

Scaling and commonality in anomalous fluctuation statistics in models for turbulence and ferromagnetism

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Abstract

Recently, Portelli *et al* (2003), *Phys. Rev. Lett.* **90** 104501 have semi-numerically obtained a functional form of the probability distribution of fluctuations in the total energy flow in a model for fluid turbulence. This follows earlier work suggesting that fluctuations in the total magnetization in the 2D XY model for a ferromagnet also follow this distribution. Here, starting from the scaling ansatz that is the basis of the turbulence model we analytically derive the functional form of this distribution and find its single control parameter that depends upon the scaling exponents and system size of the model. Our analysis allows us to identify this explicitly with that of the XY model, and suggest a possible generalization.

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1. Introduction

Scaling is an important feature of natural phenomena, arising in many degree of freedom systems that are highly correlated. In reality these systems support a finite range of scales, from the microscopic to the system size. If the number of degrees of freedom is sufficiently large, these systems will still fall into the framework of critical phenomena [1]. Such ‘inertial’ [2] systems include a disparate range of phenomena and, non-intuitively, have recently been suggested to have a common signature in the statistics of fluctuations in global measures of activity. This probability distribution function (PDF) has been compared numerically [2] for a range of models including some for out of equilibrium critical phenomena, notably a sandpile, a forest fire model, a depinning model and a stacking model for granular media. After normalization to the first two moments these PDF were found to collapse onto that of fluctuations in models for equilibrium critical phenomena.

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The functional form of this curve has been identified semi-numerically [3] with the distribution [2]

$$P(y) = K e^{a_s(u - e^u)}, \quad u = b(y - s) \quad (1)$$

where the constants K , b and s are fixed by setting the moments $M_0 = 1$, $M_1 = 0$ and $M_2 = 1$, leaving a single parameter, a_g .

The statistics of fluctuations in global quantities have been explored both experimentally and theoretically for fluid turbulence in closed systems. In the experiments of Labbé *et al* [4, 5], the normalized PDF of fluctuations in the power provided to both rotor blades stirring a closed cylinder of gas at constant angular frequency was also found to have collapsed onto (1) over a range of Reynolds number R_e . These results have been compared with the PDF of fluctuations in the total magnetization in the 2D XY model for a ferromagnet [6] which also has been identified with (1) [3]. In the experiment reported by [7] (their figure 2), the normalized PDF of fluctuations in wall pressure, rather than injected power, appears to follow these non-Gaussian statistics with insensitivity to Reynolds number [7]. Recently, [8] treated a model for closed turbulence and obtained a family of curves of form (1) semi-numerically for global fluctuations of kinetic energy, for which within a sign the experimentally measured total power is believed to act as a good proxy [8].

There is a continuing debate (see [9–16] and references therein) concerning the origin of this apparent universality [6, 2]. The aim of this paper is not to establish the existence, or lack thereof, of a universality class. Furthermore, we do not address the correspondence between experimentally measured quantities and those captured by models for turbulence (see [8]).

Here, we analytically derive (1) for the model for intermittent turbulence in a finite sized system treated by [8]. We obtain a_g as a function of the model parameters. The analysis then leads to a direct identification with results obtained previously for the 2D XY model [17], elucidating the origin of the value $a_g \sim \pi/2$ obtained for that system [6, 2]. We then suggest that the features of the model that are intrinsic to this calculation are rather generic and discuss how they may encompass the wide variety of systems which have also been previously identified as exhibiting the same functional form for the fluctuation PDF [2].

2. Model for turbulence in a finite sized system

Portelli *et al* [8] obtained (1) semi-numerically for intermittent turbulence in the framework of the KO62 hypothesis which models fluctuations in the energy in the flow.

The essential features of this model are structures on a range of length scales $l_1 \dots l_j \dots l_N$ from a smallest size $l_1 = \eta$ to the system size $l_N = L$, corresponding to the dissipation and driving length scales, respectively. The Reynolds number of the flow is then defined as $R_e = (L/\eta)^{4/3}$ [18], and the ratio between successive lengthscales $(l_j/l_{j-1}) = \lambda^{1/3}$ so that $\lambda^N = (L/\eta)^3$.

