{2T}{mR} = \frac{v_{th}^2}{R} \tag{5.7}$$

In roughly one transit time R/v_{th} the ExB velocity will be accelerated to have an outward radial component equal to the thermal velocity. Integrating the equation twice toward time to obtain the radial distance over which the plasma has been shifted yields

$$\Delta R = \frac{1}{2}\frac{v_{th}^2}{R}t^2 \tag{5.8}$$

and putting for simplicity the distance to the wall equal to R (it is of course smaller in reality) one obtains the typical timescale on which the plasma is lost

$$\tau = \sqrt{\frac{2R^2}{v_{th}^2}} \tag{5.9}$$

The confinement time is roughly equal to the transit time, and consequently far too small to be interesting for a working reactor. Although the magnetic field prevents the individual particles from moving across the magnetic field lines, it is their collective behavior (i.e. through the generation of an electric field) which allows the particles to escape the magnetic field. Unfortunately, this a problem more often encountered in the physics of fusion. One might even say that this defines the research area. In general one deals with a many body problem of charged particles, and their interaction with the electro-magnetic field. It is the collective behavior and the many degrees of freedom that makes this area of research rich in physics phenomena. From the viewpoint of a fusion reactor many of these phenomena can be classified as 'not good'. The plasma necessarily has a strong pressure gradient (the pressure must be high in the centre to generate enough fusion reactions, and low at the edge due to the low temperature) and furthermore is confined in a magnetic field configurations that has a gradient in the magnetic field strength (note that the mirror force is $-\mu\nabla B$). The plasma wants to expand and escape the magnetic field, and it often finds ways to do so. Much research is aimed at understanding and reducing the channels through which energy and particles are lost.

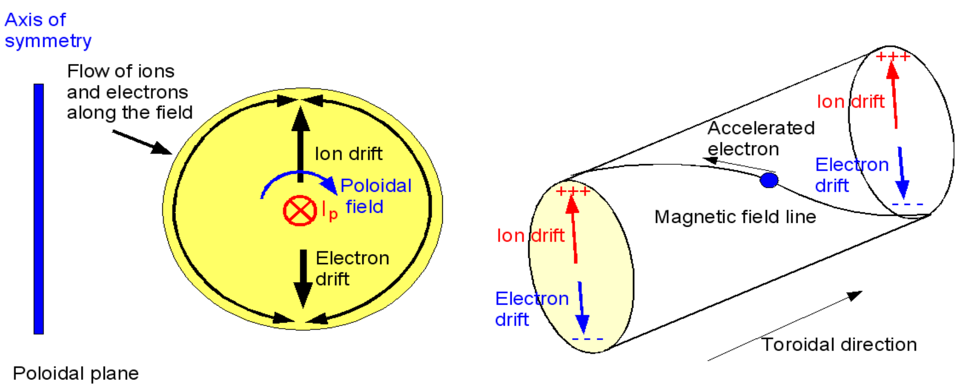

Figure 5.3: Drift direction in the tokamak, helical field lines and the parallel return current

5.2 The plasma current

For the tokamak a simple remedy has been found to remove this rapid loss of confinement. A remedy that has far reaching consequences as we will see below.

The most efficient way to compensate a current generated by the drifts is by a current along the magnetic field lines. Along the field the particles can be freely accelerated and, in the absence of collisions, the current can flow without resistivity, i.e. for zero electric field. If the field is purely toroidal then top and bottom of the plasma are not connected by the magnetic field lines. Introducing a poloidal magnetic field as shown in Fig.5.3, however, would connect the top and the bottom allowing the vertical drift current to be closed by a current along the field lines. The poloidal field can be generated by a current in the toroidal direction inside the plasma. The combination of the toroidal and poloidal field makes that the field lines wind around the donut shaped plasma hellically as shown in Fig. 5.3, and maybe more clearly in Fig. 5.4. One can think of the setup of the parallel flow in the following way. The drifts of ions and electrons lead to charge separation with the top of the tokamak being postively charged, whereas the bottom is negatively charged. The field line however connects the top and the bottom and therefore the electric field connected with the charge separation has a component parallel to the magnetic field. Along the field the electrons and ions can be efficiently accelarated, and this accelaration will lead to a flow of the electrons along the field line from bottom to top, and similarly for the ions from top to bottom. These flows represent again a current, but unlike the drift motion this current runs along the magnetic field. It is this current that will close the vertical drift current as schematically shown in Fig. 5.3 in the left diagram. In the case of zero resistivity, the parallel current can flow without any electric field, and no vertical electric field will build up.

