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Generation of residual energy in the turbulent solar wind

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In situ observations of the fluctuating solar wind flow show that the energy of magnetic field fluctuations always exceeds that of the kinetic energy, and therefore the difference between the kinetic and magnetic energies, known as the residual energy, is always negative. The same behaviour is found in numerical simulations of magnetohydrodynamic turbulence. We study the dynamics of the residual energy for strong, anisotropic, critically balanced magnetohydrodynamic turbulence using the eddy damped quasi-normal Markovian approximation. Our analysis shows that for stationary critically balanced magnetohydrodynamic turbulence, negative residual energy will always be generated by nonlinear interacting Alfvén waves. This offers a general explanation for the observation of negative residual energy in solar wind turbulence and in the numerical simulations. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4764469]

I. INTRODUCTION

Plasma turbulence plays an important role in many astrophysical problems, such as heating of the solar wind, transport in accretion disks, and scattering of cosmic rays (for a review, see, e.g., Refs. 1 and 2). Magnetohydrodynamics is a suitable approach to study relatively large scale, low frequency phenomena. Moreover, incompressible Alfvénic fluctuations in a collisionless plasma (such as in the solar wind) are well described by ideal magnetohydrodynamic (MHD) equations despite the collisionless nature of the plasma.[3] The first model of incompressible MHD turbulence was proposed by Iroshnikov[4] and Kraichnan[5] who realized one of the key features of the Alfvénic turbulence–nonlinear interaction is possible only between Alfvén waves propagating in opposite directions along the mean magnetic field. The energy cascade in MHD turbulence then occurs as a result of collisions between these oppositely propagating Alfvén waves. Ideal MHD equations conserve separately the energies of both interacting waves that are propagating parallel and anti-parallel w.r.t. the mean magnetic field. In the Alfvén wave there is an equipartition between magnetic ($E_m = b^2/8\pi$, where $\mathbf{b}$ is the magnetic field perturbation) and kinetic ($E_k = \rho v^2/2$, where $\mathbf{v}$ is the velocity perturbation and ρ is the density) energies. Hence one might expect that this equipartition should hold for developed MHD turbulence as well. Surprisingly, both solar wind observations[6–11] and numerical simulations[12,15,16] show that in the inertial interval of MHD turbulence, the difference between the kinetic and magnetic energies, the residual energy $E^R = E_k - E_m$, is always negative, i.e., the magnetic energy always exceeds the kinetic energy. Therefore, negative residual energy in the entire inertial interval seems to be generic feature of MHD turbulence, and any complete model of MHD turbulence should be consistent with this fact.

The dynamics of the residual energy in MHD turbulence has been studied theoretically by different authors in the past.[1,13–17] These studies were focused on understanding the origin of negative residual energy and on determination of its self-similar spectrum in the inertial interval of MHD turbulence. Grappin *et al.*[14] studied the dynamics of the residual energy for isotropic MHD turbulence in the framework of the eddy damped quasi-normal Markovian (EDQNM) approximation (for application of the EDQNM to hydrodynamic turbulence, see Refs. 18 and 19. This approach was first applied to MHD turbulence in Refs. 20 and 21). They concluded that the stationary solution of the residual energy spectrum $E^R(k)$ is the result of the balance between the "dissipation term" provided by the local nonlinear interactions and the "generation term" related to the nonlocal interactions of Alfvén waves. Müller and Grappin[15] extended these results to the case of globally isotropic MHD turbulence in the presence of strong background magnetic field. Recently Wang *et al.*[16] studied the dynamics of the residual energy in the framework of *weak* turbulence theory. The authors derived the governing equation of the residual energy and suggested (but did not show) that the interaction of Alfvén waves spontaneously generates negative residual energy even if it is absent initially. This suggestion was supported by numerical simulations. Using a simple model of weakly colliding Alfvén waves it has been demonstrated that the magnetic energy is generated more effectively than the kinetic energy.[17]

In this paper we show for the first time how negative residual energy must arise generally from *strong* MHD turbulence. We study the dynamics of residual energy for strong, anisotropic critically balanced MHD turbulence in the framework of the EDQNM approximation. We derive the governing equation for residual energy and show that *even if the residual energy is absent initially*, negative residual energy is generated by nonlinearly interacting Alfvén waves. This provides a natural explanation for the observed properties of the residual energy in the solar wind and in numerical simulations of strong MHD turbulence.