Following [8] we wish to calculate the statistics of the total energy in the flow

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \dots + \varepsilon_j(t) \dots + \varepsilon_N(t) \quad (2)$$

from a model expressed in terms of an intermittent energy transfer rate ε_j on lengthscale l_j drawn from a PDF which has moments

$$\langle \varepsilon_j^q \rangle = \varepsilon_0^q \left(\frac{l_j}{L} \right)^{-\mu(q)} \quad (3)$$

with the condition $\langle \varepsilon_j \rangle = \varepsilon_0$ which fixes $\mu(1) = 0$. It follows that the standard deviation

$$\sigma_j^2 = \frac{\varepsilon_0^2}{v_j} = \varepsilon_0^2 [(l_j/L)^{-\mu(2)} - 1]. \quad (4)$$

The individual ϵ_j are assumed independent, each with PDF $P_j(\epsilon_j)$, giving

$$\begin{aligned}
 P(\epsilon) &= \int \delta\left(\epsilon - \sum_{j=1}^N \epsilon_j\right) \prod_{l=1}^N P_l(\epsilon_l) d\epsilon_l \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\epsilon} dk \prod_{l=1}^N \hat{P}_l(k)
 \end{aligned}
 \tag{5}$$

where $\hat{P}_l(k) = \int P_l(\epsilon_l) d\epsilon_l \exp(-ik\epsilon_l)$.

In the framework of KO62 this scaling system supports a cascade from large to small scales, with the intermittency parameter $\tau(2) = -\mu(2)$. Importantly, although we can envisage a cascade, the above model does not explicitly require one and will map onto other models provided that the basic assumptions, namely of ϵ_j that are independent and drawn from a PDF with the scaling property (3), (4), hold.

The P_j will depend upon the details of the system, to make progress we first consider a tractable choice, the χ^2 distribution

$$P_j(\epsilon_j) = A_j \epsilon_j^{v_j-1} e^{-a_j \epsilon_j}
 \tag{6}$$

and later explore how this may be generalized. The influence of the microscopic distribution P_j has been explored in the context of the 2D XY model in [19]. This choice for $P(\epsilon_j)$ was used to evaluate (5) semi-numerically in [8]; we will now evaluate it analytically and as a corollary directly obtain the solution previously obtained semi-numerically for the 2D XY model [3].

3. Evaluating the characteristic function

The constants A_j , v_j and a_j of (6) are fixed through (4) so that equation (5) can be written as [8]

$$P(\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\epsilon} dk e^{-S_N}
 \tag{7}$$

where

$$S_N = \sum_{j=1}^N \frac{1}{f_j} \ln(1 + ik\epsilon_0 f_j) = \sum_{j=1}^N \frac{1}{f_j} \ln\left(1 + ik f_j \frac{\beta\sigma}{N}\right)
 \tag{8}$$

where

$$f_j = 1/v_j = \exp(j\bar{a}) - 1
 \tag{9}$$

$\bar{a} = (\mu(2)/3) \ln \lambda$ and where we define total variance as

$$\sigma^2 = \sum_{j=1}^N \sigma_j^2 = \epsilon_0^2 \sum_{j=1}^N f_j
 \tag{10}$$

and $\beta = N\epsilon_0/\sigma$. For a specific system, if one has the values of $\mu(2)$ and λ , and that (6) is a good approximation for $P_j(\epsilon_j)$, one can evaluate (8) numerically [8]. Here, however, we wish to establish why (1) appears to also describe the 2D XY model. We proceed by analytically evaluating (8), and begin with S_N . By formal expansion

$$\frac{dS_N}{dk} = i\frac{\beta\sigma}{N} \sum_{n=0}^{\infty} \left(-ik\frac{\beta\sigma}{N}\right)^n \sum_{j=1}^N f_j^n.
 \tag{11}$$