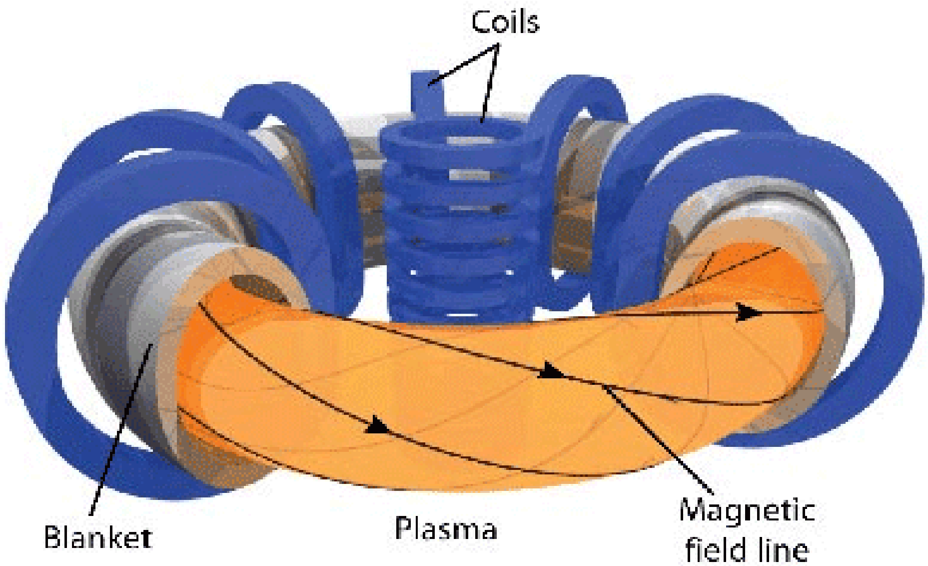

Figure 5.4: Picture of the tokamak that clearly shows the helical winding of the field lines

In order for the tokamak to have an equilibrium it must have a poloidal field and, therefore, a plasma current in the toroidal direction. In general the resistivity ($\eta = 1/\sigma$ where σ is the electric conductivity) of a plasma is very small, but nevertheless finite. A continiuous current must therefore be driven by an electric field.

$$\eta J = E_\phi = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{\partial A_\phi}{\partial t} = -\frac{\partial A_\phi}{\partial t} \tag{5.10}$$

In the equations above Φ is the electro-static potential, and A_ϕ is the toroidal component of the vector potential. Because of toroidal symmetry, all quantities are independent of the toroidal angle ϕ. The electrostatic part of the electric field is therefore zero, and the current can only be driven over a time dependent vector potential. Taking the integral over the circular surface in the $z = 0$ plane using again Stokes theorem one directly obtains

$$\int \mathrm{d}^2 A \, B_z = 2\pi R A_\phi \tag{5.11}$$

And consequently the generation of a current demands an continuously increasing magnetic flux through the $z = 0$ plane. In the tokamak such a flux is provided by a transformer coil as shown in Fig. 5.5. By chaning the current in the transformer coil (primary winding) the magnetic field in the iron core is increased. This leads to the generation of a toroidal electric field. One can think of the plasma as the secondary winding of a normal transformer.