II. MAIN FORMALISM

We consider incompressible MHD turbulence in the presence of a constant magnetic field $\mathbf{B}_0$ directed along z axis. The Elsasser variables $\mathbf{w}^{\pm} = \mathbf{v} \pm \mathbf{b}/\sqrt{4\pi\rho}$, the eigenfunctions of counter propagating Alfvén waves, are usually considered as the most fundamental variables to study MHD turbulence.[1] The dynamics of the Elsasser variables in the absence of dissipative effects is governed by the incompressible MHD equations

$$\left(\frac{\partial}{\partial t} \mp \mathbf{V}_A \cdot \nabla\right)\mathbf{w}^{\pm} + (\mathbf{w}^{\mp} \cdot \nabla)\mathbf{w}^{\pm} + \nabla P = 0, \quad (1)$$

$$\nabla \cdot \mathbf{w}^{\pm} = 0. \quad (2)$$

Here P is the total (hydrodynamic plus magnetic) pressure and $\mathbf{V}_A \equiv \mathbf{B}_0/\sqrt{4\pi\rho}$ is the Alfvén velocity.

We Fourier transform and eliminate the pressure terms to obtain[22]

$$\left(\frac{\partial}{\partial t} \mp i\omega_{\mathbf{k}}\right)\tilde{\mathbf{w}}^{\pm} = -i\int d\mathcal{F}^{k}_{1,2}[\tilde{\mathbf{w}}^{\pm}_1 - \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot \tilde{\mathbf{w}}^{\pm}_1)](\mathbf{k} \cdot \tilde{\mathbf{w}}^{\mp}_2), \quad (3)$$

where the caret denotes the unit vector, $\tilde{\mathbf{w}}^{\pm}(\mathbf{k})$ is the Fourier transform of $\mathbf{w}^{\pm}$, $\tilde{\mathbf{w}}^{\pm}_1$ denotes $\tilde{\mathbf{w}}^{\pm}(\mathbf{k}_1)$, $\omega_{\mathbf{k}} = V_A k_z$ is the frequency of the Alfvén wave, $d\mathcal{F}^{k}_{1,2} \equiv d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}$, and $\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \equiv \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$ is the Dirac delta function.

Incompressible anisotropic MHD turbulence is governed by the interaction of shear Alfvén waves, whereas the pseudo Alfvén waves (incompressible limit of the slow magnetosonic wave) play a passive role.[22] Therefore, here we will consider shear Alfvénic turbulence. We define the unit polarization vector of the shear Alfvén wave as $\hat{\mathbf{e}}_{\mathbf{k}} = \hat{\mathbf{k}}_{\perp} \times \hat{\mathbf{z}}$ (here $\hat{\mathbf{z}}$ denoted unit vector in z direction and $\hat{\mathbf{k}}_{\perp}$ is the unit vector parallel to the perpendicular component of the wave vector $\mathbf{k}$) and introduce the amplitudes of the shear Alfvén waves $\phi_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$ as

$$\tilde{\mathbf{w}}^{-}_{\mathbf{k}} = i\phi_{\mathbf{k}}\hat{\mathbf{e}}_{\mathbf{k}}, \quad \tilde{\mathbf{w}}^{+}_{\mathbf{k}} = i\psi_{\mathbf{k}}\hat{\mathbf{e}}_{\mathbf{k}}. \quad (4)$$

Equation (3) than reduces to the following equations:

$$\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{k}}\right)\phi_{\mathbf{k}} = \int T^{k}_{1,2}\phi_1\psi_2 d\mathcal{F}^{k}_{1,2}, \quad (5)$$

$$\left(\frac{\partial}{\partial t} - i\omega_{\mathbf{k}}\right)\psi_{\mathbf{k}} = \int T^{k}_{1,2}\psi_1\phi_2 d\mathcal{F}^{k}_{1,2}, \quad (6)$$

where $T^{k}_{1,2} \equiv (\hat{\mathbf{e}}_{\mathbf{k}} \cdot \hat{\mathbf{e}}_{\mathbf{k}_1})(\mathbf{k} \cdot \hat{\mathbf{e}}_{\mathbf{k}_2})$ is the matrix element of the interaction.