We need to find $F_n = \sum_{j=1}^N f_j^n$ with the condition that $F_0 = N$ and $F_1 = \sum_{j=1}^N f_j = (N/\beta)^2$. From our definition of the f_j we can evaluate F_1 and obtain

$$\exp(N\bar{a}) = 1 + (N + N^2/\beta^2)(1 - \exp(-\bar{a})). \quad (12)$$

In the limit of $N/\beta^2 > 1$, that is, $N \gg 1$, so that $|\bar{a}| \ll 1$ for finite system size L (i.e. $\lambda \rightarrow 1$) this gives

$$e^{N\bar{a}} \approx 1 + \frac{N^2\bar{a}}{\beta^2}. \quad (13)$$

We can now expand the F_n in $e^{N\bar{a}}$ and substitute (13)

$$\begin{aligned} F_n &= \sum_{j=1}^N (e^{j\bar{a}} - 1)^n = \sum_{j=1}^N e^{nj\bar{a}} - n \sum_{j=1}^N e^{(n-1)j\bar{a}} + \dots \\ &= \frac{e^{nN\bar{a}} - 1}{1 - e^{-n\bar{a}}} - n \frac{e^{(n-1)N\bar{a}}}{1 - e^{-(n-1)\bar{a}}} + \dots \\ &\approx \frac{N^n}{n\bar{a}} \left(\frac{N\bar{a}}{\beta^2} \right)^n + \mathcal{O} \frac{N^{n-1}}{n\bar{a}} \left(\frac{N\bar{a}}{\beta^2} \right)^{n-1} + \dots \end{aligned} \quad (14)$$

and in our limit $N \gg 1$, $|\bar{a}| \ll 1$ such that $N/\beta^2 > 1$ we can take $N|\bar{a}|/\beta^2 \sim 1$ to give to lowest order $F_n \approx (N^n/(n\bar{a}))((N\bar{a})/\beta^2)^n$ where $n \geq 1$ ($F_0 = N$). Using this in (11) gives

$$\frac{dS_N}{dk} = i\sigma\beta - i\sigma\mathcal{Z} \ln(1 + ik\sigma/\mathcal{Z}) \quad (15)$$

where $\mathcal{Z} = \beta/(N\bar{a})$. This now readily integrates to give

$$S_N(k) = ik\sigma\beta + \psi(k) \quad (16)$$

where

$$\psi(k) = ik\sigma\mathcal{Z} - \mathcal{Z}^2 \left(1 + ik \frac{\sigma}{\mathcal{Z}} \right) \ln \left(1 + ik \frac{\sigma}{\mathcal{Z}} \right). \quad (17)$$

With $\phi = \varepsilon - \epsilon_0 N = \varepsilon - \beta\sigma$ we can write (8) as

$$P(\varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\phi} e^{-\psi(k)} dk. \quad (18)$$

The limit $N\bar{a} \rightarrow 0$, $\mathcal{Z} \rightarrow \infty$, corresponds to retaining terms up to k^2 in (17) and immediately gives a Gaussian distribution for $P(\varepsilon)$. It is tempting to take this as the Gaussian limit of $P(\varepsilon)$, $\bar{a} \rightarrow 0$, or $\mu(2) \rightarrow 0$. However, since we insisted that the PDF $P_j(\epsilon_j)$ are scaling, this limit would yield $\sigma_j \rightarrow 0$ in (4). The case where the $P_j(\epsilon_j)$ all have the same (that is, non-scaling) σ_j , corresponding to $N\bar{a} \rightarrow 0$ as this gives $F_{2\dots N} \rightarrow 0$ above. To evaluate (18) in general we need to retain the property of scaling σ_j , thus excluding this limit.

