Stationarity and stochastic processes

Stationarity

Implies that different realisations/samples are equivalent, useful since it follows that:

$\Rightarrow$ many realisations can be drawn in sequence from a single time series

- for stochastic processes this equivalence is statistical
- for deterministic processes this equivalence is repetition under time shift (periodicity)

$\Rightarrow$ implies time independence of power spectrum

a) Strong/strict stationarity
for a sequence x_k at two times k_1, k_2 define

$$F_{k_1,k_2}(a_1, a_2) = P[x_{k1} \le a_1, x_{k2} \le a_2]$$

P is just the joint CDF – expresses the correlation structure of x_k

Strong stationarity is then:

$$F_{k_1,k_2 \ldots k_N}(a_1, a_2 \ldots a_N) = F_{k_1+\tau, k_2+\tau} \ldots (a_1 \ldots a_N)$$

- completely stationary under time shift τ

- not a practical definition!

b) Weak/2nd order stationarity
all the joint moments of x_k up to order 2 are same as that of $x_{k+\tau}$

ie: $\langle x_k \rangle, \langle x_k^2 \rangle$ are independent of τ.

Alternatively, a stationarity test is provided by the covariance:

$$\text{cov}(x,y) = \frac{1}{N}\sum_{l}^{N}(x_k - \bar{x})(y_k - \bar{y}) = \langle (x_k - \bar{x})(y_k - \bar{y}) \rangle.$$

Now consider $\text{cov}(x_{k_1}, x_{k_2})$ where k_2 is just k_1 shifted by τ

$$= \text{cov}(x_0, x_{k_2-k_1}) = \text{cov}(x_0, x_\tau)$$

This is only a function of τ, not position in sequence.

This is usually normalized to the autocovariance : $\dfrac{\text{cov}(x_0, x_\tau)}{\text{cov}(x_0, x_0)}$

Spectral representation of stochastic processes (outline)

Issues are (i) defining an integral over a stochastic process and (ii) convergence of the integral for random walks (sums of stochastic steps) which grow without bound (outline not formal proof!).

Consider a finite interval of a discrete stochastic process x_k.
recall DFT

$$x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} S_m e^{2\pi imk/N} \qquad \text{- Not stochastic}$$

Write instead for our stochastic process

$$x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} S_m e^{\left[2\pi i\frac{mk}{N}+i\phi_m\right]}.$$

This is a 'random phase' model for the stochastic process x_k. The S_m are real constants and the ϕ_m is a stochastic, iid random variable.

Write this as $x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} D_m e^{2\pi imk/N}$ where $D_m = S_m e^{i\phi_m}$

It follows that the statistical properties of the D_m are that of the ϕ_m, ie:
if $\langle \phi_m \rangle = 0$ $\langle D_m \rangle = 0$ and $\langle D_m^2 \rangle = S_m^2$

This follows since $D_m D_m^* = S_m^2$ for each *m*, and the ϕ_m are uncorrelated.

We can think of the D_m as a stationary stochastic process. We can build a random walk (which is not stationary) from stationary stochastic steps ie:

$$z(f) = \sum_{m=0}^{N-1} D_m \qquad f_m < f \le f_{m+1}$$

which defines stochastic increment $dz(f) = z(f+df) - z(f)$,
then $dz(f_m) = D_m$, this is the "power" in f_m, so in an appropriate continuous limit is equivalent to $z(f)df$. The continuous limit is via the definition of a Riemann-Stieltjes integral.

Then the spectral representation theorem is:

$$x(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi ift}\; dz(f) \quad \text{formally equivalent to } x_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} D_m e^{2\pi if_m t_k}$$

where dz is stochastic – this is a Riemann-Stieltjes integral.

All the usual results of Fourier theory follow (since we have an expansion in an orthogonal set). In particular, we are interested in scaling – random walks (coloured noises) with power law power spectra.