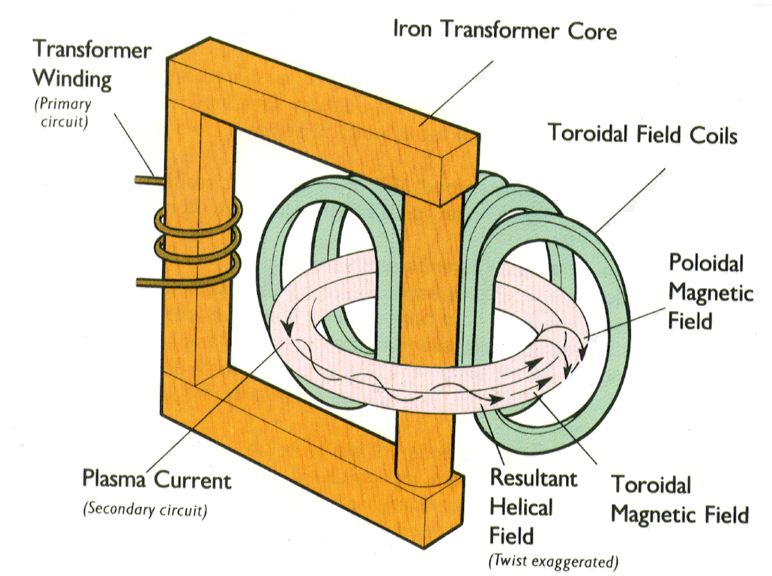

Figure 5.5: Transformer coil that drives the toroidal current in the plasma

So far so good, but the use of the plasma current has a major drawback: It prevents a continuous operation of the tokamak. The flux in the transformer can not be increased indefinitely, and therefore a permanent toroidal current is not possible through this method. This does not largely hinder current experiments, which are for other reasons (cooling !) often designed to run on timescales of several seconds. Also the next step experimental reactor ITER will not be hindered in testing the essential physics ingredients. But it is clear that an electrical power-station which can not operate continuously is a major drawback. The pulsed operation is not only inconvenient, it also puts hard constrains on the design, since the reactor will undergo a pulsed heating / cooling and the thermal stresses are an issue. A further drawback of the plasma current is that it is a source of free energy and can, therefore, drive instabilities. These instabilities are briefly discussed below.

5.3 Magnetic surfaces

The magnetic field due to the plasma current together with the toroidal magnetic field due to the coils make that the field lines wind around the torus helically. If one follows the field line around the torus for many turns it will map out a surface (if it does not close upon itself). These surfaces are known as magnetic surfaces since by definition the magnetic field vector lies in the surface. Using the equilibrium equation

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{5.12}$$

and taking the inner product with the magnetic field one finds

$$\mathbf{B} \cdot \nabla p = 0 \tag{5.13}$$

i.e. there exists no gradient of the pressure along the field lines, and hence in the surface. The magnetic surfaces will be surfaces of constant pressure, with the gradient of the pressure being perpendicular to the surfaces. A more detailed description (considering also the evolution of the temperature) would show that both density as well as temperature are constant on a magnetic surface. The physics picture is that the motion along the magnetic field is not hindered by the Lorentz force and the particles move with the thermal velocity over the surface. Inside the surface any density or temperature perturbation will therefore be quickly smeared out. Profiles of density and temperature can therefore be considered to be one dimensional, only depending on the 'radial' coordinate, i.e. the coordinate perpendicular to the surfaces.

Taking the cross product with the magnetic field of equation 5.12 yields

$$\mathbf{J}_\perp = \frac{\mathbf{B} \times \nabla p}{B^2} \tag{5.14}$$

The current perpendicular to the magnetic field is perpendicular to the pressure gradient and, therefore. also lies in the surface. The total current density can be written as the sum of the perpendicular current and the parallel current, but the latter obviously lies inside the surface as well, and therefore the magnetic surface is also the surface to which the current density is tangent.