We evaluate (18) by the method of steepest descent

$$P(\varepsilon) = \frac{e^{\mathcal{Z}^2}}{\sigma\sqrt{2\pi}} e^{-\frac{\phi}{\sigma}(\mathcal{Z} + \frac{1}{2\mathcal{Z}}) - \mathcal{Z}^2 e^{-\frac{\phi}{\sigma\mathcal{Z}}}} \quad (19)$$

which is of form (1) with $a_g = \mathcal{Z}^2 + 1/2$, that is,

$$a_g = \frac{1}{2} + \frac{1}{\bar{a}(e^{N\bar{a}} - 1)} \quad (20)$$

and $u = -\varepsilon(\bar{a}/\epsilon_0) + (N\bar{a} + A)$ with A just given by the normalization constant K .

The present choice of $P_j(\epsilon_j)$, equation (6), gives exactly

$$\hat{P}_j(k) = \frac{1}{(1 + ik\kappa_j)^{\gamma_j}} \quad (21)$$

with the κ_j and γ_j fixed by the moments. Other forms of $\hat{P}_j(k)$ for which (21) is a good (Padé type) approximant will yield a $P(\varepsilon)$ of form (1). We write (21) as

$$\begin{aligned} \hat{P}_j(k) &= \sum_{p=0}^{\infty} \frac{(-ik)^p}{p!} \langle \epsilon_j^p \rangle \\ &\simeq 1 + ik\epsilon_0 - \frac{k^2\epsilon_0^2}{2} \left(1 + \frac{1}{\gamma_j} \right) + \dots \end{aligned} \tag{22}$$

given that $\langle \epsilon_j \rangle = \epsilon_0 = \kappa_j \gamma_j$. All the coefficients in this expansion are fixed if we insist that any P_j that we consider has the same scaling (4) for σ_j , so that $\exp(j\bar{a}) = (1 + 1/\gamma_j)$. For the χ^2 PDF (6) this gives $\gamma_j = \nu_j$ which is exact. For example, one might anticipate that in a correlated system that local fluctuations may be multiplicative. An appropriate model for multiplicative noise is a lognormal PDF

$$P_j(\epsilon_j) = \frac{1}{\sqrt{2\pi}} \frac{1}{\bar{\sigma}_j \epsilon_j} \exp \left[-\frac{(\ln(\frac{\epsilon_j}{\bar{\epsilon}_j}))^2}{2\bar{\sigma}_j^2} \right] \tag{23}$$

which has Padé-type approximant of form (21) fixed by

$$\langle \epsilon_j^p \rangle = \bar{\epsilon}^p e^{\frac{1}{2} p^2 \bar{\sigma}_j^2}, \quad 1 + \frac{1}{\gamma_j} = e^{\bar{\sigma}_j^2}. \tag{24}$$

Thus if this is a good approximant, the lognormal also yields a curve of form (1).

4. Results for the turbulence model

To make a direct comparison with the results of [6, 8] we write (20) in terms of the parameters relevant to the turbulence model

$$a_g = \frac{1}{2} + \frac{3}{\mu(2) \ln(\lambda)} \left(R_e^{\frac{3\mu(2)}{4}} - 1 \right)^{-1} \tag{25}$$

so that a_g depends weakly on the Reynolds number of the flow, on (experimentally determined) $\tau(2)$ and through the logarithm, on the free parameter λ . In figure 1 we show normalized curves for the range $R_e = [10^4, 10^5, 10^6]$ corresponding to $a_g = [7.7 \dots 3.6]$ explored by [5], $\lambda = 2$ [8] and typical values of $\tau(2)$ [18]. Curves for $\tau(2) = -0.25$ are shown in the main plot and for $\tau(2) = -0.2$ (used [8]) in the inset. In both cases these show good correspondence with the solutions from the model (see figures 1 and 2 of [8]). The curves also fall close to each other explaining the relative insensitivity to Reynolds number [8], and also fall close to that for $a_g = \pi/2$ identified by [6] for the XY model, which is also plotted and which we discuss next.