As it is known,[19] for homogeneous turbulence correlation functions of turbulent fields are anti-diagonal in the wave number space [i.e., $\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle \sim \delta(\mathbf{k}_1 + \mathbf{k}_2)$], and therefore for second order correlation functions we have

$$\langle\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\rangle = 4E^{-}_{\mathbf{k}}\delta_{\mathbf{k}+\mathbf{k}'}, \quad (7)$$

$$\langle\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\rangle = 4E^{+}_{\mathbf{k}}\delta_{\mathbf{k}+\mathbf{k}'}, \quad (8)$$

$$\langle\phi_{\mathbf{k}}\psi_{\mathbf{k}'}\rangle = 2Q_{\mathbf{k}}\delta_{\mathbf{k}+\mathbf{k}'}. \quad (9)$$

Here $E^{\pm}_{\mathbf{k}}$ are the spectral energy densities of the counterpropagating Alfvén waves (henceforth we omit background density, ρ, in all equations) and angle brackets denote ensemble averages. The residual energy spectral density is the real part of the cross correlation $E^{R}_{\mathbf{k}} = \mathrm{Re}[Q_{\mathbf{k}}]$, and the imaginary part $E^{S}_{\mathbf{k}} = \mathrm{Im}[Q_{\mathbf{k}}]$ represents the antisymmetric part of the cross correlation.[9]

A dynamical equation for $E^{\pm}_{\mathbf{k}}$ derived using Eqs. (5) and (6) will contain the terms of the form $\langle\phi_1\psi_2\psi_{\mathbf{k}}\rangle$, i.e., the third order correlators. The time evolution equation for the third order correlators in turn will contain the forth order correlators and so on. Therefore some closure approximation is needed to derive the closed set of the dynamical equations. Goldreich and Sridhar[22] studied strong critically balanced MHD turbulence using EDQNM[19] approximation. The key idea of this method is the quasi-normality assumption,[18] which implies that the relation between the forth order correlators of the turbulent fields with the second order correlations are that of Gaussian random variables. In addition these authors assumed that the residual energy is zero, i.e., that there is no correlation between ϕ and ψ fields. Under these assumptions Goldreich and Sridhar[22] showed that strong, critically balanced MHD turbulence in the plane perpendicular to the mean magnetic field follows Kolmogorov's scaling, i.e., two dimensional perpendicular energy spectrum $\mathcal{E}^{\pm}(\mathbf{k}_{\perp}) = \int E^{\pm}_{\mathbf{k}} \mathrm{d}k_{\parallel}$ in the inertial interval is $\mathcal{E}^{\pm}(\mathbf{k}_{\perp}) \sim k_{\perp}^{-8/3}$.

Our approach is quite different from that of Ref. 22. We derive the governing equation for the residual energy and study its dynamics by treating the residual energy as a passive admixture, i.e., neglecting its feedback on the dynamics of $E^{\pm}_{\mathbf{k}}$. This approach is valid at least for the situations where residual energy spectral density is small relative to that of the wavefield $E^{R}_{\mathbf{k}} \ll E^{\pm}_{\mathbf{k}}$.