Before discussing the instability due to the toroidal current it is useful to introduce the so-called saftey factor (q). This safety factor is defined to be the number of toroidal turns a field line makes during one complete poloidal turn. It, therefore, measures the pitch of the field. The equation of the field line can be written as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \mathbf{B} \tag{5.15}$$

where $\mathbf{x}$ is the position vector and τ is the parameter of the curve. Taking the toroidal and poloidal components of this equation one obtains

$$\frac{\mathrm{d}l_t}{\mathrm{d}\tau} = B_t \qquad \frac{\mathrm{d}l_p}{\mathrm{d}\tau} = B_p \tag{5.16}$$

where l_t (l_p) is the toroidal (poloidal) length of the field line and B_t (B_p) is the toroidal (poloidal) component of the magnetic field. The length the field line moves in the toroidal direction can be expressed in the length it moves in the poloidal direction using the equations above

$$\mathrm{d}l_t = \frac{B_t}{B_p}\mathrm{d}l_p. \tag{5.17}$$

Here we will assume circular magnetic surfaces with a minor radius r. We will also assume that the ratio of the magnetic fields is constant. The equation above can then be integrated to obtain

$$l_t = \frac{2\pi r B_t}{B_p} \tag{5.18}$$

To obtain the safety factor ome must devide the toroidal length by the length of one turn. The latter length is $2\pi R$. Note though that R is not constant on the surface. Here it is implicity assumed that $r << R$ such that the variation in R can be neglected. This approximation will be more often used below in order to obtain analytically tractable results. For a tokamak $a/R \approx 1/3$, where a is the plasma radius.

The approximation is, therefore, not all that accurate, but it does save a lot of mathematics. Using the approximation that R is roughly constant one obtains

$$q = \frac{l_t}{2\pi R} = \frac{rB_t}{RB_p} \tag{5.19}$$

The safety factor is therefore proportional to the ratio of the toroidal and poloidal field strength as well as the ratio of the minor and major radius. One can also use the relation between the poloidal field and the current

$$B_p = \frac{\mu_0 I}{2\pi r} \tag{5.20}$$

(which is obtained from integrating Ampere's law over the flux surface), where I is the total toroidal current through the surface, to express the safety factor as

$$q = \frac{2\pi r^2 B_t}{\mu_0 RI} = \frac{2AB_t}{\mu_0 I} = \frac{2B_t}{\mu_0 \langle J \rangle} \tag{5.21}$$

In the equation above A is the area of the surface, and $\langle J \rangle$ is the averaged current density

$$\langle J \rangle = \frac{I}{A} \tag{5.22}$$

The saftey factor plays an important role as critical parameter for several large scale plasma instabilities. Here only one of them is discussed. In chapter 3 the stability of the screw pinch against the kink was discussed on the bassis of the physical picture that the current leads to an instability whereas the vertical field stabilizes the mode. The tokamak can be thought of as a bend screw pinch, rather than a bend theta pinch. The current in the plasma can therefore lead to a kink instability. whereas the toroidal field stabilizes the mode. It turns out that the safety factor q yields a good critical parameter for this instability. The equation for q given abobe hints at this since q is proportional to the ratio of the toroidal field and the current. The smaller q the larger the current is compared with the toroidal magnetic field, and the more likely it is that the kink will be unstable. It turns out that for $q < 1$ the plasma is always unstable against the kink. For larger values it depends on the details of the profiles but as a rule of the tumb $q = 3$ is used as a critical parameter for tokamak operation (i.e. $q > 3$ implies stability). This kink limit has direct consequences for the ratio of the poloidal and toroidal magnetic field strength. For $q > 3$

$$\frac{B_p}{B_t} = \frac{r}{qR} < \frac{r}{3R} \approx 0.1 \tag{5.23}$$

where $r/R \approx 1/3$ was used. The maximum attainable poloidal field strength is therefore one order of magnitude smaller than the imposed toroidal field.