A more sensitive method for identifying a_g from data is to calculate the third moment of (1) directly [20] rather than compare curves. With $M_0 = 1, M_1 = 0, M_2 = 1$:

$$M_3 = -\frac{N\bar{a}}{\beta} = -\sqrt{-\frac{\tau \ln \lambda}{3} \left(R_e^{-\frac{3\tau}{4}} - 1 \right)} = -\left(a_g - \frac{1}{2} \right)^{-\frac{1}{2}} \tag{26}$$

so its magnitude increases with decreasing a_g which from (25) corresponds to increasing R_e , consistent with data. The handedness is not determined here; this depends upon how the global quantity measured experimentally relates to that of the model, namely the energy flow within the fluid [8].

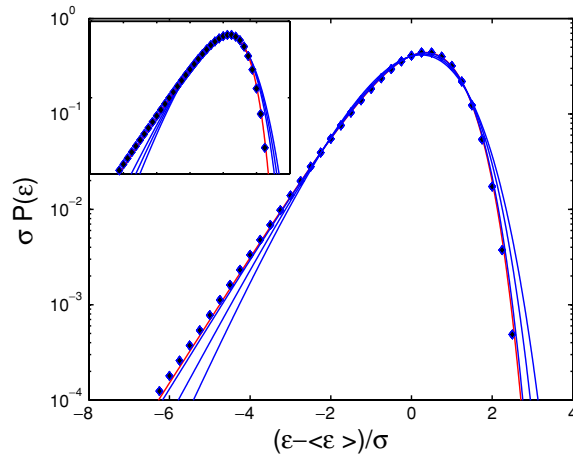


Figure 1. Function (1) is plotted for $a_g = [4.24, 2.76, 1.90]$ (blue lines), $a_g = 1.74$ (red line) and $a_g = \pi/2$ (diamonds) relevant to the model [8] for a closed turbulence experiment, the 2D XY model and [2], respectively. The inset shows the same axis ranges and replaces the blue curves with $a_g = [7.76, 5.18, 3.62]$ relevant to [8]. The handedness is chosen for comparison with [2, 6, 8].

(This figure is in colour only in the electronic version)

5. The 2D XY model

The total magnetization in the spin-wave approximation to the 2D XY model was also shown for a range of system sizes [6] to collapse onto (1), with $a_g \simeq \pi/2$ (for a study of the full 2D XY model see [21]). The PDF of total magnetization M can be written as [17]

$$Q(M) = \int_{-\infty}^{\infty} \frac{dt}{2\pi\sigma} e^{-it \frac{M-\langle M \rangle}{\sigma}} e^{\mathcal{S}} \tag{27}$$

where

$$\mathcal{S} = \sum_{p=2}^{\infty} \left(-it \sqrt{\frac{2}{a_2}} \right)^p \frac{a_p}{2p}, \quad a_p = \frac{1}{N_s^p} \sum_{q \neq 0} \frac{1}{\gamma_q^p}$$

and γ_q specifies the lattice Green's function. Bramwell *et al* [3] evaluated this numerically to demonstrate that it is well described by (1). We now show that the sum \mathcal{S} is related to the sum S_N (8) by writing

$$\begin{aligned} -S_N &= -\sum_{p=1}^N \frac{\ln(1 + ik\epsilon_0 f_p)}{f_p} = \sum_{p=1}^N \frac{1}{f_p} \sum_{m=1}^{\infty} \frac{(-ik\epsilon_0 f_p)^m}{m} \\ &= \sum_{m=1}^{\infty} \frac{(-ik\epsilon_0)^m}{m} F_{m-1} = -ik\langle \epsilon \rangle + \sum_{p=2}^{\infty} \frac{(-ik\epsilon_0)^p}{p} F_{p-1} \end{aligned} \tag{28}$$

where $\langle \epsilon \rangle = N\epsilon_0$. If we then make the identification

$$F_{p-1}\epsilon_0^p \equiv \frac{a_p}{2} \left(\sqrt{\frac{2}{a_2}} \right)^p \tag{29}$$

then $Q(M)$ (27) has the same functional form as $P(\epsilon)$ so that it also shares the same distribution (1) to within the approximations made here, namely, that following expansion (14) we have

neglected terms of order $1/N$, and that (27) is also an approximation, good for N large. Importantly, from the definition of a_p in (27), the rhs of (29) is independent of N_s . In this sense we have approximately evaluated integral (27) in the thermodynamic limit.