According to Eqs. (5) and (6) we have the following equation for the mixed second order correlation function:

$$\partial_t\langle\phi_{\mathbf{k}}\psi_{\mathbf{k}'}\rangle = 2\partial_t Q_{\mathbf{k}}\delta_{\mathbf{k}+\mathbf{k}'} = -i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\langle\phi_{\mathbf{k}}\psi_{\mathbf{k}'}\rangle + \int T^{k}_{1,2}\langle\phi_1\psi_2\psi_{\mathbf{k}'}\rangle d\mathcal{F}^{k}_{1,2} + \int T^{k'}_{1,2}\langle\phi_2\psi_1\phi_{\mathbf{k}}\rangle d\mathcal{F}^{k'}_{1,2}. \quad (10)$$

Combining Eqs. (5) and (6) for third order correlation function we have

$$\begin{aligned}\partial_t\langle\phi_1\psi_2\psi_{\mathbf{k}}\rangle = {}& -i(\omega_1 - \omega_2 - \omega_{\mathbf{k}})\langle\phi_1\psi_2\psi_{\mathbf{k}}\rangle \\ &+ \int T^{1}_{3,4}\langle\phi_3\psi_4\psi_2\psi_{\mathbf{k}}\rangle d\mathcal{F}^{1}_{3,4} \\ &+ \int T^{2}_{3,4}\langle\psi_3\phi_4\phi_1\psi_{\mathbf{k}}\rangle d\mathcal{F}^{2}_{3,4} \\ &+ \int T^{k}_{3,4}\langle\psi_3\phi_4\phi_1\psi_2\rangle d\mathcal{F}^{k}_{3,4}. \end{aligned} \quad (11)$$

Using quasi-normal approximation to express fourth order correlation functions by the second order correlations ($\langle\phi_1\phi_2\phi_3\phi_4\rangle = \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle + \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle + \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle$), and using Eqs. (7)–(9) and taking into account that for homogeneous turbulence $\langle\phi_1\psi_2\psi_{\mathbf{k}}\rangle \sim \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}}$, we have

$$\begin{aligned}(\partial_t + 2i\omega_1)\langle\phi_1\psi_2\psi_{\mathbf{k}}\rangle = {}& \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}}[16T^{k}_{2,1}E^{-}_1(E^{+}_2 - E^{+}_{\mathbf{k}}) \\ &+ 8(T^{1}_{2,k}Q^{*}_2E^{+}_{\mathbf{k}} - T^{k}_{1,2}Q^{*}_{\mathbf{k}}E^{+}_2) \\ &+ NL(Q,Q). \end{aligned} \quad (12)$$

Here $NL(Q, Q)$ denotes all nonlinear terms proportional to $Q_{\mathbf{k}} Q_{1,2}$ and asteric denotes complex conjugation. Because here we study the case $|Q_{\mathbf{k}}| \ll E^{\pm}_{\mathbf{k}}$ this term will be omitted in further analysis.

In accordance with standard EDQNM technique (e.g., Refs. 19 and 22) we add linear demping term $\eta^-_1 \langle \phi_1 \psi_2 \psi_{\mathbf{k}} \rangle$ (where η^-_1 is so-called eddy damping rate) to the left hand side of Eq. (12). Stationary solution of the obtained equation is

$$\langle \phi_1 \psi_2 \psi_{\mathbf{k}} \rangle = 16 \Theta^-_1 T^k_{2,1} E^-_1 (E^+_2 - E^+_{\mathbf{k}}) \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}} + 8 \Theta^-_1 (T^1_{2,k} Q^*_2 E^+_{\mathbf{k}} - T^k_{1,2} Q^*_{\mathbf{k}} E^+_2) \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}}. \quad (13)$$

Here $\Theta^{\pm}_{\mathbf{k}} = 1/(\eta^{\pm}_{\mathbf{k}} \mp 2i\omega_{\mathbf{k}})$. The first term on the right hand side of Eq. (13) is identical to the expression of the third order mixed moment derived in Ref. 22, where MHD turbulence with zero cross correlation ($Q_{\mathbf{k}} = 0$) was studied. The second term on the right hand side of Eq. (13) is the term related to the nonzero cross correlation.