5.4 Outward shift of the surfaces

Consider a circular surface in the $Z = 0$ plane with its centre on the axis of symmetry as shown in Fig. 5.6 and calculate the magnetic flux through this surface. It is obvious that only the poloidal magnetic field will contribute. This flux is therefore referred to as the poloidal flux. Now consider a second surface in the Z plane for $Z > 0$ (indicated by the index 2), with a different radius such that it again touches

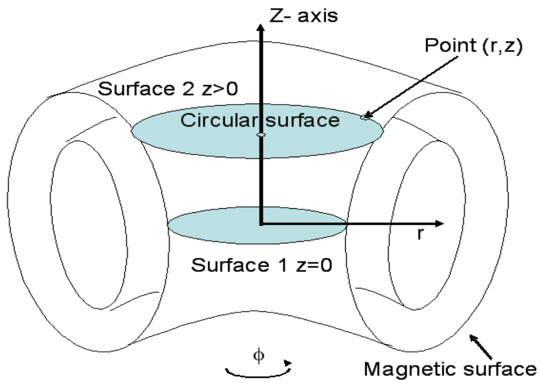

Figure 5.6: Drawing of the surfaces through which the poloidal flux is calculated

the same magnetic surface. Integrating $\nabla \cdot \mathbf{B} = 0$ over the volume enclosed by the two surfaces and the magnetic surfaces yields

$$\int \mathrm{d}^2 A_1 \mathbf{B} \cdot \mathbf{n} = \int \mathrm{d}^2 A_2 \mathbf{B} \cdot \mathbf{n} \tag{5.24}$$

(where we have used the Gauss theorem, and the fact that the flux through the magnetic surface is zero, since the magnetic field lies inside the surface and, therefore has no component through the surface). The poloidal flux on any point of the magnetic surface is therefore the same. The magnetic surfaces are surfaces of constant poloidal flux and are also referred to as flux surfaces.

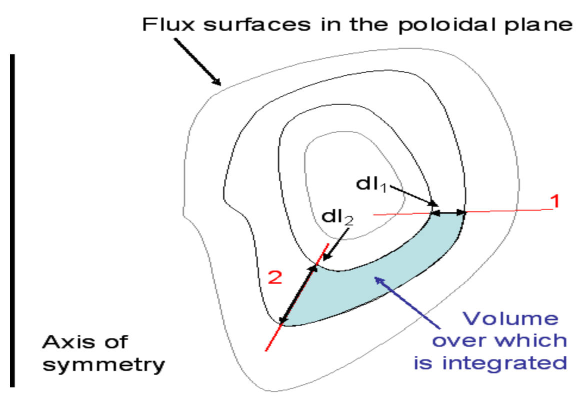

Figure 5.7: Drawing of the volume over which is integrated to relate the poloidal field at two points 1 and 2 on the surface

One can also use the divergence free nature of the magnetic field to derive an expression for the variation of the poloidal magnetic field over the surface. Consider two flux surfaces, and take a toroidally symmetric volume that is bounded by the flux surfaces as well as two surfaces (1 and 2) that cut the flux surfaces at right angles as shown in Fig. 5.7. Again the flux through the bounding flux surfaces is zero and the flux through the surfaces 1 and 2 must be the same. In the limit of a small distance between the surface, this immediately yields

$$B_{p2} = B_{p1} \frac{R_1}{R_2} \frac{\mathrm{d}l_1}{\mathrm{d}l_2} \tag{5.25}$$

where $\mathrm{d}l$ is the distance between the surfaces. If this distance is constant, which is the case for circular concentric surfaces, one finds that the poloidal field, like the toroidal field, varies as $1/R$. Magnetic surfaces in a tokamak have a tendency to be circular, because of the tension in the magnetic field. However, external coils allow for a shaping of the surfaces, and the relation above allows for a direct evaluation of the poloidal field dependence by eye. Where the surfaces are far apart, the poloidal field is weak, where the surfaces are pushed together the poloidal magnetic field is high. The relation above can also be expressed in the poloidal flux. Since the flux is constant on each of the surfaces