It then just remains to estimate a_g for the 2D XY model. In [3] this was approximated asymptotically, here we simply note that insisting that the normalized $Q(M)$ and $P(\varepsilon)$ share the first three moments yields $a_g = 1/2 + (a_2/2)^3/a_3^2$ from (26) and equation (21) of [3]. They also calculated the normalized third moment for a square lattice. Their value of $M_3 = -0.8907$ gives $a_g = 1.7428$ which will give curves close to those for $a_g = \pi/2$ as shown in figure 1. Our analysis is thus consistent with both a value of $a_g \approx \pi/2$ [2], and the asymptotic exponent of [3].

6. Generalization

A variety of disparate systems have recently been shown numerically [2, 6] to have a common signature in the statistics of fluctuations in a global measure of activity which is of form (1). These include out of equilibrium critical phenomena, notably a sandpile, a forest fire model, a depinning model and a stacking model for granular media.

We will now argue that the scaling ansatz which was our starting point for the model for fluid turbulence in section 2 and our derivation of (1) may also encompass these disparate systems.

The ansatz we chose corresponds to that of a scaling system that generates spatial structures or domains (patches) on length scales $l_1 \dots l_j \dots l_N$ from a smallest size $l_1 = \eta$ to the system size $l_N = L$. ‘Length scale’ in this more general sense means ‘appropriate characteristic measure’ i.e. length in one dimension, area in two dimensions or volume in three dimensions. In a dynamical out of equilibrium system, such as a sandpile or a forest fire model, a steady state is achieved by driving on the smallest length scale $l_1 = \eta$ and by means of open boundaries, removing structures on the system size L . In section 2 we considered a system driven on the largest scale L and dissipating on the smallest, mapping onto fluid turbulence in a closed system. In a model for a ferromagnet, the system may fluctuate about an equilibrium, but nevertheless has a minimum patch size (one spin), a maximum patch size (the system size) and scaling of patches in between.

The global quantity ε is now taken to be associated with the total number of instantaneously active sites within each patch. In a model realized numerically, such as a forest fire or avalanche model, instantaneously active sites are those seen at a given timestep in the computation. In a forest fire model, active sites correspond to burning trees, in an avalanche model, to relaxing sites in evolving avalanches [2]. The global quantity may refer to the energy dissipated by these sites, or simply refer to the time evolution of their spatial distribution as in the case of space occupied by anisotropic particles settling under gravity or magnetization of spins in a ferromagnet [2].

The common feature of these systems is that at any instant in time there will be $m_j(t)$ patches on any length scale l_j and associated with each patch, ε_j^* of this quantity. On each length scale l_j we then have $\varepsilon_j = m_j(t)\varepsilon_j^*$ and in total, ε given by (2). We now assert that in common with the turbulence model, the ε_j^* are independent and have intermittent, scaling statistics (3). We can then envisage the following generic scaling system which comprises

(I) *Non-space filling, intermittent patches.* The details of $\langle m_j^q \rangle$ depend on the system, for example the probability of patches l_{j-1} merging, and/or patches l_{j+1} breaking up to form patches on l_j . We take as a necessary condition of scaling that the moments obey

$$\langle m_j^q \rangle l_j^{\gamma(q)} = \langle m_{j-1}^q \rangle l_{j-1}^{\gamma(q)} = \langle m_N^q \rangle L^{\gamma(q)}. \quad (30)$$

If the system were space filling, $\gamma(1)$ would be 1 so $\gamma(1) < 1$ implies non-space filling patches. Allowing $\gamma = \gamma(q)$ permits intermittency.