Substituting Eq. (13) into Eq. (10) after simple manipulations we derive the following dynamical equation for the residual energy spectrum and its complex counterpart:

$$\partial_t Q_{\mathbf{k}} + 2i\omega_{\mathbf{k}} Q_{\mathbf{k}} = 8 \int d\mathcal{F}^k_{1,2} T^k_{1,2} T^k_{2,1} \times [\Theta^-_1 E^-_1 (E^+_2 - E^+_{\mathbf{k}}) + (\Theta^+_1)^* E^+_1 (E^-_2 - E^-_{\mathbf{k}})] + NL_1(E^{\pm}, Q), \quad (14)$$

where $NL_1(E^{\pm}, Q)$ denotes all nonlinear terms proportional to the products $E^{\pm} Q$ and asteric stands for the complex conjugation.

For further simplification of this equation we will make several assumptions: (i) We consider the balanced case, i.e., MHD turbulence with zero cross helicity ($E^+_{\mathbf{k}} = E^-_{\mathbf{k}}$, $\eta^+_{\mathbf{k}} = \eta^-_{\mathbf{k}} = \eta_{\mathbf{k}}$); (ii) we assume critically balanced strong MHD turbulence, i.e., assume that linear ($\tau_l \sim V_A k_\parallel$) and nonlinear ($\tau_{nl}$) timescales are of the same order of magnitude. The critical balance condition leads to the scaling relation between characteristic longitudinal and transverse scales of turbulent wave packets $k_\parallel = f(q) \sim q^\nu$, where $\mathbf{q}$ is the perpendicular component of the wave vector. For the eddy damping rate we have[22] $\eta_{\mathbf{k}} = \eta_0 q^2 [k_\parallel E(k_\parallel, \mathbf{q})]^{1/2}$. In the context of Eq. (14) this means that (for details see Eqs. (40)-(43) of Ref. 22) $\int dk_\parallel E^{\pm}_{\mathbf{k}} Re[\Theta_{\mathbf{k}}] = \alpha \mathcal{E}(\mathbf{q})/\eta(q)$, where $\eta(q) \equiv \eta(k_\parallel(q), q) = \eta_0 q^2 [f(q) E(f(q), \mathbf{q})]^{1/2} = \beta q^2 [\mathcal{E}(\mathbf{q})]^{1/2}$. Here α and β are the positive constants of order unity; (iii) in accordance with the Goldreich-Sridhar model (see Eq. (36) of Ref. 22) we assume that the Fourier spectra $E^{\pm}_{\mathbf{k}}$ are symmetric functions in the wave number space, i.e., we assume $E^{\pm}(k_\parallel, \mathbf{q}) = E^{\pm}(-k_\parallel, \mathbf{q})$; and (iv) we assume that the turbulence is axially symmetric, i.e., $\mathcal{E}(\mathbf{q}) = \mathcal{E}(q)$. Integrating Eq. (14) with respect to $k_\parallel$, taking into account that $T^k_{1,2}$ does not depend on $k_\parallel$, and noting that due to the symmetry property $E^{\pm}(k_\parallel, \mathbf{q}) = E^{\pm}(-k_\parallel, \mathbf{q})$ imaginary part of the first term on the right hand side of the equation is zero, for two dimensional spectrum $\mathcal{Q}_q = \int dk_\parallel Q_{\mathbf{k}}$ (similarly, two dimensional spectra of the residual energy $\mathcal{E}^R_q$ and its complex counterpart $\mathcal{E}^S_q$ are defined as $\mathcal{E}^{R,S}_q = \int dk_\parallel E^{R,S}_{\mathbf{k}}$) we obtain

$$\partial_t \mathcal{Q}_q + 2i \int dk_\parallel \omega_{\mathbf{k}} Q_{\mathbf{k}} = \alpha \int T^k_{1,2} T^k_{2,1} \frac{\mathcal{E}_1(\mathcal{E}_2 - \mathcal{E}_q)}{\eta_1} d\mathcal{G}^q_{1,2} + NL_2(\mathcal{E}^{\pm}, \mathcal{Q}), \quad (15)$$

where $d\mathcal{G}^q_{1,2} \equiv d^2\mathbf{q}_1 d^2\mathbf{q}_2 \delta_{\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2}$. In derivation of the first term on the right hand side of Eq. (15) we used the identity $\int dk_\parallel dk_{1\parallel} dk_{2\parallel} \delta_{k_\parallel - k_{1\parallel} - k_{2\parallel}} \Theta^-_1 E^-_1 E^+_2 = \alpha \mathcal{E}^-_1 \mathcal{E}^+_2 / \eta^-_1$. The second term on the rhs of Eq. (15) is nonlinear term obtained after the integration of the corresponding term of Eq. (14).