$$\delta\psi = |\nabla\psi(1)|\mathrm{d}l_1 = |\nabla\psi(2)|\mathrm{d}l_2 \tag{5.26}$$

and therefore

$$B_p \propto \frac{1}{R}|\nabla\psi| \tag{5.27}$$

5.5 Shaping of the surfaces

As already mentioned the the relation between the poloidal magnetic field strength and the distance between the surfaces allows one to estimate the variation of the field strength by eye when given the shape of the surfaces. It, however, can also be used the other way around. If one wants to shape a surface to have a specific form one can generate a poloidal field using a field coil as shown in Fig. 5.8. If the field generated by the coil increases the field strength it will push the surfaces closer together, whereas the opposite occurs for a poloidal field that weakens the poloidal field of the plasma (see Fig. 5.8).

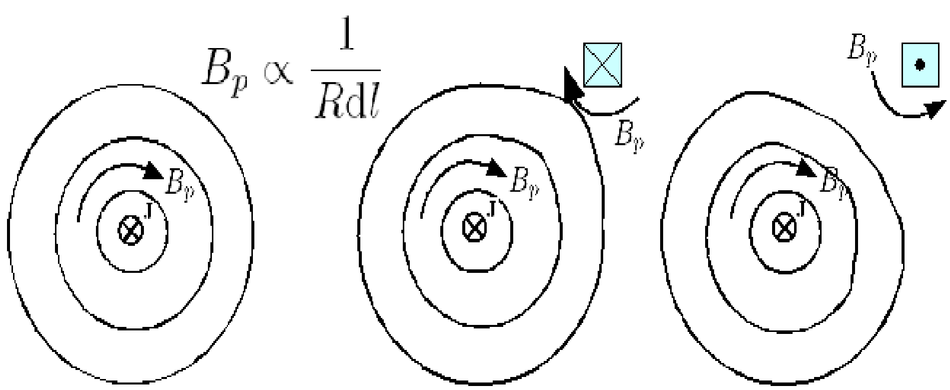

Figure 5.8: Shaping of the surfaces using a field coil

In any modern tokamak the plasma is shaped to obtain a desired shape. The most dominant shaping is the vertical elongation of the plasma. It will be discussed in the next chapter why such a shaping is benificial. The vertical elongation can be directly understood for the discussion above to be generated by two field coils on the top and bottom of the plasma, with a current in the direction of the plasma current as sketched in figure 5.9.

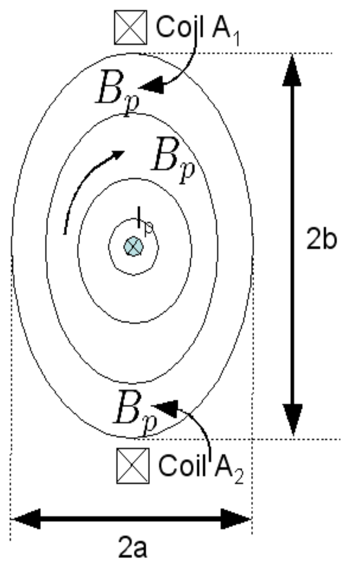

Figure 5.9: Vertical elongation of the plasma using two field coils

The current in the coils can be thought of as pulling the plasma towards them. Indeed since the current is in the direction of the plasma current, and two currents in the same direction attract each other, each of the coils exerts a net force on the plasma. One, therefore, has to consider if the new configuration is still an equilibrium. To investigate one must calculate the poloidal field generated by the coil, after which the force on the plasma can be obtained from the Lorentz force equation. The poloidal field follows from the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.28}$$

Which is hard to solve for this case since the coil is circular (toroidal symmetry applied!!). However, assumming that $b << R$, i.e. the distance between from the coil to the centre of the plasma (see Fig. 5.9) is much smaller than the radius of curvature, one can neglect the bending of the coil, and caclulate the poloidal field as if the coil is straight. This yields

$$2\pi d B_{A1} = \mu_0 I_{A1} \qquad \rightarrow \qquad B_{A1} = \frac{\mu_0 I_{A1}}{2\pi d} \tag{5.29}$$

where d is the distance to the coil, and I_{A1} is the total current in the coil. The force density on the plasma is given by the Lorentz force