(II) *Fractal support.* On any patch there will be a density of active sites ϵ_j^*/l_j which in general can vary with l_j ; for a system which is scaling we can however take

$$\frac{\epsilon_j^*}{l_j^\alpha} = \frac{\epsilon_{j-1}^*}{l_{j-1}^\alpha} = \frac{\epsilon_N^*}{L^\alpha} \quad (31)$$

where $\alpha = 1$ is the special case of uniform density on all patches, and patches that do not have fractal boundaries.

(III) *Conservation.* Scaling implies that there is no preferred l_j on which the active sites accumulate so that the mean will be just the ensemble average determined on any length scale. This is consistent with conservation of active sites when patches merge ($l_j \rightarrow l_{j+1}$) or break up ($l_j \rightarrow l_{j-1}$).

It follows from (I), (II) and (III):

$$\langle \epsilon_j^q \rangle = (\epsilon_N^*)^q \langle m_N^q \rangle \left(\frac{l_j}{L} \right)^{(\alpha q - \gamma(q))} = \epsilon_0^q \left(\frac{l_j}{L} \right)^{-\mu(q)}. \quad (32)$$

The condition $\langle \epsilon_j \rangle = \epsilon_0$ fixes $\gamma(1) = \alpha$ or $\mu(1) = 0$. The details of the system specify $\mu(2)$ which then fixes the standard deviation of (6) expressed through (32) and immediately leads to (4).

(IV) *Finite size.* We finally specify the number of length scales N ; given scaling, a choice is constant $(l_j/l_{j-1}) = \lambda^{\frac{1}{3}}$ so that $\lambda^N = (L/\eta)^3$.

In summary then, this scaling ansatz is that $\langle \epsilon_j \rangle = \epsilon_0$, $\langle \epsilon_j^2 \rangle = \epsilon_0^2 (l_j/L)^{-\mu(2)}$ with $\mu(2) \neq 0$ and $(l_j/l_{j-1})^3 = \lambda$. Any system that is specified by this ansatz and is well approximated by (21) will share the same behaviour (1) in the statistics of global activity $P(\epsilon)$ that we have calculated above for the turbulence model. Importantly, these conditions may apply to more than one quantity in a given system, and any such quantity will share these same statistics.

For a given system, curve (1) is specified by a_g which is a function of the system parameters N , $\mu(2)$ and λ . This family of curves is however insensitive to a_g [20]. This, combined with the practical difficulty of obtaining good statistical resolution over fluctuations ranging over several orders of magnitude suggests a straightforward reason for the close, but not exact, curve collapse that has been reported in figure 2 of [2]. Importantly, we do not extend this argument to the 2D XY model; rather in this case we have utilized the correspondence of (7) with the result of [17] (equation (27)).

7. Summary

From the starting point of a model for fluid turbulence in a finite sized system, previously treated semi-numerically by [8], we have analytically derived the functional form of the PDF of global energy flow in the system. This yielded the dependence of its single control parameter a_g on the intermittency parameter, the ratio between lengthscales, and the smallest and largest scale lengths in the system (i.e. the Reynolds number). We then directly identified this function with that previously obtained for fluctuations in total magnetization in the 2D XY model and thus elucidated the origin of the previously identified value $a_g \sim \pi/2$ [2]. The PDF was shown to be relatively insensitive to variations in a_g , explaining the previously reported close correspondence of these curves for the turbulence model and the 2D XY model [6, 8].

Importantly, the functional form of the PDF that we derive is just that of the sums of a large but finite sets of independent numbers drawn from PDF with moments that are scaling. This corresponds to a model of intermittent turbulence in which one also envisages a cascade, but

the cascade property is not intrinsic to the calculation. We suggest that this system is rather generic and may encompass the wide variety of systems which have also been previously identified as exhibiting the same functional form for the fluctuation PDF [2].

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