Our main interest here is to study the first term on the rhs of Eq. (15) which can result in the production of residual energy even if it is zero initially. The matrix element of interactions can be expressed as

$$T^k_{1,2} \equiv (\hat{\mathbf{e}}_{\mathbf{k}} \cdot \hat{\mathbf{e}}_{\mathbf{k}_1})(\mathbf{k} \cdot \hat{\mathbf{e}}_{\mathbf{k}_2}) = q \cos\theta_1 \sin\theta_2, \quad T^k_{2,1} = -q \cos\theta_2 \sin\theta_1, \quad (16)$$

where $\theta_{1,2}$ are the angles between $(\mathbf{q}_1, \mathbf{q})$ and $(\mathbf{q}, \mathbf{q}_2)$, respectively. Noting also that

$$q_1 \sin\theta_1 = q_2 \sin\theta_2, \quad q_1 \cos\theta_1 = q - q_2 \cos\theta_2, \quad (17)$$

the first term I on the rhs of Eq. (15)

$$I \equiv 16\alpha \int T^k_{1,2} T^k_{2,1} \frac{\mathcal{E}_1(\mathcal{E}_2 - \mathcal{E}_q)}{\eta_1} d\mathcal{G}^q_{1,2}, \quad (18)$$

can be expressed as a difference of two integrals $(1/16\alpha) I = I_1 - I_2$, where

$$I_1 = \int d\mathcal{G}^q_{1,2} \sin^2\theta_2 \cos^2\theta_2 \frac{q^2 q_2^2}{q_1^2 \eta_1} \mathcal{E}_1(\mathcal{E}_2 - \mathcal{E}_q), \quad (19)$$

$$I_2 = \int d\mathcal{G}^q_{1,2} \sin^2\theta_2 \cos\theta_2 \frac{q^3 q_2}{q_1^2 \eta_1} \mathcal{E}_1(\mathcal{E}_2 - \mathcal{E}_q). \quad (20)$$

Integral I_1 coincides with the integral which describes the temporal evolution of the energy density (Eq. (45) of Ref. 22) and therefore is zero for stationary critically balanced turbulence. We therefore only need to consider I_2. Performing integration with respect to the $\mathbf{q}_1$, introducing new variable $r = q_2/q$, and taking into account that $\mathcal{E}_q \sim q^{-\mu}$ and consequently $\eta_1 \sim q_1^{2-\mu/2}$, where for the GS model $\mu = 8/3$, Eq. (20) can be expressed as

$$I_2 \sim q^{2-3\mu/2} \int_0^\infty dr r^2 (r^{-\mu} - 1) \times \int_0^{2\pi} \frac{d\theta_2 \sin^2\theta_2 \cos\theta_2}{(1 + r^2 - 2r\cos\theta_2)^{2+\mu/4}}. \quad (21)$$

The transformation $r \to 1/r$ converts the integral over the interval $[1, \infty]$ to an integral over the interval [0,1]. Thus,

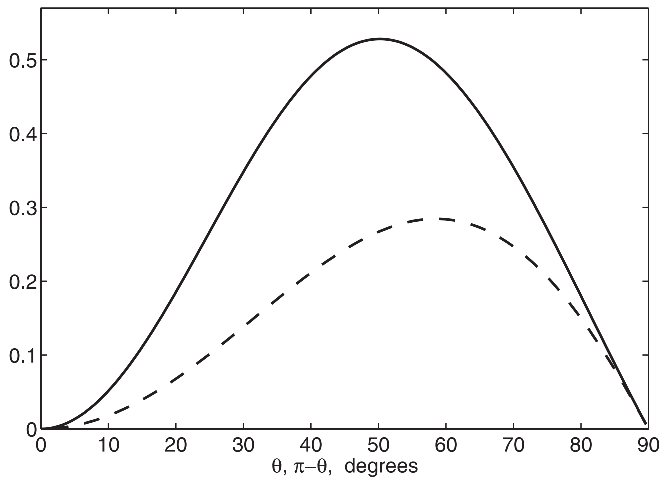

FIG. 1. The integrand of Eq. (22). See text for details.