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{5.30}$$

The total force on the plasma can then be caculated through the integral over the volume

$$\mathbf{F}_T = \int \mathrm{d}^3V\, \mathbf{J} \times \mathbf{B} = -2\pi R \int \mathrm{d}^2A\, J B_{A1} \mathbf{e}_z = -2\pi R \frac{\mu_0 I_{A1}}{2\pi b} I_p \tag{5.31}$$

where in the last step it has been assumed that all the plasma current is concentrated in the plasma centre. This, of course, is not true, but allows for a simple analytically tractable model. The two coils (indicated by the indices 1 and 2) exert forces in the vertical direction.

$$\mathbf{F}_{T1} = -\frac{\mu_0 R I_{A1} I_p}{b} \mathbf{e}_z \qquad \mathbf{F}_{T2} = \frac{\mu_0 R I_{A2} I_p}{b} \mathbf{e}_z \tag{5.32}$$

which are oppositely directed. It follows directly that for an equilibrium the forces must balance, and therefore

$$I_{A1} = I_{A2} = I_A \tag{5.33}$$

An equilibrium exists when the two currents are equal, generating a symmetricly vertically elongated plasma

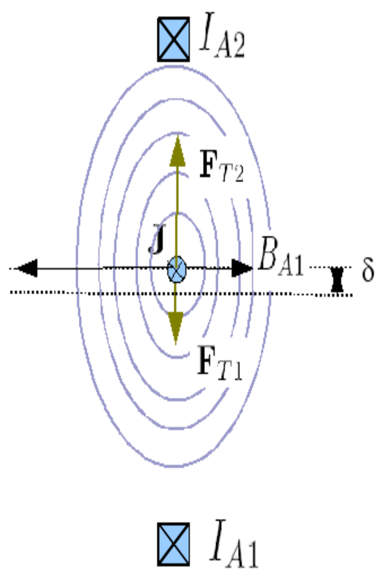

Figure 5.10: A vertical displacement δ of the plasma weakens the poloidal field of the lower coil at the plasma centre, while it strengthens the field due to the upper coil. This leads to a net force in the direction upward, and an unstable growth results.

The existence of an equilibrium, however, does not mean that the plasma is stable. In fact, it is not. A small vertical displacement of the plasma will lead to an exponential growth of the displacement, as will be shown below. Consider the plasma shifted upward by a small distance δ as shown in Fig. 5.10.

Since the magnetic field strength generated by the coil falls of as 1/d, this small displacement leads to a change in the total force on the plasma equal to

$$\mathbf{F}_T = \mu_0 R I_A I_p \left[\frac{1}{b-\delta} - \frac{1}{b+\delta}\right]\mathbf{e}_z \tag{5.34}$$

Assuming the displacement to be small, one can use the Taylor expansion

$$\frac{1}{b+x} = \frac{1}{b}\frac{1}{1+x/b} \approx \frac{1}{b}\left[1 - \frac{x}{b}\right] \tag{5.35}$$

to obtain

$$\mathbf{F}_T = \frac{2\mu_0 R I_A I_p}{b^2}\delta \mathbf{e}_z. \tag{5.36}$$

The upward displacement of the plasma makes that the poloidal field due to the lower coil, which pulls the plasma downward, is weaker. The poloidal field due to the upper coil, which pulls the plasma upward, on the other hand is stronger. A net force in the positive z-direction results, and the plasma is accelerated upward. This calculation shows that a vertical instability exists. Any small pertubation in the vertical position will lead to a rapid acceleration of the plasma after which it is lost to the wall.