$$I_2 \sim q^{2-3\mu/2}\int_0^1 \mathrm{d}r(r^2 - r^{3\mu/2})(r^{-\mu} - 1) \times \int_0^{2\pi} \frac{\mathrm{d}\theta_2 \sin^2\theta_2 \cos\theta_2}{(1 + r^2 - 2r\cos\theta_2)^{2+\mu/4}}. \quad (22)$$

It is straightforward to show that this integral is always positive. Indeed, for any fixed value of r and any $\theta_2 = \theta_0$ such that $\cos\theta_0 < 0$, the contribution to the integral of the vicinity of this point (r, θ) is always less than contribution of the vicinity of the point $(r, \pi - \theta_0)$ and therefore $I_2 > 0$. This is illustrated in Figure 1. The continuous line is the integrand of Eq. (22) as a function of θ plotted in the interval $[0, 90°]$ for $r = 0.3$. The dotted line is the absolute value of the integrand as a function of $180° - \theta$ plotted in the interval $[90°, 180°]$. We can see that the positive value of the integrand in the interval $[0, 90°]$ is always greater than the negative value at the corresponding point in the interval $[90°, 180°]$.

We have shown that the nonlinear "production" term in the equation of the residual energy for stationary critically balanced turbulence is always negative and consequently, even if the residual energy *is absent initially*, it will be generated by the strong nonlinear interactions of Alfvén waves. In the derivation above we assumed that the turbulence is balanced, i.e., assumed that cross helicity is equal to zero. The formalism developed in the previous chapter can be straightforwardly extended for the case in imbalanced MHD turbulence if the energy spectra of the counter propagating Alfvén waves follow Kolmogorov scaling as predicted by model of imbalanced MHD turbulence developed in Ref. 23. Therefore, obtained result is quite general and will thus be a feature of any study or simulation of *strong* Alfvénic MHD turbulence.

In addition, Eq. (15) has an important corollary. For symmetric spectra $[E^{\pm}(k_\parallel, \mathbf{q}) = E^{\pm}(-k_\parallel, \mathbf{q})]$ the production term is real, and therefore there can be no direct generation of the antisymmetric part of the cross correlation $E^S_{\mathbf{k}}$. Moreover, for the symmetric energy spectra, the generation term of the residual energy [the real part of the first term on the rhs of Eq. (14)] is also symmetric and therefore generated spectrum of the residual energy is also symmetric. If so, the imaginary part of Eq. (15)

$$\partial_t \mathcal{E}^S_q + 2\int \mathrm{d}k_\parallel \omega_{\mathbf{k}} E^R_{\mathbf{k}} = \mathrm{Im}[NL_2(\mathcal{E}^{\pm}, \mathcal{Q})] \quad (23)$$

shows that even after the residual energy is generated, no $\mathcal{E}^S_q$ will be generated at all. Indeed, it is straightforward to show that both second term on the lhs as well as the nonlinear term on the rhs of Eq. (23) proportional to $E^{\pm}E^R$ are identically zero for the symmetric spectra $[E^{\pm,R}(k_\parallel, \mathbf{q}) = E^{\pm,R}(-k_\parallel, \mathbf{q})]$.

After generation of the residual energy the other terms of Eq. (15) become non-zero, and the balance among different terms in Eq. (15) should lead to the stationary state of the residual energy. In Ref. 14 it was suggested that the stationary solution of the residual energy is provided by the balance between the production and "dissipation" terms [second term on the rhs of Eq. (15), $NL_2(\mathcal{E}^{\pm}, \mathcal{Q})$]. Our analysis supports this idea. Namely, it shows that this is indeed the case, because for strong critically balanced MHD turbulence the linear term [the second term on the lhs of Eq. (15)] is identically zero at least for symmetric spectra. Therefore stationary state can be reached only due to the balance between generation and dissipation terms.

It has to be noted also that because nonlinear interactions in MHD turbulence conserve total energy, it is clear that excessive magnetic energy is generated on the expense of the kinetic energy.

Obtained results are in good agreement with the solar wind observations (e.g., Ref. 9), which show that in the inertial range of the solar wind turbulence there is significant amount of the negative residual energy $[2E^R_{\mathbf{k}}/(E^+_{\mathbf{k}} + E^-_{\mathbf{k}}) \sim -0.4]$ whereas observed level of $E^S_{\mathbf{k}}$ is negligible, thus showing that the influence of the linear term (which in general case, for non-symmetric spectra, could cause mixing of the residual energy and its complex counterpart) is negligible.

III. DISCUSSION

Frisch *et al.*[13] studied solutions of ideal incompressible MHD in statistical equilibrium using standard methods of statistical mechanics. The absolute equilibrium state is described by the Gibbs' distribution. The authors derived the spectrum of the magnetic and kinetic energies in the absolute equilibrium state and showed that the magnetic energy is always greater than or equal to the kinetic energy. Therefore one can expect that this fact is related to the similar feature observed in the non-equilibrium case of developed turbulence.[1] Our analysis shows that this is not exactly the case. Indeed, in the case of the absolute equilibrium state, the magnetic energy dominates the kinetic energy as a consequence of nonzero magnetic helicity. If magnetic helicity is absent then the spectrum of the residual energy is identically zero.[13] On the other hand, our analysis shows that the generation of negative residual energy is not directly related to the presence/absence of the magnetic helicity in the system.

Grappin *et al.*[14] were first to derive the dynamical equation for the residual energy for strong isotropic MHD turbulence. The structure of their equation is quite similar to our Eq. (14) in the sense that it also contains a "generation term" (proportional to the $E_{\mathbf{k}_1}E_{\mathbf{k}_2}$) and a "dissipation term" (proportional to the $E_{\mathbf{k}_1}E^R_{\mathbf{k}}$). The stationary solution of this equation was found in Ref. 14 assuming the balance between the generation and the dissipation terms. Taking into account that in the isotropic case, the dissipation term should be dominated by nonlocal interactions (related to the Alfvén effect), this kind of analysis led these authors to the conclusion that in the stationary state, the one dimensional residual energy spectrum should be $E^R(k) \sim k^{-2}$. Applying a similar scaling analysis to Eq. (15) and noting that in the case of critically balanced MHD turbulence both the generation and the dissipation terms are dominated by the local interactions, one can readily derive that the stationary solution should have the same spectral index as the total energy, i.e., –5/3 for the one dimensional spectrum. Detailed analysis of the residual energy spectrum predicted by the Eq. (15) as well as comparison with the solar wind observations and recent numerical simulation will be presented elsewhere.

Recently, Ref. 17 studied the dynamics of the residual energy in weak MHD turbulence using a simplified model of weakly colliding Alfvén waves and demonstrated that as a result of these interactions, negative residual energy is generated. Solar wind *in situ* observations,[6–11] where MHD turbulence is known to be strong, as well as numerical simulations of strong MHD turbulence[12,15] show that in the inertial range of turbulence the residual energy is always negative. Thus, our analytical results offer a natural explanation of the observed residual energy found in the solar wind and numerical simulations of the strong MHD turbulence.

IV. CONCLUSIONS

In this paper we derive the equation governing evolution of the residual energy for strong, critically balanced MHD turbulence in the framework of the EDQNM approximation. We show that in stationary critically balanced MHD turbulence, the nonlinear interaction of Alfvén waves always generates negative residual energy. Since in strong critically balanced MHD turbulence is dominated by perpendicular cascade and follows Kolmogorov phenomenology, we can conclude that the derived feature of the residual energy is a consequence of *only* the nonlinear interaction of Alfvén waves, i.e., formally speaking the specific feature of the matrix element of interaction T_{12}.

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