The growth rate of the instability can be calculated assuming the total mass of the plasma is M. The equation for the vertical displacment then is

$$M\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = \frac{2\mu_0 R I_A I_p}{b^2}\delta \tag{5.37}$$

Solutions can be obtained assuming

$$\delta = C\exp(\gamma t) \tag{5.38}$$

which after substitution directly gives the growth rate of the mode

$$\gamma = \sqrt{\frac{2\mu_0 R I_A I_p}{M b^2}} \tag{5.39}$$

For typical values the growth rate of the instability discussed above is $10^6\ s^{-1}$, and one would loose any plasma on a very short timescale. One might therefore be led to conclude that an elongated plasma is not possible. This, however, is not true. Elongated plasmas are used in all modern tokamak devices, and an example of a poloidal cross section is shown for the ASDEX Upgrade tokamak in Fig. 5.11. The closed thin lines in this figure represent the closed magnetic surfaces, whereas the dotted lines represent the surfaces that intersect the wall of the device. It can be seen that the plamsma has a clear elongation. The vertical instability is controlled in the following way. Two passive stabilizers, indicated by the blue arrows are placed relatively close to the plasma. These stabilizers are copper coils in which no current is directly driven (note that toroidal symmetry applies, the coils are circular). If the plasma moves up or down the magnetic flux through the stabilizers changes and a current is induced such that it tries to prevent the change of flux, or in other words it pushes the plasma back. The passive stabilizers are of course not perfectly conducting and the current induced in them decays on the resistive timescale of the copper. This makes that the plasma can still be vertically unstable, but the typical timescale is related to the resistive timescales which is of the order of milli-seconds rather than micro-seconds. This slowing down of the instability is enough to allow a control system to keep the plasma in position. One simply measures the vertical position of the plasma. When it moves upward, one decreases the current in the upper coil and increases the current in the lower coil. This generates a net force downward, and the plasma can be pulled back to its original position. In general one therefore runs an unstable equilibrium, that is only helt in place through a control system.

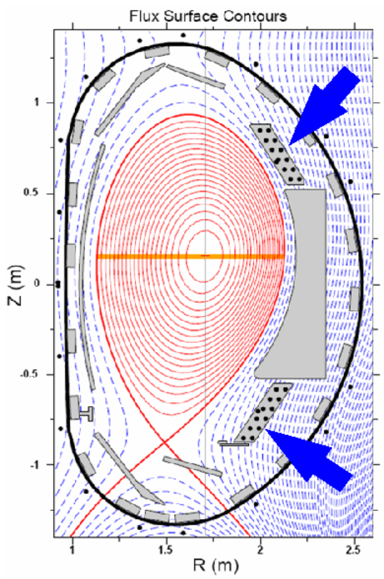

Figure 5.11: Poloidal cross section of the ASDEX Upgrade tokamak. The passive stabilizers are indicated by the arrows

Appendix A

Constants of nature used in these lecture notes

The proton mass

$$m_H = 1.6726 \cdot 10^{-27} \text{ kg}$$

The speed of light

$$c = 2.9979 \cdot 10^{8} \text{ m/s}$$

The Boltzmann Constant

$$k = 1.3806 \cdot 10^{-23} \text{ JK}^{-}1$$

Unit of energy / Temperature

$$1\ rmeV = 1.6022 \cdot 10^{-19} \text{ J}$$

Appendix B

Vector identities

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \tag{B.1}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{B.2}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \tag{B.3}$$

$$\nabla \times \nabla \psi = 0 \tag{B.4}$$

$$\nabla \cdot \nabla \times \mathbf{a} = 0 \tag{B.5}$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \tag{B.6}$$

$$\nabla \cdot \psi \mathbf{a} = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a} \tag{B.7}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a} \tag{B.8}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times \nabla \times \mathbf{b} + \mathbf{b} \times \nabla \times \mathbf{a} \tag{B.9}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b} \tag{B.10}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \nabla \cdot \mathbf{b} - \mathbf{b} \nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \tag{B.11}$$